Daniel Allcock 07-18-2005

Chevalley groups

19th century: • Lie introduced Lie groups and Lie algebras and studies their properties
• Killing identified the key structure necessary for understanding the structure of a Lie group - "Cartan subalgebra". Enumerated the simple Lie algebras:
$\mathfrak{sl}_n, \mathfrak{o}_n, \mathfrak{sp}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4, \mathfrak{g}_2$

20th century: E. Cartan identified the key structure enabling proofs of Killing's classification - the "Killing form".

19th century: Work on "Lie groups" over finite fields: $SL_n(\mathbb{F}_q)$, $O_n(\mathbb{F}_q)$
$Sp_n(\mathbb{F}_q)$
People showed that
$\begin{pmatrix}\text{commutator}\\ \text{subgroup}\end{pmatrix} \Big/ (\text{center})$
is a finite simple group except for $SL_2(\mathbb{F}_2, \mathbb{F}_3)$ and $O_3(\mathbb{F}_2, F_3)$

Early 20th century : L.E. Dickson defined analogues of G_2 and E_6 over $\mathbb{F}_q$.

$$G_2(k) = \mathrm{Aut}\begin{pmatrix}\text{Cayley algebra} \\ \text{over } k\end{pmatrix}$$

$$E_6(k) = \mathrm{Aut}\begin{pmatrix}\text{certain cubic} \\ \text{form in 27} \\ \text{variables}\end{pmatrix}$$

(ad hoc construction).

1955: Chevalley's Tohôku paper defines "Lie groups" uniformly over any commutative ring.

This started the theory of algebraic groups in earnest. This was a very active field in 1950's and 1960's. Now is a tool for applications.

Chevalley groups and slight modifications give almost all of the finite simple groups:

(FSG) = {Chevalley groups} ∪ {alt. grps} ∪ {cyclic grps of prime order} ∪ {26 sporadic simple grps}

The structure theory for algebraic groups culminates in the theorem asserting that Chevalley's groups are all the simple affine algebraic groups.

<u>Question</u>: How can one describe all affine algebraic groups (say over finite field)?

If G is a complex semi-simple Lie group with Lie algebra $\mathfrak{g}$

$$\left(\begin{array}{l} \text{e.g. } \mathfrak{g} = \mathfrak{sl}_n = \{ v \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \mathrm{tr}\, v = 0 \} \\ \qquad G = SL_n \end{array}\right)$$

Then $\mathfrak{g}$ contains a "Cartan subalgebra" - this is a special kind of a maximal abelian subalgebra $\mathfrak{t} < \mathfrak{g}$.

Under the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$:

$$x \in \mathfrak{t} \mapsto \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$$
$$y \to [x, y]$$

$\mathfrak{g}$ breaks into eigenspaces:

For any linear map $\alpha : \mathfrak{t} \to \mathbb{C}$ define

$$\mathfrak{g}_\alpha = \{ v \in \mathfrak{g} \mid [x, v] = \alpha(x) v \quad \forall x \in \mathfrak{t} \}$$

Then

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\substack{\text{finitely} \\ \text{many } \alpha \in \mathfrak{t}^*}} \mathfrak{g}_\alpha$$

<u>Example</u>: For $\mathfrak{g} = \mathfrak{sl}_n$ all the $\mathfrak{g}_\alpha$'s are 1-dimensional and are spanned by

$$e_{ij} = \begin{pmatrix} 1 \text{ in place } (ij) \\ 0 \text{ everywhere else} \end{pmatrix}$$

$$\mathfrak{t} = \begin{pmatrix} \text{all diagonal matrices} \\ \text{in } \mathfrak{sl}_n \end{pmatrix}$$

$\alpha \in \mathfrak{t}^*$ for which $\mathfrak{g}_\alpha \neq 0$ are

$$\alpha_{ij} : \operatorname{diag}(a_1, \dots, a_n) \to (a_i - a_j)$$

$$[\operatorname{diag}(a_1, \dots, a_n), e_{ij}] = (a_i - a_j) e_{ij}$$

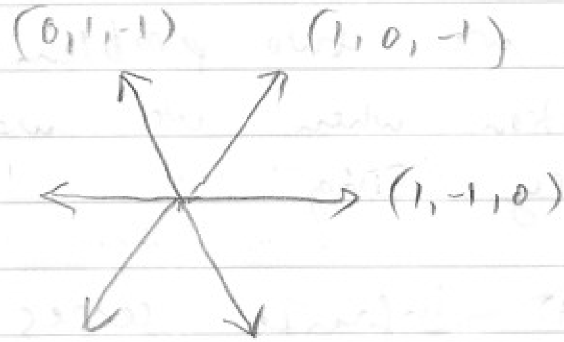
The <u>root system</u> associated to $\mathfrak{g}$ is the set of all $\alpha \neq 0$ for which

$$\mathfrak{g}_\alpha \neq 0$$

<u>Example</u>: For $\mathfrak{sl}_n$ it is

$$\{(1,-1,0,\dots,0) \text{ \& permutations}\}$$

e.g. $\mathfrak{sl}_3$:



Chevalley's idea was to write Lie groups over finite fields as automorphisms of Lie algebras over these fields

Writing down all automorphism of a Lie algebra is easy (in principle)

Indeed Jacobi's identity implies that $\forall x \in \mathfrak{g}$ the map $ad_x : \mathfrak{g} \to \mathfrak{g}$

is a derivation of $\mathfrak{g}$ over the base field.

However

$$\left(\text{derivations}\right) = \left(\begin{array}{c}\text{infinitesimal}\\ \text{automorphisms}\end{array}\right)$$

$\Rightarrow$ we can produce automorphisms as

$$\exp(ad_X) = \sum \frac{1}{k!} (ad_X)^k$$

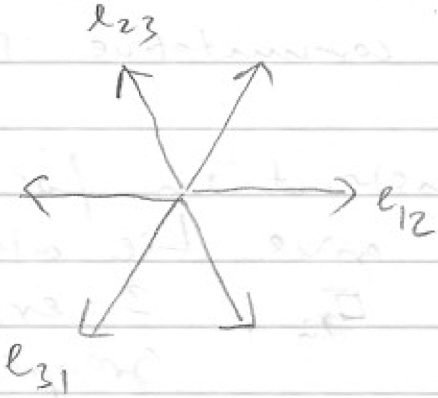
There are two problems with this prescription when we work over an arbitrary ring:

(1) infinite series do not make sense (no notion of convergence intrinsically).

This can be circumvented if we only exponentiate ad_X that happen to be nilpotent as linear transformations of $\mathfrak{g}$.

In this case the exponential will be a finite sum $\Rightarrow$ makes sense.

Example :



for $\mathfrak{sl}_3$

$$(\mathrm{ad}_{e_{ij}})^3 = 0 \qquad \forall\, i,j$$

(2) the exponential series does not make sense in finite characteristic (some denominators become zero)

This was circumvented by Chevalley who finessed the choice of generators of $\mathfrak{g}_{\mathbb{Z}}$ so that the formula for $(\mathrm{ad}_{e_\alpha})^2$ contains a factor of 2 and similarly for $(\mathrm{ad}_{e_\alpha})^n$ for $n>0$.

Example: $(\mathrm{ad}_{e_{12}})^2$ in $\mathfrak{sl}_3$ kills all of $\mathfrak{sl}_3$ except for e_{21}

Compute

$$[e_{12}, [e_{12}, e_{21}]] = -2\, e_{12}$$

$\Rightarrow$

$$\exp(\mathrm{ad}_{e_{12}}) = 1 + \mathrm{ad}_{e_{12}} + \frac{\cancel{-2}\,(e_{21} \to e_{12}, \text{residues})}{\cancel{2}}$$

This turns out to make sense over any commutative ring:

Chevalley's construction for A_n, D_n, E_n root systems give Lie algebras over $\mathbb{Z}$: $\mathfrak{sl}_n$, $\square_{2n}$, 3 exceptional groups of type E.

name	lattice L	roots
$A_{n\geq 1}$	$\{v \in \mathbb{Z}^{n+1}, \sum v_i = 0\}$	$(1,-1,0,\dots 0)$ + permutations (there are $n(n+1)$ roots)
$D_{n\geq 4}$	$\{v \in \mathbb{Z}^n, \sum v_i$ -- even$\}$	$(\pm 1, \pm 1, 0, \dots 0)$ + permutations (there are $4 \cdot \binom{n}{2}$)
E_8	$\{v \in (\frac{1}{2}\mathbb{Z})^8 \mid v_i \equiv v_j \bmod 1, \sum v_i \bmod 2\}$	(as for D_8 + $(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ even # of $-\frac{1}{2}$) (240 of those)

E_7	$(\text{root})^{\perp}$ in E_8	(126 of these)
E_6	$(A_2)^{\perp}$ in E_8	(72 of these)

Now all these roots have Eucledian norm (= self product) 2 and the inner product of two roots can be $2, 1, 0, -1, -2$

Suppose k - fixed field and look at

$$\mathfrak{t} := (L \otimes k)^* \leftarrow \text{dual}$$

Define $\mathfrak{g}_\alpha$ = a 1 dimensional v. space over k spanned by an element e_α (for each α - root)

Let $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\text{ - root}} \mathfrak{g}_\alpha$

Now we would like to define a bracket $[\,,\,]$ on $\mathfrak{g}$ so that

$$[a,b]=0 \qquad \forall a,b \in \mathfrak{t}$$

$$[a,e_\alpha]=\alpha(a)e_\alpha$$

<u>Guess</u>: Since we want $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]\subseteq\mathfrak{g}_{\alpha+\beta}$

we can try

$$[e_\alpha,e_\beta]:=e_{\alpha+\beta}$$

$\forall\ \alpha,\beta$ - roots s.t. $\alpha+\beta$ - root

This naive definition will not give $[\,,\,]$ that satisfies Jacobi.

We need to put in signs in the definition

$$[e_\alpha,e_\beta]:=\pm e_{\alpha+\beta}$$

How do we get the right signs?

Consider a new group

$$1 \to \mathbb{Z}/2 \to \hat{L} \to L \to 1$$

central element z root lattice.

To give such a central extension we need to specify an anti-symmetric pairing on L with values in $\mathbb{Z}/2$; take the inner product on L reduced modulo 2.

$\Rightarrow$ if $\hat{x}, \hat{y} \in \hat{L}$ sitting over $x, y \in L$

we have

$$\hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1} = z^{(x \cdot y \bmod 2)}$$

Take

$$\mathfrak{g} = \Big(\mathfrak{t} \oplus \bigoplus_{\hat{\alpha} \in \hat{L} \text{ over root } \alpha \in L} \mathfrak{g}_{\hat{\alpha}} \Big)$$

$$(e_{\hat{\alpha}} = - e_{z\hat{\alpha}})$$

$\mathfrak{g}_{\hat{\alpha}}$ - generated by $e_{\hat{\alpha}}$

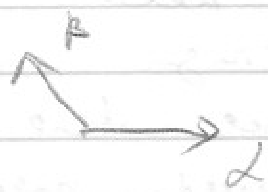
$$[\mathfrak{t}, \mathfrak{t}] = 0$$

$$[a, e_{\hat{\alpha}}] = \alpha(a)\, e_{\hat{\alpha}} \quad \forall\, a \in \mathfrak{t}$$

$$[e_{\hat{\alpha}}, e_{\hat{\beta}}] = e_{\hat{\alpha}\cdot\hat{\beta}}$$

if $\hat{\alpha} + \hat{\beta}$ is over a root

(Key step: If



$\alpha \cdot \beta = -1$

$$[e_{\hat{\beta}}, e_{\hat{\alpha}}] = e_{\hat{\beta}\cdot\hat{\alpha}} = e_{z\,\hat{\alpha}\cdot\hat{\beta}}$$
$$= -e_{\hat{\alpha}\hat{\beta}}$$
)

$[e_{\hat{\alpha}}, e_{\hat{\alpha}^{-1}}] = -\alpha \cdot$ (inner product with α)

an element in $\mathfrak{t}^{**} = \mathfrak{t}$