A talk for grad students at the Seattle Summer Institute Boot Camp

MODULI SPACES AND WHY ALGEBRAIC GEOMETERS LOVE THEM

ANGELA GIBNEY

1. Introduction

Roughly speaking, a moduli space is a variety (but often a scheme or a stack) whose (closed) points are in one to one correspondence with equivalence classes of algebro-geometric objects of one kind or another. The algebraic geometry of the moduli space reflects the way those objects live in families.

Algebraic geometers love moduli spaces because by studying a particular moduli space we can learn about the objects it parametrizes. The prospective provided by moduli theory has allowed modern algebraic geometers to solve classical problems and make sweeping statements about algebraic varieties.

Called *the siblings* by Steven Kleiman in his beautiful article written for this conference 20 years ago, intersection theory and enumerative geometry have been central to algebraic geometry since very early on. Moduli spaces have proved crucial in the development of both subjects.

A classical enumerative result that you probably learned to solve before you decided to become a math student is that two points uniquely determine a line. Said otherwise, there is exactly one rational curve of degree 1 passing through 2 points in the plane. In fact there is exactly one rational curve of degree 2 passing through five given points in general position in $\mathbb{P}^2$.

One way to generalize this problem is to ask for N_d, the number of degree d rational curves passing through $3d-1$ given points in general position in the projective plane. As stated above, N_1 and N_2 are both one. In 1848, Jacob Steiner proved that $N_3 = 12$ and in 1873 Zeuthen showed that $N_4 = 620$. In the 1980's by using computers, it was shown that $N_5 = 87,304$.

Date: June 29, 2005.

By changing perspective to moduli spaces of stable maps, in 1994, Kontsevich proved the following recursive formula to compute N_d. Namely, that

$$N_d = \sum_{\substack{0 \leq d_a, d_b \in \mathbb{Z} \\ d_a + d_b = d}} N_{d_a} N_{d_b} d_a^2 d_b (d_b \binom{3d-4}{3d_a-2} - d_a \binom{3d-4}{3d_a-1}).$$

Another problem is to count the number of plane curves of degree d of any genus passing through the appropriate number of given points in the plane. This was solved by Lucia Caparaso and Joe Harris using moduli spaces called Severi varieties which parametrize such objects. They also gave a recursive formula giving the solution.

Moduli spaces also help one to make statements about algebraic varieties. For example, by knowing enough about the moduli space of curves, one can show that it is impossible to write down, using free parameters, one single equation describing the "general" curve of genus g for $g \geq 22$. This was proved for $g \geq 24$ by Harris, Mumford and Eisenbud and very recently for $g = 22$ and 23 by Farkas who showed that in this range, the moduli space of curves of genus g is of general type.

By studying the moduli space of curves you can make other kinds of statements about families of curves. For example, Diaz proved that for $g \geq 3$ there is no family of curves of genus g parametrized by a complete (compact) variety of dimension greater than or equal to $g - 1$. On the other hand, you can also prove that for $g \geq 3$ there is always a family of curves of genus g parametrized by a complete curve. So for $g = 3$, this describes all such families. Other results along these lines are known. But the question of whether there is a family of curves parametrized by a complete surface is still completely open for $4 \leq g \leq 7$ (Zaal has constructed a family parametrized by a complete surface defined over a field of characteristic $p \neq 0, 2$).

The upshot is that, even if you didn't know you were interested in moduli spaces, by virtue of being an algebraic geometer, you really are.

My goal for this talk is to define the concepts of moduli problem and moduli space which is the "solution" to a moduli problem.

2. Moduli problems

Algebraic geometry is concerned largely with problems about the classification of algebro-geometric objects. There are three basic ingredients making up a classification problem or a moduli problem in algebraic geometry:

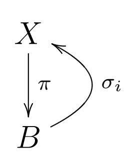
(1) a collection A of algebro-geometric objects;
(2) an equivalence relation $\sim$ on A; and
(3) the notion of an (equivalence class) of families of objects of A.

Some examples are:

(1) a collection A of algebro-geometric objects, eg:
   - smooth curves of genus g;
   - configurations of n distinct points on $\mathbb{P}^1$;
   - morphisms from $\mathbb{P}^1$ to $\mathbb{P}^r$;
   - hypersurfaces of degree d in $\mathbb{P}^r$ or rather the collection of non-zero homogeneous polynomials of degree d (up to constant multiple) or said otherwise still, the projectivisation of the vector space of all degree d homogeneous polynomials in $d+1$ variables,

(2) an equivalence relation $\sim$ on A, eg:
   - smooth curves of genus g up to isomorphism;
   - configurations of n distinct points on $\mathbb{P}^1$ either up to equality (with the trivial relation), or up to projective equivalence;
   - morphisms from $\mathbb{P}^1$ to $\mathbb{P}^r$ up to isomorphism;
   - the projectivization of the vector space V of all degree d homogeneous polynomials in $d+1$ variables with either the trivial relation or the relation given by the natural action of PGL(r+1) on $\mathbb{P}(V)$.

(3) families of objects of A parametrized by a variety B, eg:
   - isomorphism classes of morphisms $\pi : X \longrightarrow B$ where the fibers of π are all (isomorphism clsasses of) smooth curves of genus g;
   - equivalence classes of diagrams



where π is a morphism and the σ_i are n disjoint sections such that a fiber over a (closed) point $b \in B$ consists of $\pi^{-1}(b) \cong \mathbb{P}^1$ together with n disjoint points $\sigma_i(b) \in \mathbb{P}^1$;

- equivalence classes of diagrams

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{P}^r \\ \pi \downarrow & & \\ B, & & \end{array}$$

where π is a morphism such that for every (closed) point $b \in B$,

$$\mu_b = \mu|_b : \pi^{-1}(b) = \mathbb{P}^1 \longrightarrow \mathbb{P}^r,$$

is a morphism.

- Intuitively, a family of hypersurfaces in $\mathbb{P}^r$ parametrized by B should be a closed subset

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B \times \mathbb{P}^r \\ \pi \downarrow & & \\ B & & \end{array}$$

such that $\pi^{-1}(b)$ is a hypersurface in $\mathbb{P}^r$ for each closed point $b \in B$.

If $B = \mathbb{A}^n_k$, such a closed subset X would be defined by a polynomial $f \in k[y_1, \dots, y_n, x_0, \dots, x_r]$ such that

(a) f is homogeneous of degree d in the x_i, and
(b) for all $a = (a_1, \dots, a_n) \in \mathbb{A}^n_k$, the polynomial $f_a = f(a_1, \dots, a_n, x_0, \dots, x_r) \in k[y_0, \dots, y_r]$ is nonzero.

So we may try to think of defining families parametrized by B using elements of $A(B)[x_0, \dots, x_r]$ satisfying the two conditions.

But this doesn't work for all varieties B. For example, if B is projective, then $A(B) = k$ and so any family would be trivial.

To fix the situation for a projective variety B we can take the idea above locally and globalize it by allowing coefficients of the homogeneous polynomials to be sections of a line bunde $\mathcal{L}$ on B rather than as elements of $A(B)$.

Use of the word moduli goes back to Riemann in his 1857 paper on abelian functions. Only in modern times has there been a precise formulation of moduli problems and their solutions.

In order to give the definition of a moduli problem and it's solution, we need a more precise definition of families of elements in A.

2.1. **families.** Let A be a collection of algebro-geometric objects and $\sim$ an equivalence relation on A. Families of objects in A differ greatly depending on A but all of them must satisfy three formal properties which are listed in the following definition.

Definition 2.1. A family of objects of A parametrized by a variety B is a morphism of varieties $\pi : X \longrightarrow B$ which satisfies:

(1) a family parametrized by $spec(k)$ consists of a single object of A;
(2) there is a notion of equivalence of families parametrized by any given variety B, which reduces to the given equivalence relation $\sim$ on A when $B = spec(k)$;
(3) for any morphism of varieties $\phi : B' \longrightarrow B$ and any family X parametrized by B, there is an induced family $\phi^*(X)$ parametrized by B'. Moreover, this operation satisfies the following functorial properties.
   (a) If
   $$B'' \xrightarrow{\phi'} B' \xrightarrow{\phi} B$$
   are morphisms of varieties, then $(\phi \circ \phi')^* = (\phi')^* \circ \phi^*$,
   (b) $id_B^* = id_{B'}$;
   (c) and for families X and X' parametrized by B, and a morphism $\phi : B' \longrightarrow B$,
   $$X \sim X', \implies \phi^* X \sim \phi^* X'.$$

3. Fine moduli spaces

The object of moduli theory is to give the set $A/\sim$ the structure of an algebraic variety M (the moduli space itself) whose geometry reflects the structures of families of objects of A.

We'll use the following notation:

$$\begin{array}{ccccc} X_U & = & X \times_B U & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & U & \longrightarrow & B. \end{array}$$

In particular, if $\pi : X \longrightarrow B$ is a family parametrized by a variety B, then for $b = spec(k) \in B$, by X_b we mean the fiber $\pi^{-1}(b) = X \times_b spec(k)$.

More precisely, suppose M is a variety whose underlying set is $A/\sim$. For any family parametrized by B, we have a so-called classifying map from B to M

$$\begin{array}{ccc} X & & \\ \pi \downarrow & & \\ B & \xrightarrow{\eta_X} & M, \end{array}$$

where if $spec(k) = b \in B$, then $\eta_X(b) = [X_b]$, the isomorphism class of the fiber of X over B. For M to be any sort of moduli space, we want at minimum for η_X to be a morphism and ideally that η_X should define a bijective correspondence between equivalence classes of families parametrized by B and morphisms $B \longrightarrow M$.

This can be nicely expressed in categorical language.

Consider the map between the categories of algebraic varieties and sets:

$$F : Var \longrightarrow Sets,$$

where for a variety B, the set $F(B)$ consists of all equivalence classes of families parametrized by B.

By the third condition satisfied by families parametrized by B, F is a contravariant functor.

Moreover, there are natural maps

$$\phi(B) : F(B) \longrightarrow Hom(B, M),$$

such that if $X \longrightarrow B$ is a family parametrized by B, then

$$\phi(B)(X) = \eta_X : B \longrightarrow M,$$

is the classifying map from B to M. These maps determine a natural transformation of functors

$$\phi : F \longrightarrow Hom(*, M).$$

For M to be a moduli space that "solves" the moduli problem described by F, we ask that ϕ should be a natural isomorphism of functors so that the functor F is represented by (M, ϕ).

Definition 3.1. A fine moduli space for a given moduli problem described by a functor F is a pair (M, ϕ) which represents the functor F.

We should stop and notice two important consequences of this definition:

(1) In the definition of fine moduli space we didn't specify that the underlying set of points of the variety M is $A/\sim$ – we get that for free - for if (M, ϕ) represents F, we have a natural bijection

$$\phi(spec(k)) : F(spec(k)) = (A/\sim) \longrightarrow Hom(spec(k), M) = M.$$

(2) The identity morphism $id_M \in Hom(M, M)$, up to equivalence, determines a family U parametrized by M and for any family $X \longrightarrow B$ the families X and $v_X^* U$ both correspond to the same morphism $\eta_X : B \longrightarrow M$.

This leads to the following alternative definition of a fine moduli space.

Definition 3.2. A fine moduli space consists of a variety M and a family U parametrized by M such that, for every family X parametrized by a variety B, there is a unique morphism $\phi : B \longrightarrow M$ with $X \sim \phi^* U$. Such a family $U \longrightarrow M$ is called a universal family for the moduli problem.

For a fixed variety X, we consider the moduli problem of describing the set of trivial families whose fibers are configurations of n distinct points on X. A family over $B = Spec(k)$ is a configuration of n distinct points on X.

We can see for ourselves that there is a fine moduli space for this moduli problem.

By an n-tuple of points on $\mathbb{P}^1$, I mean an ordered set of n distinct points $p = (p_1, \dots, p_n)$ such that $p_i \in \mathbb{P}^1$. The set $F(\mathbb{P}^1, n)$ of these n-tuples forms an algebraic variety

$$F(\mathbb{P}^1, n) = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \setminus \mathrm{d}iagonals,$$

where the set *diagonals* is the sub-loci where the points in $(\mathbb{P}^1)^{\times n}$ coincide. As we will prove, $F(\mathbb{P}^1, n)$ is a fine moduli space for configurations of n distinct points on the projective line.

To see this we define a functor from the category of algebraic varieties to the category of sets

$$F : Var \longrightarrow Sets,$$

such that if $B \in Obj(\phi(B))$, then $F(B)$ is the set of families of n-tuples over B. We can describe $F(B)$ from two points of view.

On the one hand, we set

$$F(B) = \{\pi : B \times \mathbb{P}^1 \longrightarrow B,\ \sigma_i : B \longrightarrow B \times \mathbb{P}^1, 1 \leq i \leq n\},$$

such that π is the projection onto the first factor and the σ_i are n disjoint sections of π.

A fiber of π over a (closed) point $b \in B$ consists of a copy of $\mathbb{P}^1$ together with n distinct points $\sigma_1(b), .., \sigma_n(b) \in \mathbb{P}^1$ – i.e. an n-tuple.

If B and $B' \in Obj(\phi(B))$, and $f \in Mor_{\phi(B)}(B, B')$, then there is an inclusion of sets $F(B) \subset F(B')$ given by pulling back families over B to families over B' along f. Namely, given a family over B, say $\pi : B \times \mathbb{P}^1 \longrightarrow B$, with n sections $\sigma_i : B \longrightarrow B \times \mathbb{P}^1$, we get a family $\pi : B' \times \mathbb{P}^1 \longrightarrow B'$, with n sections $\sigma_i \circ f : B' \longrightarrow B' \times \mathbb{P}^1$.

On the other hand, since the sections are disjoint, it is straightforward to check that the definition of $F(B)$ above is equivalent to describing the set

$$Hom_{\phi(B)}(B, F(\mathbb{P}^1, n)) = \{\text{morphisms } \ \sigma : B \longrightarrow F(\mathbb{P}^1, n)\}.$$

That is, morphisms $B \longrightarrow F(\mathbb{P}^1, n)$ correspond to families of n-tuples over B.

If B and $B' \in Obj(Var)$, and $f \in Mor_{Var}(B, B')$, then there is an inclusion of sets $Hom_{Var}(B, Q) \subset Hom_{Var}(B', Q)$ given by precomposition with f.

In other words, there is a natural transformation between the functor F and the functor of points of $F(\mathbb{P}^1, n)$. Moreover, the identity map $id_{F(\mathbb{P}^1,n)} \in Hom_{Var}(F(\mathbb{P}^1, n), F(\mathbb{P}^1, n))$ corresponds to the so-called "universal family" over $F(\mathbb{P}^1, n)$. This family is just given by $\pi : F(\mathbb{P}^1, n) \times \mathbb{P}^1 \longrightarrow F(\mathbb{P}^1, n)$ with sections $\sigma_i : F(\mathbb{P}^1, n) \longrightarrow F(\mathbb{P}^1, n) \times \mathbb{P}^1$ for $i \in \{1, .., n\}$ given by

$$\sigma_i(p) = \sigma_i(p_1, .., p_n) = (p, p_i).$$

Note that the fiber of this "universal family" over a point $p \in F(\mathbb{P}^1, n)$ is just the point p. It satisfies the universal property that every family is the pullback of this family (along the identity map).

A perhaps more familiar example of a fine moduli space is $\mathbb{P}^n$ which solves the problem of parametrizing families of lines through the origin in a fixed vector space V of dimension $n+1$. Suppose $\{e_0, \dots, e_n\}$ is a basis for V and $\{f_0, \dots, f_n\}$ is a basis for $V^{\vee}$.

A family of lines through the origin in V parametrized by a variety B is a line bundle $\mathcal{L} \longrightarrow B$ such that $\mathcal{L} \subset V_B = V \times_{spec(k)} B$. Taking the dual of the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow V_B \longrightarrow \mathcal{Q} \longrightarrow 0,$$

gives the short exact sequence

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow V_B^\vee \xrightarrow{f} \mathcal{L}^\vee \longrightarrow 0.$$

Since $V_B^\vee$ is generated by global sections induced by the f_i and since f is surjective, one has that $\mathcal{L}^\vee$ is generated by global sections too. This is equivalent to the existence of a homomorphism

$$B \longrightarrow \mathbb{P}(V),$$

given by the assignment $b \mapsto [f \circ f_0(b) : \ldots : f \circ f_n(b)]$. In other words, families of lines through the origin in V are in one to one correspondence with morphisms from B to $\mathbb{P}^n$.

4. Coarse Moduli Spaces

Unfortunately fine moduli spaces are rare. More commonly one comes across varieties that are almost fine moduli spaces. The general rule of thumb when working with a moduli problem F is that the kinds of isomorphism classes of elements of $A/\sim$ determine the kind of moduli space you can expect to have. In particular, if there are no non-trivial isomorphisms of the elements $A/\sim$, then you can expect to find a fine moduli space for F, otherwise, the best you can hope for is a coarse moduli space.

If there is an element $E \in A/\sim$ that has a nontrivial finite automorphism group G, then you can use the group G to define a nontrivial family $X \longrightarrow B$ whose fibers are all isomorphic to E. This means there could not be a fine moduli space M for the problem of describing families of elements in $A/\sim$. For if such a variety M were to exist, then since the image of B under the classification map $\eta_X : B \longrightarrow M$ would then be just a point, the total space X would have to be $X = B \times_{spec(k)} U$ where U is the universal family of M.

It is much more common for M not to satisfy this "universal property" for families but that ϕ has a universal property for natural transformations $F \longrightarrow Hom(*, N)$.

Definition 4.1. A coarse moduli space for a given moduli problem described by a functor F is a variety M together with a natural transformation of functors

$$\phi : F \longrightarrow Hom(*, M),$$

such that

(1) $\phi(spec(k)) : F(pt) = A/\sim \longrightarrow Hom(spec(k), M) = M$ is bijective,
(2) For any variety N and any natural transformation $\psi : F \longrightarrow Hom(*, N)$, there is a unique natural transformation

$$\Omega : Hom(*, M) \longrightarrow Hom(*, N),$$

such that $\psi = \Omega \circ \phi$.

The variety M_g of smooth curves of genus g is an example of a coarse moduli space.

5. COMPACTIFICATIONS

The fine moduli space $F(\mathbb{P}^1, n)$ is a nice smooth variety but even for $n = 2$ isn't compact. For example, it's easy to write down a sequence of configurations of just two points on $\mathbb{P}^1$ that get closer and closer to gether and so whose limit is not a configuration of distinct points on $\mathbb{P}^1$.

It is common, when working with a variety that isn't compact, to try and find a compactification of it that is easy to work with - that way one can bring to bear the tools of projective geometry to study the variety.

In the case of moduli spaces, the compactifications that work best are modular compactifications – ones which are themselves moduli spaces. $F(\mathbb{P}^1, n)$ has been shown to have what is called a modular compactification by $P^1[n]$, the Fulton-MacPherson space of so called stable configurations of n points.

In the case of compactifying the moduli space of smooth curves, different compactifications have given different interesting results about curves. The results that M_g is of general type for $g \geq 22$ and the upper bound on the dimension of a compact subvariety parametrizing a family of curves of genus g were proved by using the Deligne-Mumford-Knudsen compactification by stable curves of genus g (nodal curves having finitely many automorphisms). While the Satake compactification was used to prove the fact that for any $g \geq 3$ there is a family of curves of genus g parametrized by a compact curve.