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Introduction to Chevalley gps

Thanks to organizers.
Looking forward to meeting my gp, & to the other members' lectures.

19th C: Lie introduces Lie gps & alg's, proves some fundamental thms, recognizes role of solvability.
Killing identifies the key structure [Cartan subalg.] that allows a uniform understanding of the simple Lie alg's, & enumerates them. Encouraged to publish despite not having proven many things. (incl. finding exceptional gps G_2, F_4, E_6, E_7, E_8)
SL_n, SO_n, Sp_{2n} & exceptionals.

Early 20th C: E. Cartan develops Killing's work & provides pf of his assertions. Key technical tool is a symmetric b.f. for a Lie alg $\mathfrak{g}$, the "Killing form". Was his Ph.D. thesis — exceptionally long & delicate computations. Not read widely.

19th C: "Lie gps" over finite fields, like $SL_n(\mathbb{F}_q)$ & orthogonal & symplectic gps over finite fields, are one of the standard objects of gp thy, which @ that time meant finite gps. No connection to Lie alg, since a finite gp has no tangent space.

Early 20th C: L.E. Dickson looks for & finds versions of two of the exceptional gp's G_2 & E_6, over finite fields. $G_2(q) = \mathrm{Aut}$ (a version of the Cayley #s over $\mathbb{F}_q$).
$E_6(q) = \mathrm{Aut}$ (a certain 27-dim cubic form over $\mathbb{F}_q$).
Proves the simplicity of these gps (except for $G_2(2)$, where index 2 subgp is simple).
Impressive but still ad hoc construction.

1955: Chevalley's Tohoku paper: unified these two threads by giving uniform construction of all the simple Lie gps over $\mathbb{F}_q$, and in fact over any commutative ring.

This set the stage for the '50's & 60's development of the thy of alg. gps. An alg gp is a (over a field k) variety G over k equipped w/ mult. map $G \times G \to G$, inversion map $G \to G$ & identity elt $\{pt\} \to G$, satisfying the usual gp axioms.
all morphisms $\mathit{defined}$ over k,

The structure thy of alg. gps gives exactly the same simple gps as Killing found.

The structure theory necessary to obtain the classification, even over alg. closed k, is much more involved than for Lie alg, essentially because the Lie alg. of an alg. gp. doesn't capture enough info about G. E.g., if char $k = 2$, $k \neq \mathbb{F}_2$, then $SL_2 k$ is a simple alg. gp. but its Lie alg $\mathfrak{sl}_2 k$ is not simple. Nevertheless the answer can be put in very clean form & this is our purpose today. ~~The description we'll give is due to Frenkel & Kac, who simplified work of~~ ~~J. Tits.~~ (later)

Suppose G is a connected simple complex Lie gp. i.e. has no normal solvable subgp of dim > 0. E.g. $G = SL_n(\mathbb{C})$. $[a,b] = ab - ba$. $\mathfrak{g}$ its Lie alg, e.g. $\mathfrak{sl}_n(\mathbb{C}) = \{$trace 0 matrices$\}$. Then $\mathfrak{g}$ contains a 'Cartan subalg' $\mathfrak{t}$, a certain sort of max'l commutative subalg; e.g. $\mathfrak{t} = \{$trace 0 diagonal matrices$\}$. What's important about $\mathfrak{t}$ is its action on $\mathfrak{g}$ under the adjoint action

$$\mathfrak{t} \times \mathfrak{g} \to \mathfrak{g} \ : \ (a,b) \mapsto \text{the Lie bracket } [a,b],$$

$\mathfrak{g}$ breaks up into eigenspaces of $\mathfrak{t}$: $\forall\ \lambda : \mathfrak{t} \to \mathbb{C}$, in $\mathfrak{t}^*$,

$$\mathfrak{g}_\lambda := \{ b \in \mathfrak{g} : [a,b] = \lambda(a)\, b \quad \forall\ a \in \mathfrak{t} \};$$

only finitely many are $\neq 0$;

$$\mathfrak{g} = \underset{\substack{\text{Cartan} \\ \text{called } \mathfrak{g}_0}}{\mathfrak{t}} \oplus \Big(\underset{\text{finitely many}}{\bigoplus} \mathfrak{g}_\lambda \Big)$$

called the root spaces of $\mathfrak{g}$

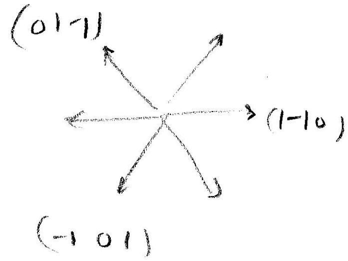
For $\mathfrak{g} = \mathfrak{sl}_n$, the $\mathfrak{g}_\lambda$'s are spanned by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $e_{ij} = \begin{cases} 1 \text{ in spot } (i,j) \\ 0\text{'s elsewhere} \end{cases}$ $i \neq j$,

$$[\operatorname{diag}(a_1, \ldots, a_n), e_{ij}] = (a_i - a_j)\, e_{ij}.$$

So $\exists$ ~~finitely many weight spaces~~ $n(n-1)$ root spaces corr. to the lin func. $\operatorname{diag}(a_1, \ldots a_n) \xmapsto{\lambda_{ij}} a_i - a_j$ on $\mathfrak{t}$. The best way to organize this data is to imagine the set of roots in $\mathfrak{t}^*$,

$$\lambda_{ij} = (0 \cdots 0\, \underset{\substack{\uparrow \\ i\text{th}}}{1}\, 0 \cdots 0\, \underset{\substack{\uparrow \\ j\text{th}}}{-1}\, 0 \cdots 0) \in \underset{\substack{\text{of coord} \\ \text{sum } 0}}{\mathbb{C}^{n-1}} \subseteq \mathbb{C}^n$$

For $\mathfrak{sl}_3$ ch:



So $\mathfrak{t}$ & the root spaces

Patterns that persist for any simple $\mathfrak{g}$:
each root space is 1-diml,

$$[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}, \text{ e.g.}$$

$$[e_{12}, e_{23}] = e_{13}$$

$$\Big[\underset{\lambda}{\longrightarrow}, \underset{\mu}{\nwarrow}\Big] \qquad \Big[\underset{\lambda+\mu}{\nearrow}\Big]$$

[Have almost complete info about mult. table.]

Chevalley took each of Killing's root systems & wrote down a Lie alg over an arbitrary field k, & then constructed his "Lie gp" as a gp of auts of it. Applying his constr. to $\mathfrak{sl}_n k$ ~~construct~~ gives $SL_n k$. In principle it's easy to build auts of a Lie alg $\mathfrak{g}$ over k, because every $a \in$ ~~elt of~~ $\mathfrak{g}$ defines a derivation $\mathrm{ad}_a : b \mapsto [a, b]$ of $\mathfrak{g}$ (this is the Jacobi identity), which is to say a infinitesimal aut of $\mathfrak{g}$. To get an auto. from an infinitesimal aut you exponentiate:

$$b \mapsto \big(\underbrace{\exp(\mathrm{ad}_a)}_{1 + \mathrm{ad}_a + \frac{\mathrm{ad}_a^2}{2} + \frac{\mathrm{ad}_a^3}{3!} + \cdots}\big)(b)$$

2 problems: ① in arbitrary k, $\nexists$ notion of convergence. Solution: only exponentiate the nilpotent derivations. e.g. $e = e_{ij} \in \mathfrak{sl}_n$ has $(\mathrm{ad}_e)^3 = 0$. So

$$(\star) \quad \exp(\mathrm{ad}_e) = 1 + \mathrm{ad}_e + \frac{\mathrm{ad}_e^2}{2} + 0 + 0 + \cdots$$

makes perfect sense.

Except that ② denoms might be 0, e.g. in char 2. Solution (& this is the amazing part of Chevalley's insight): when defining $\mathfrak{g}$ from the root system, take care that 2 divides ad_e^2, so that it can be divided by 2 & there's no denominator. e.g.

$$\mathrm{ad}_e^2 : e_{i'j'} \longrightarrow 0 \quad \text{if } (i'j') \neq (j i)$$

$$\mathfrak{t} \longrightarrow 0$$

$$e_{ji} \longrightarrow -2 e_{ij}$$

so $(\star)$ in fact has coeffs in $\mathbb{Z}$, hence gives an aut of $\mathfrak{g}$ in any char.

pictorially: e_{ji}  e_{ij}

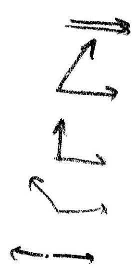
The construction

$\boxed{\text{Step 1}}$: Begin with the root system. They have $A_{n\geq 1}$, $B_{n\geq 2}$, $C_{n\geq 3}$, $D_{n\geq 4}$, E_6, E_7, E_8, F_4, G_2. We'll do only A_n, D_n & E_n since the others have complications which are more annoying than essential.

A_n we've already seen:

	L	roots (min. vecs, norm 2)	
A_n lattice	$\{v \in \mathbb{Z}^{n+1} : \sum v_i = 0\}$	$(1\,-1\,0\cdots0)$ & perm.	$(n)(n+1)$
D_n lattice	$\{v \in \mathbb{Z}^n : \sum v_i \text{ even}\}$	$(\pm1\,\pm1\,0\cdots0)$ & perm.	$\binom{n}{2}2^2$
E_8	$\{v \in (\frac{1}{2}\mathbb{Z})^8 : v_i \equiv v_j \pmod 1,\ \sum v_i \text{ even}\}$	those of D_8 together with $(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2})$, (evenly many signs)	(240)
E_7	$\text{root}^\perp \subseteq E_8$		(126)
E_6	$(\text{root, root w/ ip} = 1 \text{ with it})^\perp \subseteq E_8$		(72)

All of these have property that the auts of $\mathbb{Z}^n$ preserving the root system act trans on roots (for A_n, D_n this is easy) & any two roots have ip $= 2, 1, 0, -1, -2$ as



ip $= 2$
1 — i.e. they commute
0 — like in sl_3
-1
-2 — like in sl_3.

$\boxed{\text{Step 2}}$ Write down the Lie bracket. Here we follow Frenkel & Kac who developed an idea of J. Tits, who simplified Chevalley's own construction.

take $\mathfrak{t} := ((\text{the root lattice}) \otimes k)^*$ (root lattice $= L$)

$\mathfrak{g}_\alpha :=$ a copy of k indexed by a root α

$$\mathfrak{g} := \mathfrak{t} \oplus \bigoplus_{\text{roots } \alpha} \mathfrak{g}_\alpha$$

Using sl_n as a guide, we should define for $a \in \mathfrak{t}$, $e \in \mathfrak{g}_\alpha$

$$[a, e] = \underbrace{\alpha(a)}_{\text{recall } \alpha \in \text{root lattice defines a lin. func. on } \mathfrak{t}^* \text{ by}} \cdot e$$

we also know that for $e_\alpha \in \mathfrak{g}_\alpha$, $e_\beta \in \mathfrak{g}_\beta$ $[e_\alpha, e_\beta] \in \mathfrak{g}_{\alpha+\beta}$

In particular, since $\mathfrak{g}_{\alpha+\beta} = 0$ unless α & β are arranged a or, we know a lot of the brackets are 0.

Temptation: take a vec $e_\alpha \in \mathfrak{g}_\alpha$ & define

$$[e_\alpha, e_\beta] := \begin{cases} e_{\alpha+\beta} & \text{if } \alpha+\beta \text{ a root} \\ (\text{suitable elt of } \mathfrak{t}) & \text{if } \alpha = -\beta \\ 0 & \text{else} \end{cases}$$

Problem: bracket is antisymmetric, so $[e_\alpha, e_\beta]$ should be $-[e_\beta, e_\alpha]$. But symmetric.
$\therefore$ must introduce signs. Subtle problem, best solved by Frenkel & Kac, who developed an idea of J. Tits who simplified Chevalley's own construction.

Solution: L has a sym. bil. pairing, the inner product. Values in $\mathbb{Z}$. Reduce mod 2 gives a $\mathbb{Z}/2$-valued sym. bil. form, i.e. a $\mathbb{Z}/2$-valued antisymmetric bil. form. This is exactly what's needed to define a central ext

$$1 \to \mathbb{Z}/2 \to \hat{L} \to L \to 1$$

of gps.

Write z for the elt of ord. 2; mult. in $\hat{L}$ is just like in L: if $\hat{x}, \hat{y}$ lie over $x, y \in L$ then we know $\hat{x}\hat{y}$ is one of 2 possibilities, which differ by ~~factor of~~ z. Whatever $\hat{x}\hat{y}$ is, $\hat{y}\hat{x} = (\hat{x}\hat{y})(z^{x \cdot y})$. This lets you check if any given product of elts of $\hat{L}$ is 1, so defines the gp structure completely.

Key point: if $\hat{\alpha}, \hat{\beta}$ lie over roots α, β ($\alpha\cdot\beta = -1$) then $\hat{\alpha}\hat{\beta} = \hat{\beta}\hat{\alpha} \cdot z$.

So if we make $z = -1$ then defining $[\ ]$ using $\hat{L}$'s gp law instead of L's provides the nec. antisymmetry.

Details: let

$$\mathfrak{g} := \mathfrak{t} \oplus \Big(\bigoplus_{\substack{\hat{\alpha} \in \hat{L} \\ \text{over a root} \\ \alpha \text{ of } L}} \mathfrak{g}_{\hat{\alpha}} \Big) \Big/ \text{relations } \left(e_{z\hat{\alpha}} = -e_{\hat{\alpha}}\right)$$

where $\mathfrak{g}_{\hat{\alpha}} :=$ 1-dim over k, spanned by $e_{\hat{\alpha}}$.

bracket:

$$[\mathfrak{t}, \mathfrak{t}] = 0$$

$$[a, e_{\hat\alpha}] = \alpha(a)\cdot e_{\hat\alpha} \qquad \text{(as before)}$$

$$[e_{\hat\alpha}, e_{\hat\beta}] = e_{\hat\alpha\hat\beta} \qquad \text{if } \alpha+\beta \text{ is a root}$$

This is the key defn; $[e_{\hat\beta}, e_{\hat\alpha}] = e_{\hat\beta\hat\alpha} = e_{z\hat\alpha\hat\beta} = -e_{\hat\alpha\hat\beta}$ antisymmetric ✓.

$$[e_{\hat\alpha}, e_{\hat\alpha^{-1}}] = \text{an elt of } \mathfrak{t} = (L\otimes k)^*,$$

namely, "inner product with α"

$$[e_{\hat\alpha}, e_{\hat\beta}] = 0 \qquad \text{otherwise.}$$

Verifying Jacobi id. req's considering ~~This defines visibly inert under~~ $\cancel{\mathrm{Aut}\,\hat L = (\mathbb{Z}/2)^n . \mathrm{Aut}(L)}$, & the $e_{\hat\alpha}$'s are all ad-nilpotent, so they define derivations. The defn of $[e_{\hat\alpha}, e_{\hat\alpha^{-1}}]$ is rigged so that just like for $\mathfrak{sl}_2$

$$(\mathrm{ad}_{e_{\hat\alpha}})^2(e_{\hat\alpha^{-1}}) = -2\, e_{\hat\alpha}, \text{ so } \quad 1 + \mathrm{ad}_{e_{\hat\alpha}} + \tfrac{1}{2}(\mathrm{ad}_{e_{\hat\alpha}})^2 \text{ is an aut of } \mathfrak{g}.$$

Take gp G gen'd by $\{\exp(\mathrm{ad}_{s\cdot e_{\hat\alpha}}) : s \in k\}$

Chevalley & others proved

- G is a simple gp exc. for $A_1(\mathbb{F}_2 \& \mathbb{F}_3)$ & $A_2(\mathbb{F}_2)$
- G is the identity comp of $\mathrm{Aut}(\mathfrak{g})$ & is therefore an alg. gp.
- $A_n(k) = PSL_{n+1}(k)$
  $D_n(k) = PSO_{2n}(k)$ [The stabilizer of $\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}$ (Char $\neq 2$)]
- He found other gps like PSO_{2n+1} as fixed-pt-subgps of $A_n(k)$, e.g. $M \mapsto (M^T)^{-1}$ defines an aut of SL_n with fixed-pt gp a form of O_n.
  This could account for the B_n, C_n, F_4 & G_2 root systems.

  [For the tweaking of this idea gives some 2-gps with no analogue over $\mathbb{R}$ or $\mathbb{C}$, in char 2 & 3.]
- His $G_2(k)$ & $E_6(k)$ coincide w/ Dickson's for $k = \mathbb{F}_q$.
- These gps, & twisted forms of them, & the alternating gps, account for all but 26 of the non-abelian finite simple gps. So the scope of the thy is almost all encompassing.

[If have time:

Even if you were a pure gp-theorist, an algebro-geometric perspective clarifies a great deal about these gps, even beyond the usual "what things gul one $\mathbb{F}_q$ [illegible]" [illegible]]

E.g. $|X_n(q)|$ = polynomial in q of degree $d = \dim X_n(\mathbb{C})$.

$X = A, B, C, D, E, F, G$

This is because X_n is in fact def'd over $\mathbb{Z}$, ~~of~~ dim d over $\mathbb{Z}$.

degree E.g. The most impt. object associated to a simple alg gp is the flag mfld on which it acts. e.g. for $SL_n k$. This is G/B for ~~set of~~ a Borel subgp B; B = normalizer of max'l unipot. subgp. Over $\mathbb{F}_{q = p^n}$, the unip. subgp is a p-Sylow subgp, & one can work out its normalizer as

$$[q^d] \cdot \underbrace{(q-1)^{\text{rank}}}_{\mathbb{F}_q^*}$$

(can vary slightly with choice of SL_n, PSL_n, etc.)

while B. is alg-closed subgp in'l

$B \curvearrowright G/B$ breaks flag variety into (gen'l) cells Schubert cells, union of affine spaces ~~of~~ var. dim d_i. These become sets of size q^{d_i}; so

$$|G| = |B| \cdot |G/B|$$

$$= q^d \cdot (q-1)^{\text{rank}} \cdot \sum_{\substack{\text{Schubert} \\ \text{cells}}} q^{\dim(\text{cell})}$$

Much more can be said, e.g. e.g. conj. classes in alg gp & assoc. finite gp. but ~~now~~ I [illegible] will stop here.]