Density of rational points

July 22, 2005

Notation k field $\begin{cases} k = \mathbb{Q}(\alpha_1 \ldots \alpha_m) \text{ fin gen.} / \mathbb{Q} \\ k = \mathbb{C}(B) \quad B \text{ curve} \end{cases}$

X/k sm. var. $/k$

$= \{ f_1(x_1 \ldots x_n) = \cdots = f_r(x_1 \ldots x_n) = 0 : f_j \in k[x_1 \ldots x_n] \}$

$X(k)$ ratl. points

$= \{ (a_1 \ldots a_n) \in k^n : f_j(a_1 \ldots a_n) = 0 \}$

Guiding problem: Is $X(k)$ nonempty? Dense in X? Potential dense in X?

Defn: Rational points are potentially dense in X if there exists a finite extension L/k with $X(L)$ dense in X

Ex: • $x^2+y^2=1$ $(k=\mathbb{Q})$

$$x=\frac{-2t}{1+t^2} \quad y=\frac{1-t^2}{1+t^2} \quad t\in\mathbb{Q}$$

$\mathbb{Q}$-rational points are dense

• $x^2+y^2=-1$

$X(\mathbb{Q})=\emptyset$

$X(\mathbb{Q}(i))$ dense

Elementary Properties

$X \xrightarrow{\varphi} Y$ morphism $/k$

① φ dominant

density holds for $X \Longrightarrow$ density holds for Y

② φ étale, X, Y proper $k=\mathbb{C}(B)$, number field

potential density holds for $Y \Longrightarrow$ pot. dens. holds for X

proof Suppose $\mathbb{C}(B)$-ratl points are dense in Y (B smooth proj curve)

Extend φ to models

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\varphi"] \arrow[dr, "\text{proper}"'] & \mathcal{Y} \arrow[d, "\text{proper}"] \\ & B \end{tikzcd}$$

$\deg\varphi=d$

φ ramifies over $\{b_1 \ldots b_m\} \subset B$ July 22, 2005
(unramified elsewhere) ②

Rational points of Y $\rightsquigarrow$ $\mathbb{C}(\tilde{B})$-rational points of X

$$\tilde{B} \longrightarrow B \quad \text{degree } d, \text{ branched over } b_1 \ldots b_m$$

There are only finitely many such $\tilde{B}$, so for some $L = \mathbb{C}(\tilde{B})$, $X(L)$ is dense in X $\square$

Cor. • Density of rational points is a birational property

• X unirational $/\bar{K}$ $\Rightarrow$ potential density holds for X

• $X \subset \mathbb{P}^n$ smooth hypersurface of degree $d \ll n$
$\Rightarrow$ Potential density holds for X
[Harris Mazur Pandharipande]

d	1	2	3	4
n	≥ 2	≥ 2	≥ 3	≥ 7 ??

n grows very fast as $d \to \infty$

...n density results

Theorem (Faltings) C/k smooth curve $g(C) \geq 2$
(nonisotrivial if $k = \mathbb{C}(B)$)

Then $\# C(k) < \infty$ (and potential density fails)

Def: X smooth proj. $\dim(X) > 0$

X is of *general type* if $\dim \Gamma(X, \omega_X^{\otimes m}) \sim m^{\dim(X)}$
$m \gg 0$.

e.g. $X \subset \mathbb{P}^n$ hypersurface of degree $d > n+$

Conj (Lang, Bombieri, [redacted] Vojta)

X/k of general type
(nonisotrivial if $k = \mathbb{C}(B)$)

Then $X(k)$ is not dense in X.

Consequences:

① X dominates variety of general type
$\Rightarrow X(k)$ not dense.

②
$$\begin{tikzcd} \tilde{X} \arrow[r, "\text{dominant}"'] \arrow[d, "\text{étale}"'] & Z \text{ general type} \\ X & \end{tikzcd}$$

$\Rightarrow X(k)$ not dense

Converse to Lang Conjecture

July 22, 2005

4

X smooth proj. $/k$.

Assume there exists no diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\text{dominant}"] \arrow[d, "\text{étale}"'] & Z \\ X & \end{tikzcd}$$

Z general type nonisotrivial

Then rational points of X are potentially dense.

Ex X/k (geom) simple abelian variety

$\exists\, L/k$ finite so that

$$\operatorname{rank} X(L) > 0$$

$Y = \overline{X(L)}$ is union of translates of abelian subvarieties

$= X$ $\square$

First open case is K3 surfaces –

$K_X < 0$	$K_X = 0$	$K_X > 0$
expect many rational points		expect sparse rational points

Density for elliptic K3 surfaces

July 22, 2005

5

Theorem (Bogomolov – Tschinkel)

X/k K3 surface admitting an elliptic fibration $\varphi : X \longrightarrow \mathbb{P}^1$

Then potential density holds

Step I

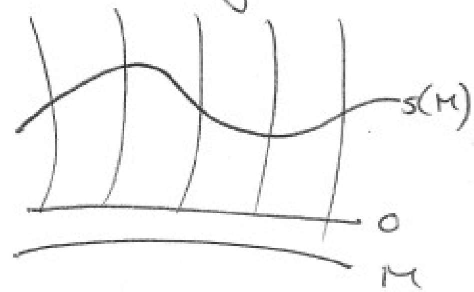
Lemma $Y \longrightarrow M$ elliptic fibration over curve M, $g(M) = 0, 1$ with 0-section

If $\operatorname{rank}(Y/M) > 0$ then potential density holds

Proof.

$s : M \longrightarrow Y$ nontorsion

$$\begin{tikzcd} Y \arrow[d, shift right] \\ M \arrow[u, shift right, "0"'] \arrow[u, bend right, "s"'] \end{tikzcd}$$



$\Longrightarrow \bigcup_{N \geq 1} N[s(M)] \subset Y$ dense

$\exists\, L/k$ finite with $M(L)$ dense

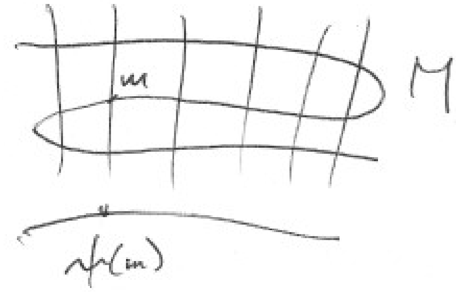
$\therefore$ $Y(L)$ dense in Y $\square$

Step II Let $M \subset X$ be a curve of genus $0, 1$ dominating $\mathbb{P}^1$

Assume that for some $m \in M$

$$\deg(M/\mathbb{P}^1) \cdot m - (M \cap X_{\varphi(m)}) \in \operatorname{Jac}(X_{\varphi(m)})$$

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \varphi} \\ \mathbb{P}^1 \end{array}$$



has infinite order

July 22, 2005
(6)

Then $$\begin{array}{c} Y = X \times_{\mathbb{P}^1} M \\ \downarrow \\ M \end{array}$$ satisfies conditions of the lemma

Step III (Hard) Produce a suitable M

Techniques ① Classification of degenerate fibers and analysis of Tate Shafarevich group of Jacobian

② Deformation theory of curves in X.

Open problem

$k = \mathbb{Q}(\alpha_1 \dots \alpha_m)$

X/k K3 surface with

$\mathrm{Pic}(X_{\bar{k}}) \simeq \mathbb{Z}$

e.g. general quartic surface $X/\mathbb{Q}$

Does potential density hold for X?

Not one example is known

[There are examples for $k = \mathbb{C}(\mathbb{P}^1)$]

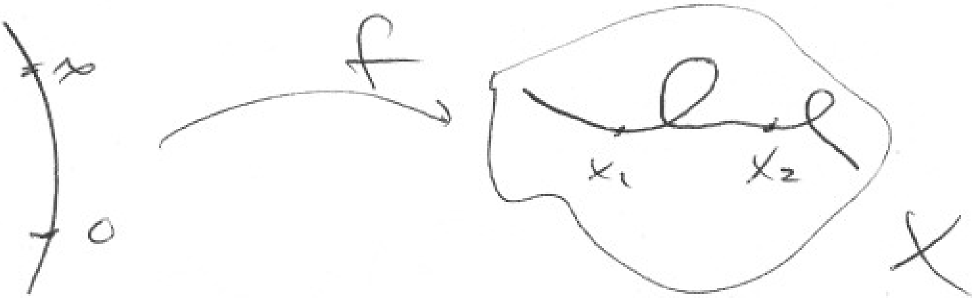
Density for rationally connected varieties July 22, 2005

(7)

X/F smooth proper variety over uncountable alg. closed field

Defn: X is rationally connected if for very general points $x_1, x_2 \in X$ there exists a rational curve joining them i.e.

$$f : \mathbb{P}^1 \longrightarrow X$$
$$f(0) = x_1 \quad f(\infty) = x_2$$



Ex:

- $X \subset \mathbb{P}^n$ smooth hypersurface of degree $d \le n$
- X rational, unirational
- X Fano ($-K_X$ ample)

July 22, 2005

(8)

Now B smooth proj curve

$K = \mathbb{C}(B)$

Existence Theorem (Graber Harris Starr)

$X/_K$ rational connected

Then $X(K) \neq \phi$

Geometric interpretation: WLOG assume X is smooth and projective

Fix a model

- π flat and projective over B
- $\mathcal{X}$ nonsingular

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ B \end{array}$$

k-rational points of X $\Longleftrightarrow$ sections $s: B \longrightarrow \mathcal{X}$

Density Theorem (Kollár Miyaoka Mori)

Fix distinct points $b_1 \dots b_r \in B$ so that $\mathcal{X}_{b_i} = \pi^{-1}(b_i)$ smooth $i = 1 \dots r$

and points $x_i \in \mathcal{X}_{b_i}$

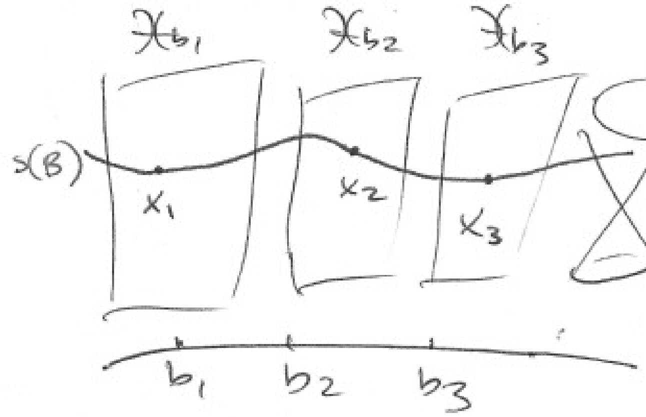
Then $\exists$ section

$$s : B \longrightarrow \mathcal{X}$$

with $s(b_i) = x_i$

$i = 1 \dots r$

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ B \end{array}$$



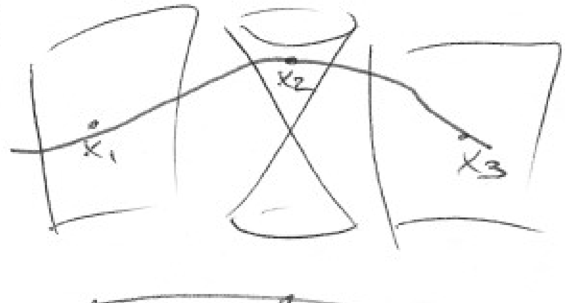
Cor $X(k)$ is dense in X.

Open problem: Approximation

Fix distinct points $b_1 \dots b_r \in B$

and smooth points $x_i \in \mathcal{X}_{b_i}$ $i = 1 \dots r$

($\mathcal{X}_{b_i}$ possibly singular)



Is there a section

$$s : B \longrightarrow \mathcal{X}$$

with $s(b_i) = x_i$?

Example: Fibers ~~[illegible]~~ cubic surfaces