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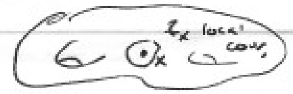
What's Up with Geometric Langlands

X - smooth projective algebraic curve $/ \mathbb{C}$

Start w/ $\operatorname{Pic} X$ = moduli space of line bundles on X

$= \operatorname{Jac} X \times \mathbb{Z}$

$$= {}_{\substack{\text{principal}\\ \text{divisors}}} \backslash \operatorname{Div} X = \mathbb{C}(X)^* \Big\backslash \prod_{x \in X}{}' \mathbb{Z} \quad \leftarrow \text{all but fin. } 0$$



$$= \mathbb{C}(X)^* \Big\backslash \prod{}' \mathbb{C}((z_x))^* \Big/ \prod \mathbb{C}[[z_x]]^*$$

$$= \mathbb{C}(X)^* \backslash \mathbb{A}_X^* / \mathcal{O}_{\mathbb{A}_X} = GL_1(\mathbb{C}(X)) \backslash GL_1(\mathbb{A}_X) / K$$

"shimura variety"

Weil: Moduli space of rank n vector bundles on X $\operatorname{Bun}_n X$

$$= GL_n(\mathbb{C}(X)) \backslash GL_n(\mathbb{A}_X = \prod{}' \mathbb{C}((z_x))) / GL_n(\mathcal{O}_{\mathbb{A}_X})$$

why? trivialize in local coor. near every x
& trivialize generically away from x & then the data
is the pasting up to equivalence of vector bundles.

Similarly for $\operatorname{Bun}_G X$ = G-bundles on X

rep. theory

"Geometric Langlands": "Harmonic analysis" for symmetries acting on $Bun_n X$.

· GL_1 $\quad x \in X \quad \mathcal{L} \mapsto \mathcal{L}(x)$

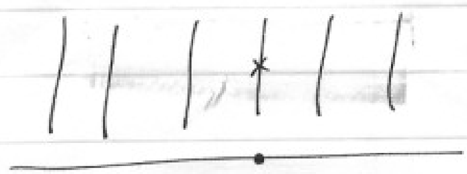
$\qquad\qquad\qquad\qquad \mathcal{L} \mapsto L(-x)$

adds 1 or -1 to $\pi_1' \mathbb{Z}$

$\mathcal{L}(x)$

· GL_2 $\quad x \in X$ elementary transformations

$\mathbb{P}(V)$



blow up at x

& then blow down fiber

called elementary transformation.

This is not a map from $Bun_2 X \to Bun_2 X$

but it does give us a corr.

$$\begin{tikzcd} & Hecke_{2,x} \arrow[dl] \arrow[dr] & \\ Bun_2 X & & Bun_2 X \end{tikzcd}$$

$\cdot \qquad\qquad\qquad\qquad \mathcal{O}\ \mathbb{P}'$

· Hecke correspondences: modify a bundle at a single point.

Geometric Langlands: algebraic geometers "harmonic analysis" of Hecke operators on "functions" on $Bun_n X$

for instance: simultaneous diag. operators

Roughly: eigenfunctions $\leftrightarrow$ Galois representations into GL_n

$\pi_1(X) \longrightarrow GL_n$

$\Leftrightarrow$ rank n local system / locally constant sheaf

$\Leftrightarrow$ vector bundle w/ connection.

we will call the moduli space of these by $\mathcal{L}oc_n X$

Prototype of harmonic analysis / rep. theory:

Fourier transform.

$\underline{\qquad}\ \mathbb{R} \qquad\qquad \underline{\qquad}\ \mathbb{R}$

Fourier transform $\quad \mathrm{Fun}(\mathbb{R}) \longrightarrow \mathrm{Fun}(\mathbb{R})$

i.e. $L^2(\mathbb{R})$, $\mathcal{S}$, distributions

$$f(x) \longmapsto \hat{f}(t) = \int e^{-itx} f(x)\, dx$$

$$\begin{tikzcd} & \boxed{e^{itx}}\ \ \mathbb{R}\times\mathbb{R} \arrow[dl] \arrow[dr] & \\ \underline{\qquad}\ \mathbb{R} & & \underline{\qquad} \end{tikzcd}$$

so we pull back $f(x)$ to $\mathbb{R}\times\mathbb{R}$ & integrate to push down the function back to $\mathbb{R}$.

i.e. integral transform

- $e^{itx}: \mathbb{R} \longrightarrow U(1) \subset \mathbb{C}^*$: 1-dim representation characters of $\mathbb{R}$

Writes any function as a con't linear combo of e^{itx}.

Fourier $f(x) = c\int e^{itx} \hat{f}(t)\, dt$ c some constant

· turns action of $\frac{\partial}{\partial x}$ into mult. by t.

$\cap$ $\mathrm{Lie}(\mathbb{R})$

Analog of This in Algebraic geometry?

· functions : · should be rich

· would like Fourier transforms

"Fun" on an algebraic variety X := The derived category of X

what usually mean: $D^b(\mathrm{Coh}\,X)$ bounded derived category of coherent sheaves

made out of complexes of coherent sheaves:

$$\to A^{i-1} \to A^{i} \to A^{i+1} \to$$

arises in physics as "D-branes"

" " non-commutative geometry

" " birational geometry

Now: P (sheaf) $\in D(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$$\pi_{2*}(P \otimes \pi_1^{*} F) \in D(Y)$$

$$\left[\begin{tikzcd} & \text{Hecke} \arrow[dl] \arrow[dr] & \\ X = Bun_2 & & X = Bun_2 \end{tikzcd}\right]$$

act by correspondences on $D(X)$

Fourier-Mukai transform takes place of Fourier trans.

A : abelian variety (Think of Jac X)

$A^\vee$ = dual abelian variety (again Jac X)

= moduli of deg (0) line bundles on $A = Pic^0 A$

~~$A \times A^\vee$~~

P line bundle : Poincare sheaf

$$\begin{tikzcd} & P \arrow[d] & \\ & A \times A^\vee \arrow[dl] \arrow[dr] & \\ A & & A^\vee \end{tikzcd}$$

$$D(A) \ni F \longmapsto \pi_{2*}(P \otimes \pi_1^* F) \in D(A^\vee)$$

all push forwards & pull backs

Theorem (Mukai): This defines an equiv $D(A) \cong D(A^\vee)$

This tells us: any sheaf on A is a direct integral of line bundles on A

$\otimes \Longleftrightarrow *$ (convolution) $\quad A \times A \to A$

$$F \xrightarrow{\mathbb{F}} \mathbb{F}(F) \to \mathbb{F}^{-1}\mathbb{F}(F)) = F$$

Now, e^{itx} not algebraic, but does satisfy algebraic equation:

$$(\partial_x - t) e^{itx} = 0$$

ring of polynomial diff operators

$$\mathcal{D} = \mathbb{C}[x]\langle \partial_x \rangle \quad \text{which satisfy } \partial x - x\partial = 1$$

$$e^{itx} \in \mathrm{fun}_\sigma \mathbb{R} \circlearrowleft \mathcal{D}$$

apply $\mathcal{D}$ to e^{itx} & look at span

$M = \mathcal{D} e^{itx}$, This is a $\mathcal{D}$ module

$$M \simeq \mathcal{D} / \mathcal{D}(\partial - it)$$

Thus we go from functions (not algebraic) $\longrightarrow$ $\mathcal{D}$-modules

adding to "Fun" X = $\mathcal{D}_X$-modules $\to$ $\mathcal{O}_X$ module

$\mathcal{D}_X = \mathcal{O}_X \langle T_X \rangle$

$$D(\mathcal{D}_A\text{-modules}) \longleftrightarrow D\left(\begin{array}{ll} A^\natural & = \text{line bundles} \oplus \text{connection} \\ \downarrow & \\ A^\vee & = \text{line bundles on } A \end{array} \right)$$

$$D(A) \longleftrightarrow D(A^\vee)$$

which says any $\mathcal{D}$-module is a direct integral of line bundles $\oplus$ diff.

Now: Geometric Langlands (Conjecture)

~~Fourier transform~~

rank n v.s. + connection on X

$$D(\mathcal{D}_{Bun_n X}) \simeq D_{coh}(\mathcal{L}oc_n X) \quad \Leftrightarrow \pi_1(X) \to GL_n$$

"diff functions" "functions w/ $\otimes$ mult"

"automorphic function" associated to ρ $\leftarrow \rho : (\pi_1 X \to GL_n) =$ points $\in \mathcal{L}oc_n X$

$\downarrow$

sky scraper sheaf

$\mathcal{F}$ Hecke

$$\mathcal{F} \searrow Bun_2 \swarrow Hecke \searrow Bun_2 \qquad \mathcal{F} \otimes \mathbb{C}^n$$