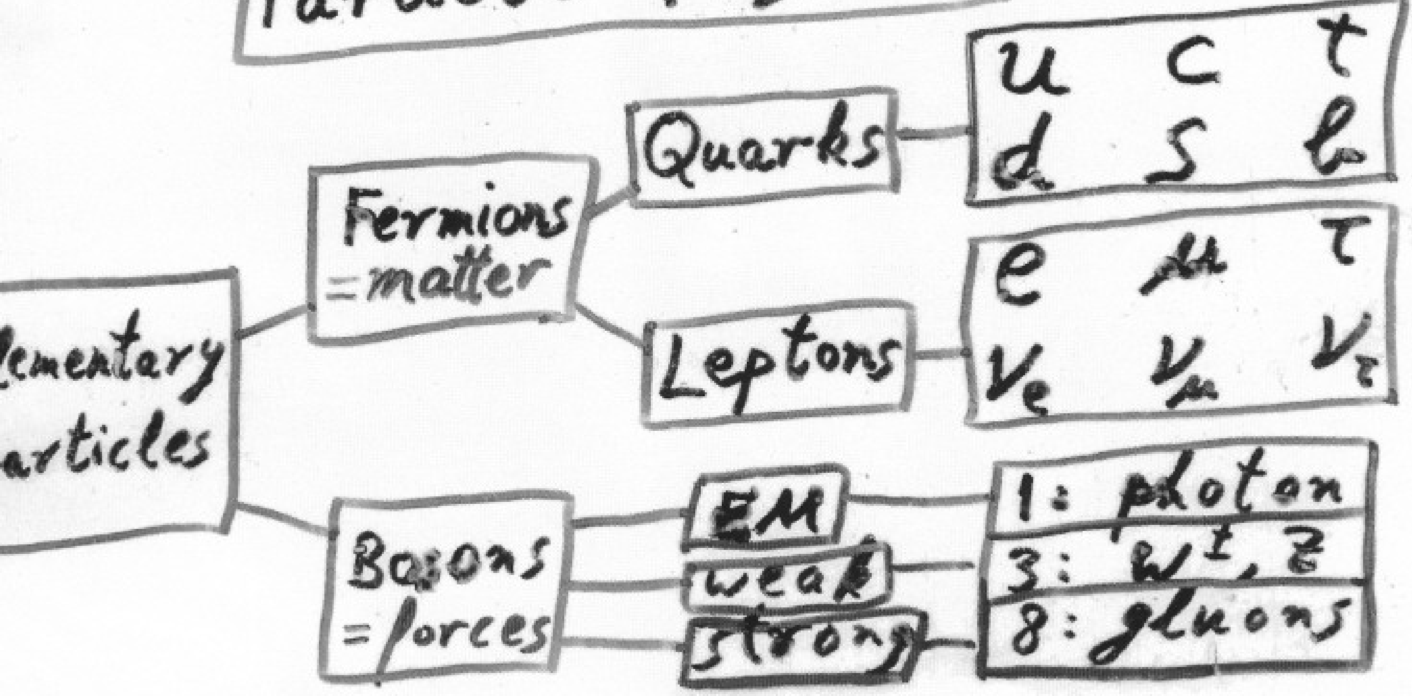
From Strings to Standard Models via Algebraic Geometry

A. G. Bootcamp

July 2005

particle physics
standard model + extensions,
GUT's.
strings, CY's
heterotic strings & vacua
particle spectrum
SU(n) vacua (vector bundles)
Wilson lines
standard model spectrum
Bundles on elliptic fibrations
Genus 1 fibrations
Schoen's CY's
Bundles
Calculations

Particle Physics



2/3

-1/3

-1

0

0

±1, 0

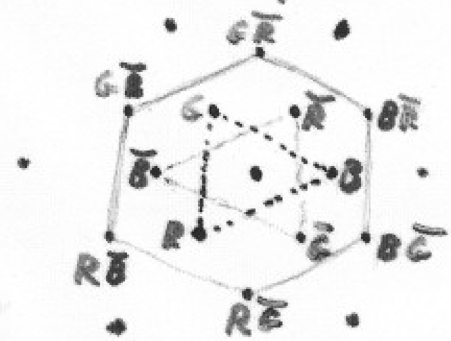
0

composite particles: proton (uud), neutron (udd)
+1 0

allowed interactions are restricted by charge conservation laws, ~~or~~: invariance under group action. e.g.:

* electric charge $\in \frac{1}{3}\mathbb{Z}$ ~ characters of $U(1)$

* strong (= color) charge $\in$ hexagonal lattice $\approx \mathbb{Z}^2$.



* mass, momentum, spin ~ representations of

$$\underset{\text{Poincare}}{P} = \underset{\text{translations}}{\mathbb{R}^{3,1}} \rtimes \underset{\text{Lorentz}}{Spin(3,1)}$$

The $\boxed{\text{Standard Model}}$ is the <u>QFT</u> of particle physics.

A <u>field theory</u> is specified by $\boxed{(M, \mathcal{F}, G, \mathcal{L})}$

* M = manifold (= spacetime)
* $\mathcal{F}$ = vector bundle on M

  (particles ~ fields = sections of $\mathcal{F}$)
* G = symmetry group acting on M, $\mathcal{F}$.

  (<u>allowed</u> <u>interaction</u> = G-invariant monomial on $\mathcal{F}$;
* $\mathcal{L}$ = Lagrangian = sum of all allowed interactions, w/ coefficients = <u>interaction</u> <u>strengths</u> = coupling "constants"

<u>Classical</u> field theory: solutions are field configurations which extremize $\mathcal{L}$.

<u>Quantum</u> field theory: "integrate over all field configurations". (coupling "constants" depend on energy to avoid divergences.)

$\boxed{SM:}$

M = Minkowski = $\mathbb{R}^{3,1}$

$G = SU(3) \times SU(2) \times U(1) \times P \quad = SM \times P$

$\mathcal{F} = \mathcal{F}_b \oplus \mathcal{F}_f$, $\quad \mathcal{F}_b$ = connections on trivial SM-bundle over M

$\mathcal{F}_f = \bigoplus$ 3 copies (= generations) of rank 16 bundle:

$Q : (3,2)_{1/6} \qquad L : (1,2)_{-1/2}$

$\bar{D} : (\bar{3},1)_{1/3} \qquad \bar{E} : (1,1)_{1}$

$\bar{u} : (\bar{3},1)_{-2/3} \qquad \bar{\nu} : (1,1)_{0}$

Extensions of the SM

The SM is fantastically successful, but <u>incomplete</u>:

* need masses for particles $\rightsquigarrow$ add Higgs particle
* 19 or so arbitrary parameters
* 3 generations unexplained
* 3 coupling constants unify at high energy $\rightsquigarrow$ GUT, SUSY
* fine issues: proton decay, ...
* no gravity $\rightsquigarrow$ strings.

$\boxed{\text{SUSY}}$: doubles the particle spectrum
makes room for Higgs ($\Rightarrow$ mass)
but also for color triplets ($\Rightarrow$ proton decay)

$\boxed{\text{GUT}}$: embed SM $\subset$ SU(5), SO(10), E6, ...
GUT symmetry at high energy
breaks to SM symmetry at low energy.

e.g.

e.g.	SU(5): 	$16 = \underset{V}{5} + \underset{\Lambda^2 V}{10} + \underset{\Lambda^5 V}{1}$
or	Spin(10):	$16 = S^+$

$\boxed{\text{Gravity}}$ is difficult to quantize: grows w/ distance^{-2}, or energy^2, diverges
$\Rightarrow$ introduce minimal length scale.
point particle $\rightsquigarrow$ string.

String theory

Bosonic strings, consistent in : 26 D
Superstrings : 10 D
M-theory : 11 D

"

hypothetical theory including the 5 known string theories as limits:

I : open strings

II : closed strings < A, B } mirror symmetry

Heterotic : < $Spin(32)/\mathbb{Z}_2$, $E8 \times E8$

"

[consistent combination of bosonic right movers (in 10 D) + super left movers (in 26 D = 10 + 16).
16 D space = $\mathbb{R}^{16}/\Lambda$
Λ = unimodular even rk = 16 lattice]

To get 4D theory: compactify. I.e. take 10 D space = $M^{3,1} \times X$, X: compact.
consistency: X = CY3, holonomy = SU(3)
($\Longleftrightarrow$ complex, Kahler, global 3 forms $\neq 0$)

$\boxed{\text{Goal: get SM from a string compactification}}$

$\boxed{\text{E8 heterotic vacua}}$

Specified by $(X, \bar{V})$:

X: CY3
$\bar{V}$: stable $E8 \times E8$ bundle on X
$c_2(\bar{V}) = c_2(X)$: (GS anomaly)

or in het M-theory:

$$c_2(\bar{V}) + [C] = c_2(X) \text{ in } H^4(X, \mathbb{Z})$$

$C \subset X$: effective curve.

$\bar{V} \Leftrightarrow V, V'$ 2 E8-bundles
usually V' trivial, so:

(X, V)
X : CY3 $\qquad c_2(V) + [C] = c_2(X)$
V: stable E8 bundle

cf. Candelas, Horowitz, Strominger, Witten ('84)

$c_i(V) \in H^{2i}(X, \mathbb{Z})$: i-th Chern class

$c_1(V) = 0$ because E8 is simple
$c_2(V) = c_2(X) - [C]$
$c_3(V)$: # generations. should be 3.

$\boxed{\text{Particle spectrum}}$

Kernel of:

$$\boxed{\not{D} : \mathrm{ad}\, V \otimes \mathrm{Spin}^+_{10} \longrightarrow \mathrm{ad}\, V \otimes \mathrm{Spin}^-_{10}}$$

* decompose $\mathrm{Spin}_{10} = \mathrm{Spin}_4 \otimes \mathrm{Spin}_X$
* identify $\mathrm{Spin}_X = \oplus_{q=0}^{3} \mathcal{A}^{0,q}(X)$
* use duality

$\Rightarrow$

$$\boxed{\ker \not{D} = \oplus_{q=0,1} H^q(X, \mathrm{ad}\, V).}$$

$\boxed{SU(n) \text{ vacua}}$

Often: $G \subset E8$, e.g. $G = SU(n)$, $n = 3,4,5$

V: G-bundle ($\Leftrightarrow$ rank-n VB, $c_1(V) = 0$)

$\mathcal{V} := V \times^G E8$

Resulting compactification has "low energy gauge group":

$$\boxed{H := Z_{E8}(G)}$$

Decompose $248 = \text{ad}\, E8$ under $G \times H \subset E8$:

$$\boxed{\text{ad}\, E8 = \oplus_i\, \mathcal{U}_i \otimes R_i}$$

($\mathcal{U}_i$: G-irreps, R_i: H-reps)

$\Rightarrow$

$$\boxed{\text{ad}\, \mathcal{V} = \oplus_i\, \mathcal{U}_i(V) \otimes R_i}$$

($\mathcal{U}_i(V)$: vector bundles associated to G-bundle V)

$$\boxed{\text{Spec} = \text{Ker}\, \not{D} = \oplus_{q=0,1} \oplus_i\, H^q(X, \mathcal{U}_i(V)) \otimes R_i}$$

* The R_i describe particle types.
* The cohomology dimensions count numbers of particles of each type.

$$\boxed{G = SU(3), \quad H = E6}$$

$$\boxed{248 = (8,1) \oplus (1,78) \oplus (3,27) \oplus (\bar{3},\overline{27})}$$

$$\left(\begin{array}{l} 8 = \mathrm{ad}\, SU(3) \qquad 78 = \mathrm{ad}\, E6 \\ 3,\bar{3} : \text{(anti) fundamental of } SU(3) \end{array}\right)$$

$$\boxed{Spec = H^1(X, \mathrm{ad}\, V) \oplus H^0(X,\theta) \otimes 78 \oplus H^1(X,V) \otimes 27 \oplus H^1(X,V^*) \otimes \overline{27}}$$

$$\boxed{G = SU(5), \quad H = SU(5)}$$

$$\boxed{248 = (24,1) \oplus (1,24) \oplus (10,5) \oplus (\overline{10},\bar{5}) \oplus (5,\overline{10}) \oplus (\bar{5},10)}$$

$$\boxed{\begin{array}{l} Spec = H^1(X, \mathrm{ad}\, V) \oplus H^0(X,\theta) \otimes 24 \oplus \\ \qquad \oplus H^1(X, \Lambda^2 V) \otimes 5 \oplus H^1(X, \Lambda^2 V^*) \otimes \bar{5} \oplus \\ \qquad \oplus H^1(X, V) \otimes \overline{10} \oplus H^1(X, V^*) \otimes 10. \end{array}}$$

$$\text{Note:} \left[\begin{array}{l} h^0(X,V) = h^0(X,V^*) = 0 : \text{stability.} \\ h^1(X,V) - h^1(X,V^*) = \text{index} = \chi(V^*) \\ \qquad \Rightarrow \#\ \text{generations.} \\ \text{calculation shows: } \chi(V) = \chi(\Lambda^2 V) \end{array}\right.$$

$\boxed{\text{Wilson lines}}$

$F \subset H$ finite subgroup
$F \times \tilde{X} \to \tilde{X}$ free action on CY3 $\tilde{X}$
$\pi : \tilde{X} \to \tilde{X}/F =: X$ quotient CY3.

V on $X \rightsquigarrow \tilde{V} := \pi^* V$, G-bundle on $\tilde{X}$
$\mathcal{V} := V \overset{G}{\times} E8 \rightsquigarrow \tilde{\mathcal{V}} := \pi^* \mathcal{V} = \tilde{V} \overset{G}{\times} E8$

F action on $\tilde{X}$ lifts to $\tilde{V}, \tilde{\mathcal{V}}$, their cohomologies.

Wilson line:

$$\boxed{W := \tilde{X} \overset{F}{\times} H,}$$

flat H-bundle on X induced from F-bundle $\tilde{X} \to X$.

The $(G \times H)$-bundle $V \oplus W$ induces new $E8$-bundle on X:

$$\boxed{\mathcal{V}' := (V \oplus W) \overset{(G \times H)}{\times} E8.}$$

$\boxed{\text{Goal: particle spectrum of } (X, \mathcal{V}').}$

Structure group of $\mathcal{V}'$ is $\boxed{G \times F}$ (not G)
$\Rightarrow$ low energy gauge group of $(X, \mathcal{V}')$ is

$$\boxed{S := Z_{E8}(G \times F) = Z_H(F).}$$

strings	GUT	SM
E8	H	S

$\boxed{\text{SM spectrum}}$

$\boxed{\pi^* V' \approx \pi^* V =: \tilde{V}} \Rightarrow$

$\exists$ 2 actions of F on $\tilde{V}$ lifting action on $\tilde{X}$:

ρ: quotient $= V$

ρ': quotient $= V'$.

$\boxed{ad\, \tilde{V} = \oplus_i\, \mathcal{U}_i(\tilde{V}) \otimes R_i} \Rightarrow$

$$H^q(X, ad\, V) = \oplus_i\, H^q(\tilde{X}, \mathcal{U}_i(\tilde{V}))^{\rho(F)} \otimes R_i \leftarrow H \text{ acts}$$

$$H^q(X, ad\, V') = \oplus_i \left(H^q(\tilde{X}, \mathcal{U}_i(\tilde{V})) \otimes R_i\right)^{\rho'(F)} . \leftarrow S \text{ acts}$$

Decompose each H-rep R_i into F-irreps A_j:

$R_i = \oplus_j (A_j \otimes B_{ij})$	$B_{ij} = Hom_F(A_j, R_i)$

$\Rightarrow$

$$\boxed{Spec' = \oplus_{q=0,1} H^q(X, ad\, V') = \oplus_{q,i,j} \left(H^q(\tilde{X}, \mathcal{U}_i(\tilde{V})) \otimes A_j\right)^F \otimes B_{ij}}$$

$B_{ij} \leftrightarrow$ bunch of particles, each with multiplicity $m_{ij} := \dim \left(H^q(\tilde{X}, \mathcal{U}_i(\tilde{V})) \otimes A_j\right)^F$.

$\boxed{G = SU(3),\ H = E6}$

H has max subgroup

$$\boxed{H_0 = SU(3)_C \times SU(3)_L \times SU(3)_R}$$

Take $F = F(n, \hat{n}) = \mathbb{Z}/n \oplus \mathbb{Z}/\hat{n}$ with generators in H_0:

$1_C \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_L \times 1_R$
$1_C \times 1_L \times \begin{pmatrix} \hat{\alpha} & & \\ & \hat{\alpha} & \\ & & \hat{\alpha}^{-2} \end{pmatrix}_R$

where $\alpha, \hat{\alpha}$ are roots of 1 of order $n, \hat{n}$.

Or: $F_0 = F_0(n) \subset F(n,n)$, $F_0 \approx \mathbb{Z}/n$, generator

$$1_C \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_L \times \begin{pmatrix} \alpha & & \\ & \alpha & \\ & & \alpha^{-2} \end{pmatrix}_R .$$

Either F or F_0 break $H = E6$ to

$$\boxed{S = SU(3)_C \times \left(\frac{SU(2) \times U(1)}{\mathbb{Z}/2}\right)_L \times \left(\frac{SU(2) \times U(1)}{\mathbb{Z}/2}\right)_R} .$$

Recall $R_i = 1, 78, 27, \overline{27}$. Decompose R_i under H_0:

$$1 = (1,1,1)$$
$$78 = (8,1,1) \oplus (1,8,1) \oplus (1,1,8) \oplus (3,3,3) \oplus (\overline{3},\overline{3},\overline{3})$$
$$27 = (3,\overline{3},1) \oplus (1,3,\overline{3}) \oplus (\overline{3},1,3)$$
$$\overline{27} = (\overline{3},3,1) \oplus (1,\overline{3},3) \oplus (3,1,\overline{3}),$$

then decompose further under $S \subset H_0$:

$\mathcal{U}_i$	$H^q(\tilde{X}, \mathcal{U}_i(\tilde{V}))$	R_i	A_j	B_{ij}
8	$H^1(\tilde{X}, \mathrm{ad}\,\tilde{V})$	1	$0,0$	$(1,1,1)$
1	$H^0(\tilde{X}, \mathcal{O})$	78	$0,0$	$(8,1,1)+(1,3,1)+(1,1,3)+2\times(1,1,1)$
			$3,0$	$(1,2,1)$
			$-3,0$	$(1,2,1)$
			$0,3$	$(1,1,2)$
			$0,-3$	$(1,1,2)$
			$1,1$	$(3,2,2)$
			$-2,-2$	$(3,1,1)$
			$-1,-1$	$(\bar{3},\bar{2},\bar{2})$
			$2,2$	$(\bar{3},1,1)$
			$1,-2$	$(3,2,1)$
			$-2,1$	$(3,1,2)$
			$-1,2$	$(\bar{3},\bar{2},1)$
			$2,-1$	$(\bar{3},1,\bar{2})$
3	$H^1(\tilde{X}, \tilde{V})$	27	$-1,0$	$(3,\bar{2},1)$
			$2,0$	$(3,1,1)$
			$1,-1$	$(1,2,\bar{2})$
			$-2,-1$	$(1,1,\bar{2})$
			$1,2$	$(1,2,1)$
			$-2,2$	$(1,1,1)$
			$0,1$	$(\bar{3},1,2)$
			$0,-2$	$(\bar{3},1,1)$
$\bar{3}$	$H^1(\tilde{X}, \tilde{V}^*)$	$\overline{27}$	$\vdots$	$\vdots$

$G = SU(5)$, $H = SU(5)$

$F = \mathbb{Z}/2 \subset H$: $\begin{pmatrix} - & & & & \\ & - & & & \\ & & - & & \\ & & & + & \\ & & & & + \end{pmatrix} \Rightarrow S = SM = SU(3) \times SU(2) \times U(1)$

U_i	$H^q(\tilde{X}, U_i(\tilde{V}))$	R_i	A_j	B_{ij}	
24	$H^1(\tilde{X}, \mathrm{ad}\,\tilde{V})$	1	0	$(1,1)$	
1	$H^0(\tilde{X}, \mathcal{O})$	24	0	$(8,1)_0 \oplus (1,3)_0 \oplus (1,1)_0$	
			1	$\cancel{(3,\bar{2})_5 \oplus (\bar{3},2)_{-5}}$	
10	$H^1(\tilde{X}, \Lambda^2 V)$	5	0	$(3,1)_2$	D
			1	$(1,2)_{-3}$	$\bar{L}$
$\overline{10}$	$H^1(\tilde{X}, \Lambda^2 V^*)$	$\bar{5}$	0	$(\bar{3},1)_{-2}$	$\bar{D}$
			1	$(1,\bar{2})_3$	L
5	$H^1(\tilde{X}, V)$	$\overline{10}$	0	$(3,1)_{-4} \oplus (1,1)_6$	u, E
			1	$(\bar{3},\bar{2})_1$	$\bar{Q}$
$\bar{5}$	$H^1(\tilde{X}, V^*)$	10	0	$(\bar{3},1)_4 \oplus (1,1)_{-6}$	$\bar{u}, \bar{E}$
			1	$(3,2)_{-1}$	Q

Bundles on elliptic fibrations

$\boxed{\begin{tikzcd} \pi : X \arrow[r] & B \arrow[l, bend left, dashed, "\sigma"] \end{tikzcd}}$ elliptic fibration (with section σ)

$\pi^\vee : X^\vee \to B$ dual fibration

Spectral construction: [Friedman, Morgan, Witten]
[D]

$$\boxed{\text{Bundle on } X \leftrightarrow \text{"spectral cover" } \begin{tikzcd} C \arrow[r, hook] \arrow[dr] & X^\vee \arrow[d] \\ & B \end{tikzcd} \quad + \text{ line bundle } L \text{ on } C.}$$

($\exists$ versions for vector bundles, G-bundles, ...)

$\exists$ "Poincare bundle" P on $X \times_B X^\vee$

$\Rightarrow$ integral transform

$$\boxed{\begin{aligned} FM : D^b(X) &\xrightarrow{\sim} D^b(X^\vee) \\ \mathcal{F} &\mapsto Rp^\vee_*(p^*\mathcal{F} \otimes P) \end{aligned}}$$

$$\left(\begin{tikzcd} & X \times_B X^\vee \arrow[dl, "p"'] \arrow[dr, "p^\vee"] & \\ X \arrow[dr, "\pi"'] & & X^\vee \arrow[dl, "\pi^\vee"] \\ & B & \end{tikzcd} \right)$$

FM interchanges spectral data

(C, L) (C irreducible, finite$/B$, ...)

with vector bundles V (stable, ...)

$\boxed{\text{Genus 1 fibrations}}$

Often $\tilde{X} \to \tilde{B}$ has elliptic fibration

F acts on $\tilde{X}, \tilde{B}$ but does not preserve $\sigma \Rightarrow$

$X \to B$ is a genus 1 fibration with no section.

$\boxed{X^\vee := \mathrm{Pic}^0(X/B)}$ the corresponding elliptic fibration: X is an $X^\vee$-torsor.

[Caldararu], [D & Pantev]:

Bundle on $X \leftrightarrow$ spectral cover $C \subset X^\vee$
+ line bundle L on a gerbe on C

Explicit formulas for this gerby FM have not been worked out. So alternative:

Construct bundles $\tilde{V}$ on $\tilde{X}$

Lift the F action to $\tilde{V}$

Descend to X

Schoen's CY3's

B, B' rational elliptic surfaces = "dP_9"

$$\boxed{\tilde{X} := B \times_{\mathbb{P}^1} B' \ : \ CY3.}$$

Schoen: examples where $\mathbb{Z}/2$ (& some other groups) act freely on $\tilde{X} \Rightarrow CY3\ X$.

D, Ovrut, Pantev, Waldram:

- explicit geometric construction of Schoen's
- actions of $\mathbb{Z}/2$ and FM on cohomology
- Conditions on C, L for $V = FM(C, L)$ to be $\mathbb{Z}/2$-invariant

$\Rightarrow SU(5)$ stable bundles on X

$$\left| \begin{array}{l} c_1(V) = 0 \\ c_2(V) = c_2(X) - [C_0] \\ c_3(V) = 3 \end{array} \right.$$

Hence E8 bundles V, V' giving strong vacua with LEGG = SU(5), SM.

$\boxed{\text{Our bundles}}$

The spectral data for our $V = V_5$ is reducible.

So instead:

$$\boxed{0 \longrightarrow V_2 \longrightarrow V \longrightarrow V_3 \longrightarrow 0}$$

construct V as an extension between bundles of ranks $= 2, 3$. Each V_i constructed by FM. actually:

$$\begin{tikzcd}
C_i \arrow[r, hook] \arrow[dr] & B \arrow[d] \\
& \mathbb{P}^1
\end{tikzcd}$$

of degree i

over $\mathbb{P}^1$, with line bundle N_i.

$$\boxed{\text{FM} : (C_i, N_i) \longleftrightarrow W_i \quad \text{on } B \neq B^\vee}$$

For appropriate $L_i \in Pic(B^\vee)$,

$$\left[\begin{tikzcd}
& X \arrow[dl, "q"'] \arrow[dr, "q'"] & \\
B \arrow[dr] & & B^\vee \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}\right]$$

$$\boxed{V_i := q^* W_i \otimes q'^* L_i .}$$

need: $\mathbb{Z}_2$ preserves V_2, V_3 and the extension class $\in Ext^1(V_3, V_2)$.

ohomological calculations

Need : $H^1(X, \cdot)$ for $\boxed{\cdot = V, V^*, \Lambda^2 V, \Lambda^2 V^*, \mathrm{ad}\, V}$.

Index $\Rightarrow$ $\boxed{\begin{array}{l} H^1(X, V) - H^1(X, V^*) = 3 \\ H^1(X, \Lambda^2 V) - H^1(X, \Lambda^2 V^*) = 3. \end{array}}$

For the rest: Use Leray for $X \to B$ and $B \to \mathbb{P}^1$ to convert $H^q(X, \cdot)$ for $\cdot = V_i, V_i^*, \Lambda^2 V_3, \Lambda^2 V_3^*, V_2 \otimes V_3$, etc. to cohomology of various VB's on $\mathbb{P}^1$! (Hard work.) Then calculate explicitly (sometimes by computer) the ranks of the coboundary maps.

$\mathbb{Z}_2$ - action

Again, very long calculations.

(3) Physics: results & desiderata

$\boxed{SU(5)\ GUT}$

Using similar techniques, we constructed $SU(5)$-bundles on elliptic CY3 $X \to B$ where B are various Hirzebruch surfaces.

$\Rightarrow$ • Anomaly free heterotic vacua

• LEGG $= SU(5)$

• 3 generations of fermions

$\boxed{\text{Behaviour in families}}$

For $G = SU(3)$ have "standard embedding": $V = T_X$. If you vary complex structure of X, the spectrum remains fixed: $H^q(X, \Lambda^p T_X)$ etc. are deformation invariant.

$\boxed{\text{Surprise}}$: for our families, the spectrum jumps up for V in various closed loci in moduli space.

$\boxed{\text{SM vacua}}$

Our bundles on Schoen threefolds gives

- Anomaly free heterotic vacua
- LEGG = SM = $SU(3) \times SU(2) \times U(1)$
- 3 generations of quarks & leptons
- Various combinations of scalars, e.g. 8/9 Higgs, 10/9 color triplets
- Spectrum jumps in families
- Cubic couplings: probably $\neq 0$, probably vary continuously with moduli

Want: region in moduli where two cubic couplings ("Yukawas") remain large, others vanish.

$\Rightarrow$ * Higgs mechanism can give mass to quarks, leptons

* Dangerous cubic couplings causing nucleon decay are suppressed.

Also exploring: other CY's

$G = SU(4)$

...