Tom Graber, <u>Intro to Stacks</u>

<u>Dictionary</u>:

Stacks	Schemes
stack	scheme
morphism	morphism
sheaf	sheaf
⋮	⋮

Similar "dictionary" for Varieties & Schemes.

But you would make some mistakes if you think of schemes just as varieties.

<u>E.g. (Mistake)</u>: $f, g : S \to T$ (true for varieties but not schemes)

Then $f = g \iff f(s) = g(s) \ \forall s \in S$

<u>E.g. (Non-mistake)</u> $X, Y \subseteq Z$, $p \in X \cap Y$

Then $T_p(X \cap Y) = T_p X \cap T_p Y$ (false for varieties but <u>true</u> for schemes)

<u>E.g. (Non-mistake)</u> $X = X_1 \cup X_2$, $X_1 \cap X_2 = Z$.

Then $\mathrm{Hom}(X, Y) = \mathrm{Hom}(X_1, Y) \times_{\mathrm{Hom}(Z,Y)} \mathrm{Hom}(X_2, Y)$

(false for varieties, true for schemes)

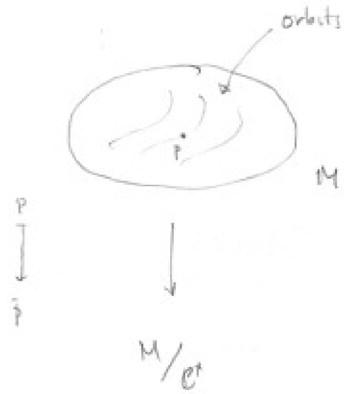
$\Rightarrow$ perhaps schemes better than varieties.

Similarly, we'll see why perhaps stacks better than schemes

<u>Example</u>: M smooth scheme

$\mathbb{C}^* \circlearrowright M$ w/o fixed pts.

Picture:



Consider $M/\mathbb{C}^*$

Guess: $\underbrace{T_{\bar{p}}(M/\mathbb{C}^*) \cong T_p(M) / T_p(\text{orbit})}_{\text{true for stacks, not for schemes}}$

$\boxed{\text{Origin of Stacks}}$

<u>Moduli problems</u>: Given a class of geometric objects

(e.g. degree d plane curves, or
line bundles on a scheme X, or
smooth connected curves of genus g)

Find a scheme whose closed points naturally parameterize the objects

(e.g. projective space of homo. deg. d polynomials, or
$\mathrm{Pic}(X)$, or
$\boxed{\mathcal{M}_g}$)

To "solve" this problem is to exhibit a scheme, $\mathcal{M}_g$, s.t.
the closed pts are <u>naturally</u> in bijection w/ the set of iso. classes of curves

E.g. $g=1 \longleftrightarrow$ elliptic curves, are determined (up to $\cong$) by the j-invariant

So: isom. classes $\longleftrightarrow$ $\mathbb{C} = \mathbb{A}^1$

Becomes more precise by considering families of curves:

A morphism of schemes $\begin{array}{c} X \\ \downarrow \pi \\ S \end{array}$ (flat, proper, etc.)

s.t. $\pi^{-1}(s)$ is a curve $\forall s$.

Picture:

For every $s \in S$, we have a curve $C_s := \pi^{-1}(s)$

Suppose $\mathcal{M}_g$ is our moduli space (of curves).

Then we have a map of sets:

$$\begin{aligned} f: S &\longrightarrow \mathcal{M}_g \\ s &\longmapsto [C_s] \end{aligned}$$

This should actually be a morphism of schemes

$\rightsquigarrow$ $\{\text{families of curves}/S\}/_{\cong} \longrightarrow \mathrm{Hom}(S, \mathcal{M}_g)$

$\rightsquigarrow$ $\{\text{families of curves}/\,\cdot\,\}/_{\cong} \xrightarrow{\ \Phi\ } \mathrm{Hom}(\cdot, \mathcal{M}_g)$

both sides are functors $\underline{Sch} \to \underline{Sets}$

Φ is a natural transformation of functors

Dream: Φ should be an isomorphism

If we have such $(\mathcal{M}_g, \Phi)$, then there is some universal family

$$\begin{array}{c} C \\ \downarrow \\ \mathcal{M}_g \end{array} \quad \rightsquigarrow \quad id \in \mathrm{Hom}(\mathcal{M}_g, \mathcal{M}_g)$$

Sadly, this is not possible. :(

Reason: Suppose we have $\mathcal{M}_g$.

Consider two curves meeting in a pt:

$S =$  $= S_1 \cup S_2$, $\quad S_1 \cap S_2 = pt$.

Then should have

$$\underset{\{\text{families of curves}/S\}/\cong}{\underset{\parallel}{\mathrm{Hom}(S, \mathcal{M}_g)}} = \underset{\{\text{fam.}/S_1\}/\cong}{\underset{\parallel}{\mathrm{Hom}(S_1, \mathcal{M}_g)}} \underset{\underset{\{\text{curves}\}}{\underset{\parallel}{\mathcal{M}_g}}}{\times} \underset{\{\text{fam.}/S_2\}/\cong}{\underset{\parallel}{\mathrm{Hom}(S_2, \mathcal{M}_g)}}$$

Is this true?

We have a map $\psi : LHS \to RHS$

ψ is surjective but not injective (gluing together families requires knowing iso. data on overlap)

Remarks: (1) $\nexists$ moduli scheme $\mathcal{M}_g$ (in this sense)

(2) To understand (gluing of) families of curves, we need to remember honest families and choices of isos.

Better:

$\rightsquigarrow$ (1) A moduli problem is a category

(2) The stack is the moduli problem

Note: A scheme X is determined by the functor $\mathrm{Hom}(\cdot, X)$

<u>Def</u>: A <u>stack</u> is a category F (w/ additional properties) and a functor to <u>Sch</u>

<u>Example</u>: The stack $\mathcal{M}_g$

objects: families of curves $\begin{array}{c} C \\ \downarrow \\ S \end{array}$

arrows: cartesian diagrams
$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_2 \arrow[d] \\ S_1 \arrow[r] & S_2 \end{tikzcd}$$

A morphism of stacks $F_1 \to F_2$ is a functor (compatible w/ maps to <u>Sch</u>)

Now: $\mathrm{Hom}(S, \mathcal{M}_g)$ = families of curves

a category, not a set!

$\Rightarrow$ category of stacks is a 2-category

<u>Simplest Example</u> $B\mathbb{Z}/2$ (Moduli space of $:$)

objects: $\begin{array}{c} P \\ \downarrow \\ S \end{array}$, a finite, étale deg 2 morphism

arrows: cartesian diagrams

(Note: $\mathrm{Hom}(S, F)$ = preimage of S in F under $F \to$ <u>Sch</u>)

Now consider $S =$ 

& two maps. $f: S \to B\mathbb{Z}/2$ via 

$g: S \to B\mathbb{Z}/2$ 

We can cover S by open sets

$U = S \setminus p$

$V = S \setminus q$

Observe that:

$f|_U =$  , $f|_V =$ 

& $g|_U =$  , $g|_V =$ 

$\leadsto$ families not determined by open cover

<u>Simple Example</u> BG (G a finite group) (<u>Classify stack for G</u>)

objects: $\begin{matrix} P \\ \downarrow \\ S \end{matrix}$ finite étale morphism w/ free G-action whose quotient is S (i.e. principal G-bundle)

arrows: cartesian diagrams

<u>Turns out</u>: quasi-coherent sheaves on BG $\longleftrightarrow$ representations of G

sheaf cohomology $\longleftrightarrow$ group cohomology