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What sorts of algebro-geometric objects arise in physics and why should we (as pure mathematicians) care?

For the most part "physics" will mean high energy theoretical physics i.e. quantum field theory or string theory.

Why does algebraic geometry appear in physics?

(I) Romantic reason - the "unreasonable effectiveness of math and science"

(II) "Drunk looking for keys in the dark" reason - algebraic geometry occurs in some specific limits of physical considerations. These could be far away from reality but are the ones we can see and do lead to deep and interesting mathematics.

Why should we care?

The math-physics connections have similar payoffs to other connections – various intuitions, techniques (ideas), paradigms from physics provide many new ideas when ported to algebraic geometry.

E.g.: we have gotten

- new structures on known mathematical invariants (quantum product on cohomology)

- (mathematically) unexpected relationships, e.g. (intersection theory on the moduli of curves) <-> (solutions to certain PDEs) (Witten's conjecture and Kontsevich's theorem)

Do we need to "know physics" to take advantage of this stuff?

Not really but some knowledge helps!

A main character in the algebraic geometry related to physics is the notion of a moduli space.

This notion will be covered in detail in Angela Gibney's talk.

Briefly: A Moduli space is an algebraic variety / scheme / stack parameterizing isomorphism classes of some algebraic objects or structures.

Understanding geometry of moduli spaces gives insight into the objects themselves.

E.g. "integrals on moduli spaces"
$\Leftrightarrow$ intersection in the moduli space
$\Leftrightarrow$ counting objects satisfying some set of conditions

The physics (in broad strokes) is usually encoded in a <u>theory</u>.

Loosely a <u>theory</u> is specified by

(1) a manifold M (Riemannian or with a metric with signature $(n,1)$)

(2) a collection of fields e.g.

- C^∞-sections of a bundle on M (matter fields)
- connections on a bundle on M (gauge fields)
- maps $f : M \to N$ for some fixed N - target manifold (might be a Kähler manifold)

(3) An action S on the space of fields (almost always infinite dimensional). S is a real-valued function on the space of fields

Typically the action S is written as

$$S(\text{field}) = \int_M L(\text{field})$$

where L is an expression which takes as inputs a field and its derivatives and produces an output a function or a measure on M.

<u>Terminology</u>: L is called the Lagrangian of the theory.

Partition function of the theory: given by a "path integral"

$$Z := \int_{\text{all fields}} e^{iS} \cdot \text{"measure"}$$

a measure on the space of fields.

The main difficulty in specifying the quantum theory is in making sense of the path integral.

In practice the "mythical path integral" is evaluated by various stuff by localization:

In specific situations the path integral localizes onto a finite dimensional moduli space consisting of all fields that are critical points for the action S.

A theory has parameters, e.g.:

- constants entering the definition of the Lagrangian

- the choice of a complex or a Kähler structure on the target manifold N, e.g. N might be a Calabi-Yau 3-fold – a projective 3-fold with $c_1(TN)=0$.

($\Leftarrow$) N has a Ricci flat Kähler metric by Yau's theorem)

• D-branes (= boundary conditions for $f : M \to N$)

These are of the form:

$$\left(\begin{matrix} L \subset N \\ \text{submanifold} \end{matrix} \; , \; \begin{matrix} \text{gauge field} \\ \text{on } L \end{matrix} \right)$$

and usually satisfy some special PDE (= critical pt equation).

Duality: Physics word for isomorphism. Two theories are considered dual or equivalent after some interesting identification of the parameters

Flesh out some examples:

(1) Gauge theory on a Kähler surface (M, ω)

Fix $E \to M$ - complex C^∞ bundle on a rank r $\mathbb{C}$-bundle

A gauge field is a connection A on E, i.e.

$$d_A : \Omega^0(M, E) \to \Omega^1(M, E)$$

$\mathbb{C}$-linear map satisfying the Leibnitz rule w.r.t. multiplication by functions.

We will assume $c_1(E) = 0$ and also that E is equipped with fixed Hermitian metric $|\cdot|$ preserved by d_A:

$$d \langle s, t\rangle = \langle d_A s, t\rangle + \langle s, d_A t\rangle$$

$\forall$ $s, t \in \Omega^0(M, E)$

The curvature of A is the operator

$$d_A^2 : \Omega^0(M, E) \to \Omega^2(M, E)$$

which turns out to be linear w.r.t. multiplication by functions and so corresponds to a section

$$F_A \in \Omega^0(M, \mathrm{End}\, E)$$

Now

$$\begin{pmatrix} \text{Space of} \\ \text{fields} \end{pmatrix} = \begin{pmatrix} \text{Space of} \\ \text{connections} \\ \text{on } E \text{ preserving} \\ \langle \cdot, \cdot \rangle \end{pmatrix}$$

Define the action S to be

$$S(A) := \int_M |F_A|^2$$

The critical points are all solutions to

$$F_A \wedge \omega = 0$$

This is the Hermitian - Yang - Mills equation and the solutions are called instantons.

Theorem of Donaldson

$$\begin{pmatrix} \text{solutions to} \\ \text{HYM} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{holo structures on} \\ E \text{ which are} \\ \text{polystable w.r.t. } \omega \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} \text{moduli space of} \\ \text{semistable bundles} \end{pmatrix}$$

The integrals over the moduli of holomorphic semistable bundles are the main objects of study of Donaldson theory.

(2) Sigma models on Calabi-Yau 3-folds (N, ω).

One considers the fields to be C^∞ - maps

$$f : \Sigma_g \to (N, \omega)$$

↑ Riemann surface of genus g

The string action is given by

$$S(f) = \int_{\Sigma_g} |df|^2$$

(this assumes that part of the data specifying our field is a metric on Σ_g).

This functional is minimized by holomorphic maps locally

(The constant maps are global minima of S)

Suppose $\beta \in H_2(N, \mathbb{Z})$ and let

$M_g(N, \beta)$ - moduli space of all holo maps

$$f : \Sigma_g \to N$$

s.t. $f_*[\Sigma_g] = \beta$.

Now the localization procedure mentioned before reduces the computation of the quantum partition function

$$Z = \int_{\text{all maps}} e^{iS} \text{ "measure"}$$

to integrals on the finite dimensional moduli space

$$M_g(N, \beta)$$

The study of such integrals is the subject of GW theory

The parameters of the sigma model are the Kähler moduli or rather the area

$$\int_\Sigma f^* \omega$$

and the string coupling constant allowing us to enumerate g.

<u>Moral</u>: All GW integrals should be put together in a single

function involving all the parameters.