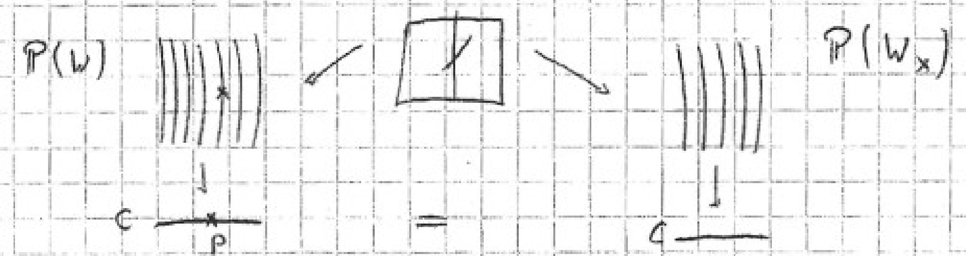
Why rational curves ?

Answer: Because they are there & because they can help us study varieties that contain them.

## § 1 Examples of varieties with rational curves

- Hypersurfaces in $\mathbb{P}_n$ of low degree, complete intersections with few equations of low degree
- some moduli spaces
  - moduli of curves of low genus (can be written by equations with free parameters)
  - moduli of vector bundles on curves.

  Basic idea: elementary transformation




gives map fiber over p → Space of vector bundles over C

- FANO-manifolds (⇝ big result of S. MORI)
- exceptional loci in the minimal model program

## § 2 Varieties covered by lines

Let $X \subset \mathbb{P}_N$ be a hypersurface which is covered by lines.

(classically studied in complex differential geometry)

Interesting object: Variety of tangent directions to lines

Def Given $x \in X$, let $\mathcal{C}_x = \{ \vec{v} \in \mathbb{P}(T_X|_x) \mid \vec{v}$ is tangent to a line in ~~P~~ $X$ through $x \}$

$$\mathcal{C} = \bigcup_{x \in X} \mathcal{C}_x$$

Idea: these are directions of minimal curvature.

attribution debatable

Cartan / Fubini:

If $X_1, X_2$ are as above, $U_i \subset X_i$ analytic open sets, $\varphi : U_1 \to U_2$ an isomorphism that respects $\mathcal{C}$, ~~#~~ and if some technical conditions hold, then $\varphi$ extends to a global isomorphism. □

Sometimes enough that $\mathcal{C}_i|_{U_i}$ are isomorphic.

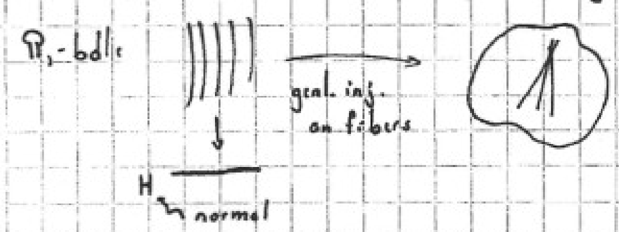
## §3 Varieties not covered by lines

Problem: The cubic surface is not covered by lines. What to do?

Idea: The cubic surface is covered by conics. Try to use these

Fact There exists a parameter space for irreducible, generically reduced rational curves on $X$, $\mathrm{RatCurves}^n(X)$, similar to Grass.

A family of ratl. curves is a diagramm




Can think of $\mathrm{RatCurves}^n(X)$ as a fine moduli space.

Definition $H \subset \mathrm{Rat\,Curves}^n(X)$ an irreducible component whose members dominate $X$ such that no other component of curves of lower degree has this property, is called dom. family of rational curves of min. degree.

Fix $H$ throughout

Idea: These curves are a good substitute for lines.

(„like lines" or „like geodesics")

Definition Let $x \in X$ be a genl. point. Set

$$H_x = \{ \text{curves } \ell \in H \mid x \in \ell \}$$

<u>Fact:</u> Let $x \in X$ be a genl. point.

- $H$ need not be compact, but $H_x$ is compact.
- If $y \in X$ is any other point, only finitely many curves in $H$ contain both $x$ and $y$.
- Only finitely many curves are singular at $x$, and their singularities are immersed.
- The rational map

$$\begin{array}{cccc} \tau_x : & H_x & \dashrightarrow & \mathbb{P}(T_X|_x) \\ & \ell & \longmapsto & \mathbb{P}(T_\ell|_x) \end{array}$$

is birational, becomes a morphism after going to the normalization of $H_x$.

<u>Thus</u> $V_x :=$ Image of $\tau_x$ is „variety of min. ratl. tgts. at $x$".

Again, this is ~~the~~ a very important invariant.

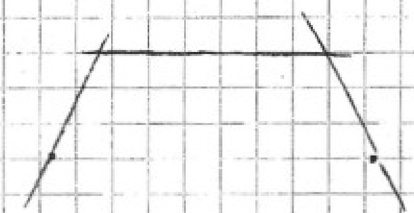
Fact (Hwang-Mok) Statement of Cartan-Fubini holds in our setup.

Many important properties of X can be read off $\mathcal{C}_x$: stability of $T_X$, length, deformation rigidity, ..

## §4 Application Examples

### Length of a rationally connected manifold

Definition: X is rationally connected by H-curves $\Leftrightarrow$ any two general point $x, y \in X$ can be joined by a chain of rational curves from H.



Question: ~~How~~ Given X, how many curves are needed to connect x and y?

[If I know how many, I can make a number of effective statements about X. For instance, using Nadel's product theorem, one can give a bound how singular zero loci of section in a vector bundle ~~B~~ can be if one knows the restriction of the vector bundle to the curve.]

Stronger Question: Given $X \ni x$ look at

$loc^1 x =$ locus of $H_x$-curves

$loc^2 x =$ locus of $H_y$-curves where $y \in loc^1$

$\vdots$

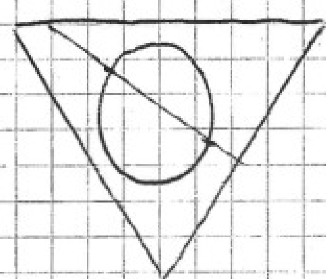
$loc^n x =$ locus of length-$n$ chains through $x$

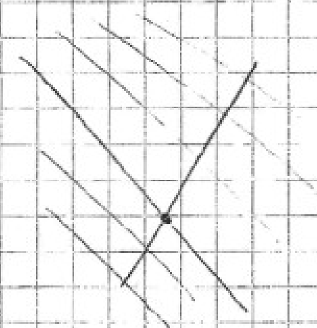
How to compute $d_k =$ ~~loc~~ $\dim loc^k(x)$

Answer: Look at secant varieties of $\mathcal{C}_x$

Idea

$\mathbb{P}(T_x|_x)$





(1) Deformation Rigidity

Given a smooth proj. morphism

$$\begin{array}{c} X \\ \pi \downarrow \\ \Delta \end{array}$$

where $\pi^{-1}(0)$ is unknown and $\pi^{-1}(t)$ are all isomorphic ~~and~~ to $S$, where $S$ is rational hom. of Picard number 1. Then $\pi^{-1}(0) \cong S$.

Hong-Mok

Due to Siu for $\mathbb{P}_n$

<u>Idea:</u> If $S$ is ~~on~~ of this special form, $\ell_x$ will also be rather special. Use induction: since $\ell_x$ cannot deform, $S$ cannot deform.

Other: classification problems, stability...