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An Introduction to equivariant Cohomology

(and Why geometers love it)

X top space. $\rightsquigarrow H^*(X)$

G top group $\quad G \curvearrowright X \rightsquigarrow H^*_G(X)$ "equivariant cohomology"

larger ring
Perhaps easier to describe

Def 1: (Borel) Given EG, a contractible space with a free G-action, get a classifying space

$$BG = EG/G \quad \text{i.e.} \quad \begin{array}{c} EG \\ \downarrow \\ BG \end{array} \quad \text{Principal } G\text{-bundle}$$

Then the equivariant coh. ring is

$$H^*_G(X) := H^*\left((EG \times X)/G \right)$$

This is indep. of choice of EG

Fact: For $n > 0$, if a principal G-bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$

is n-connected, i.e. maps $S^m \to E$ are contractible

For $m \leq n$, then $\quad H^m_G(X) = H^m\left((E \times X)/G \right)$

In other words, we can compute

$H^*_G(X)$ using (Finite-dim'l) approximations of EG.

There is also an algebraic approximation theory.

Example ① $G = \mathbb{C}^*$ (or S^1)

S^1 acts freely on $S^{2n-1} \subseteq \mathbb{C}^n$ (i.e. $\overset{alg}{\mathbb{C}^*}$ acts freely on $\mathbb{C}^n \setminus \{0\}$)
with quotient $\mathbb{C}\mathbb{P}^{n-1}$

$$\begin{array}{c} S^{2n-1} \\ \downarrow \\ \mathbb{C}\mathbb{P}^{n-1} \end{array} \quad \text{principal } S^1 \text{ bundle}$$

$S^{2n-1} \to \mathbb{C}\mathbb{P}^{n-1}$ is $2n-3$ connected : S^{2n-1} is the equator.

So taking
$$ES^1 = \lim S^{2n-1} = S^\infty$$
$$BS^1 = \lim \mathbb{C}\mathbb{P}^{n-1} = \mathbb{C}\mathbb{P}^\infty$$

$$H^*_{S^1}(pt) = H^*(BS^1) = \mathbb{R}[t] \quad \deg t = 2$$

(2) $G = (\mathbb{C}^*)^\ell = (S^1)^\ell$ taking $EG = (S^\infty)^\ell$, $BG = (\mathbb{C}\mathbb{P}^\infty)^\ell$

$$\rightsquigarrow H^*_{(\mathbb{C}^*)^\ell}(pt) = \mathbb{R}[t_1, \dots, t_\ell] = \mathrm{Sym}(\mathfrak{t})$$

<u>Prop</u>: G is connected compact Lie group acting on X, T max. torus with Weyl group $N(T)/T$.
Then $H^*_G(X) = H^*_T(X)^W$

<u>Facts</u>: • If G acts freely on X then
$H^*_G(X) = H^*(X/G)$

From now on, $G = T = (\mathbb{C}^*)^\ell$ or $(S^1)^\ell$
and $T \curvearrowright X$ nice enough

• $H^*_T(X) \underset{\text{modules}}{=} H^*(X) \otimes H^*_T(pt)$

• $H^*_T(X) \xrightarrow[t_1 = \dots = t_\ell = 0]{\text{ring surj}} H^*(X)$ recover the ordinary cohomology, can use to simplify computations in $H^*(X)$

Dictionary

X	$T \curvearrowright X$
cohomology	equiv. coh
maps	equiv. maps

Some properties/concepts in equiv. coh.

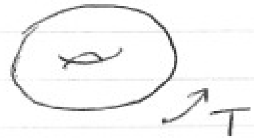
- functoriality
- Künneth
- Leray spectral sequ.
- Chern classes

<u>Definition 2</u>: First, we define equivariant homology

<u>Def'n</u>: An <u>equivariant i-cycle</u> is a G equivariant map $f: E \to X$, where E has a free G-action, $P := E/G$ is a pseudomanifold of dim i.

$$P \leftarrow E \xrightarrow{f} X$$

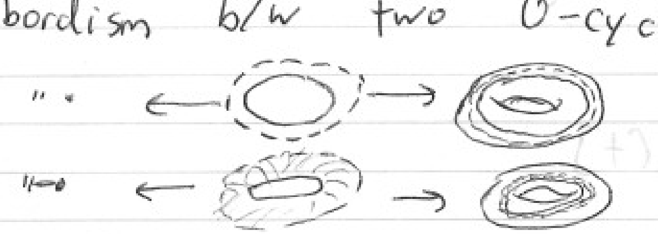
<u>Example</u>: $X = S^1 \times S^1$, $T = S^1$



0-cycle

1-cycle

A cobordism b/w two 0-cycles



the cycles

the cobordism

<u>Def'n</u> $H_i^G(X) = \left\{ \begin{array}{l} \text{cobordism classes of} \\ \text{equiv. } i\text{-cycles} \end{array} \right\}$

group: $-$ is the opposite orientation
adding = union

$$H_0^{S^1_\bullet}(S^1_\bullet \times S^1) = \mathbb{R}$$
$$H_1^{S^1}(S^1 \times S^1) = \mathbb{R}$$

In this case, the action is free so we expect that

$$H_*^{S^1}(S^1 \times S^1) = \underset{\substack{\| \\ S^1 \times S^1/S^1}}{H_*(S^1)}, \text{ which is the case.}$$

<u>Def'n</u>: As vector spaces $H^i_G(X) = (H_i^G(X))^* = \mathrm{Hom}(H_i^G(X), \mathbb{R})$

multiplication is dual to

$$\begin{array}{c} H_i^G(X) \otimes H_j^G(X) \simeq \\ H_{i+j}^{G \times G}(X \times X) \\ \uparrow \Delta \\ H_{i+j}^G(X) \end{array}$$

<u>Example 2</u>: $X = pt$, $T = S^1$

R cycle $\quad \mathbb{CP}^k \leftarrow S^{2k+1} \rightarrow pt$

these are all non-zero + in fact generate

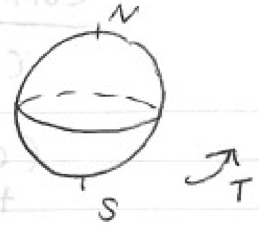
$H_{2k}^T(pt) = \mathbb{R}$ generated by $\mathbb{CP}^k$

$H_{odd}^T(pt) = 0$

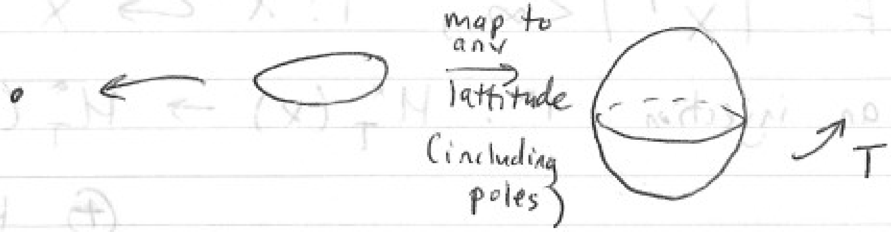
$H_{T}^{*}(pt) = \mathbb{R}[t]$

Example 3: $X = S^2 = \mathbb{CP}^1$, $T = S^1$

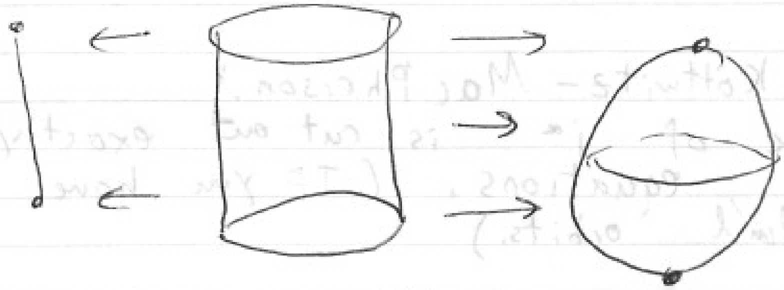
Two fixed pts, so action is not free



0 cycles



A cobordism b/w N + S



gives a relation $[\mathbb{CP}^0_N] = [\mathbb{CP}^0_S]$ in

$$H_0^T(N \cup S) \subset H_0^T(\mathbb{CP}^1)$$
$$\parallel$$
$$H_0^T(N) \oplus H_0^T(S)$$

In fact, this is the only relation

$$0 \to \mathbb{R} \to H_*^T(N \cup S) \to H_*^T(S^2) \to 0$$
$$1 \mapsto [N] - [S]$$

Dualizing,

$$0 \leftarrow \mathbb{R} \leftarrow H_T^*(N \cup S) \leftarrow H_T^*(\mathbb{CP}^1) \leftarrow 0$$
$$\parallel$$
$$\{(f,g) : f,g \in \mathbb{R}[t]\}$$

$$\Rightarrow H_T^*(\mathbb{CP}^1) = \{(f,g) : f,g \in \mathbb{R}[t],\ f(0) = g(0)\}$$

Common theme: "we can compute the equivariant cohomology using fixed points"

Can make eq. coh. descriptions/ calculations in terms of fixed loci.

e.g. if $|X^T| < \infty$, $i : X^T \hookrightarrow X$

gives an injection $i^* : H^*_T(X) \to H^*_T(X^T) =$

$$\bigoplus_{p \in X^T} \underbrace{H^*_T(p)}_{\text{polynomial rings}}$$

Goresky-Kottwitz-MacPherson: image of i^* is cut out exactly by (explicit) linear equations. (If you have finitely many 1 dim'l orbits.)

(2) Localization

(?) Q+R: There is also progress on when fixed pts are not isolated.