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Wed 9:45 (Mantovan):

Langlands' Conjecture and Shimura Varieties

Langlands' Program: for a field F, study

$$\mathcal{G}_F = \mathrm{Gal}(\bar{F}/F)$$

in particular, F a number field (or $F = \mathbb{Q}$)

In particular, study n-dim rep^{n}s of $\mathcal{G}_{\mathbb{Q}}$.

When $n=1$, this is just <u>class field theory</u>

characters of $\mathcal{G}_{\mathbb{Q}} \longleftrightarrow$ some characters of $\mathbb{A}^\times$, where

$$\mathbb{A} = \prod_p' \mathbb{Q}_p$$

$\Downarrow$ generalize

n-dim rep^{n}s of $\mathcal{G}_{\mathbb{Q}} \longleftrightarrow$ <u>automorphic</u> rep^{n}s of $GL_n(\mathbb{A})$

But can do automorphic rep^{n}s of $G(\mathbb{A})$, G any linear algebraic group. ($G = Sp_{2n}, SO_{2n+1}, \dots$). Should correspond to $\mathrm{Hom}(\mathcal{G}_{\mathbb{Q}}, {}^LG)$ (${}^LG$ = Langlands dual)

– What to encode?

$n=1$: class field theory

$$\pi: \mathbb{A}^\times \to \mathbb{C}^\times \rightsquigarrow \chi_\pi : \mathcal{G}_{\mathbb{Q}} \to \mathbb{C}^*$$

for all p, $\mathrm{frob}_p \in \mathcal{G}_{\mathbb{Q}}$ <u>lifts</u> $x \mapsto x^p$ in pos. char $/\mathbb{F}_p$.

$\langle \mathrm{Frob}_p \rangle = W_p \subseteq \mathcal{G}_{\mathbb{Q}}$ is <u>dense</u> in $\mathcal{G}_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$

<u>Chebotarev's Density Theorem:</u> $\langle \mathrm{frob}_p : \text{almost all } p \rangle \subseteq \mathcal{G}_{\mathbb{Q}}$ dense

for almost all p, $\chi_\pi(\mathrm{frob}_p) = \pi(p)$ only depends on p, unramified @ p.

Joe Rabinoff

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What should the Langlands correspondence encode?

$$\mathrm{Rep}_{aut}(GL_n(\mathbb{A})) \longleftrightarrow n\text{-dim rep}^{\underline{n}}\text{s of } \mathcal{G}_{\mathbb{Q}}$$

$$\pi \longleftrightarrow \Sigma_\pi$$

almost all p, $L_p(\Sigma_\pi)$ = inverse of characteristic poly of frob_p

$$\shortparallel$$

$L_p(\pi)$ = inverse of π_p, the p-adic component of π

where $GL_n(\mathbb{A}) \curvearrowright \pi = \bigotimes_{\pi_p} \pi_p \curvearrowleft GL_n(\mathbb{Q}_p)$

eg. $\pi: \mathbb{A}^\times \to \mathbb{C}^\times$, $\prod_p \pi|_{\mathbb{Q}_p^\times} = \pi$

- Goal is to study L-functions: $L = \prod_p L_p$ $L(\pi) \longleftrightarrow L(\Sigma)$
Supposed to satisfy amazing properties: Artin's Conjecture.
(the $L(\pi)$ relatively well understood)

When $n=1$ (Cl $\overline{\text{Global}}$ field theory)

$$\begin{array}{ccc} \text{char. of } \mathbb{A}^\times & \xrightarrow[\text{CFT}]{\text{global}} & \text{char of } \mathcal{G}_{\mathbb{Q}} \\ \downarrow \text{restriction} & \circlearrowright & \downarrow \text{restriction} \\ \text{char of } \mathbb{Q}_p^\times & \xleftrightarrow[\text{CFT}]{\text{local}} & \text{char of } W_p \end{array}$$

commutes

Compatibility between global + local class field theory

Langlands' Conjecture: (generalization)

$$\begin{array}{ccc} \mathrm{Rep}(GL_n(\mathbb{A})) & \xrightarrow[\text{corresp.}]{\text{Global}} & n\text{-dim'l rep of } \mathcal{G}_{\mathbb{Q}} \\ \pi & \longmapsto & \Sigma \\ \downarrow & & \downarrow \\ \pi_p & \longmapsto & \Sigma|_{W_p} \\ \mathrm{Rep}(GL_n(\mathbb{Q}_p)) & \xrightarrow[\text{corresp}]{\text{Local}} & n\text{-dim'l rep of } W_p \end{array}$$

should commute

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- Next natural case to look at: $n=2$.
  What is known?
  GL_2: modularity of elliptic curves/$\mathbb{Q}$ (Wiles 1995, et al)
  this says: for an elliptic curve $E/\mathbb{Q}$, $\exists f$ a modular form of wt. 2; then $L(E)=L(f)$.

global Langlands: $\mathrm{Rep}(GL_2(\mathbb{A})) \longleftrightarrow$ 2-dim rep^n of $\mathcal{G}_\mathbb{Q}$

$H^1(E,\mathbb{Q}_\ell)=\rho_E$ $\circlearrowleft$ $L(\rho_E)=L(E$

classical result $\Vert\!\!\rightarrow L(f)=L(\pi_f)$ π_f f

modular form $\xrightarrow[\text{Deligne}]{\text{Eichler-Shimura}}$ E elliptic/$\mathbb{Q}$

- More results for local Langlands:
  any n, $GL_n/\mathbb{Q}_p$ in 1998 by Harris-Taylor
  $G/\mathbb{Q} \rightsquigarrow X^{an}$ $\mathbb{C}$-manifold with action by $G(\mathbb{A})$
  then $G(\mathbb{A}) \curvearrowright H^\circ(X^{an},\mathbb{C}) \longrightarrow$ this is an understood automorphic rep^n.

  <u>Aside</u>: $X^{an}_K = {}_{G(\mathbb{Q})}\backslash G(\mathbb{A})/K$, $K \subseteq G(\mathbb{A})$ a compact subgp
  $K_N = \{A \equiv 1(N)\}$ $N \geq 1$

<u>Thm</u> (Shimura): for certain $G/\mathbb{Q}$, $\exists$ algebraic models for X^{an}_K
i.e., $\exists$ alg. variety $X_{K/\mathbb{Q}}$ st. $X_K(\mathbb{C}) = X^{an}_K$
$\circlearrowleft G(\mathbb{A})$ $G(\mathbb{A})$-equivariant

So get different cohomologies:

$$H^\circ(X^{an},\mathbb{C}) = H^\circ_{et}(X\otimes_\mathbb{Q}\overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \qquad (\mathbb{C}\cong\overline{\mathbb{Q}}_\ell)$$

$\circlearrowleft$ $G(\mathbb{A})$ $\circlearrowleft$ $\mathcal{G}_\mathbb{Q} \ni \sigma : (1\times\sigma)^*$ $\circlearrowleft$ $G(\mathbb{A}) \ni g : (g\times 1)^*$

<u>certain</u> $G/\mathbb{Q} \rightsquigarrow H^{\bullet}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \curvearrowleft G(\mathbb{A}) \times \mathcal{G}_{\mathbb{Q}}$

ok for GL_2, not GL_n, $n \geq 3$

and ~~ok~~ for some $G/\mathbb{Q}$, ~~[illegible]~~ $G|_{\mathbb{Q}_p} \cong GL_2|_{\mathbb{Q}_p}$.

If $\pi \curvearrowleft G(\mathbb{A})$ an autom. rep^n $\longmapsto$ π-isotypic component $\circlearrowleft \mathcal{G}_{\mathbb{Q}}$

i.e., $$H^{\bullet}(X) \overset{s.s.}{=} \sum_{\pi} \underset{\circlearrowleft G(\mathbb{A})}{\pi} \otimes \Sigma_\pi \curvearrowleft \mathcal{G}_{\mathbb{Q}}$$

$$\boxed{\text{Hope } \Pi^{G(\mathbb{A})} \longmapsto \Sigma_\pi \curvearrowleft \mathcal{G}_{\mathbb{Q}} \text{ is a Langlands correspondence}}$$

For GL_2, $X_K^{an} = \Gamma \backslash \mathfrak{h}$ (half plane) $\Gamma \subseteq SL_2(\mathbb{Z})$

$K = \{A \equiv \mathbb{1}_2 (N)\}$

$\Gamma = \{A \equiv \mathbb{1}_2 (N\} \subseteq SL_2(\mathbb{Z})$

use moduli spaces of elliptic curves $/\mathbb{C}$

$\rightsquigarrow$ points of $X_K^{an} \ni \tau \longmapsto \mathbb{C}/\langle 1, \tau \rangle = E_\tau/\mathbb{C}$

Since the mod spc of ell. curves $/\mathbb{C}$ is algebraic,

$$X_K^{an} = G(\mathbb{Q}) \backslash G(\mathbb{A})/K = \text{some moduli space of abelian varieties } /\mathbb{C}$$

$\rightsquigarrow X_K =$ moduli space of abel. varieties $/\mathbb{Q}$.

Now (use alg geom) $\rightsquigarrow$ info about

$$H^{\bullet}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)\big|_{W_p} \curvearrowleft G(\mathbb{A}) \times \mathcal{G}_{\mathbb{Q}}$$ ~~[illegible]~~

$$\Big\| \qquad \Pi \longmapsto \Sigma_\Pi\big|_{W_p}$$

$$H^{\bullet}(X_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell) \curvearrowleft G(\mathbb{A}) \times W_p$$

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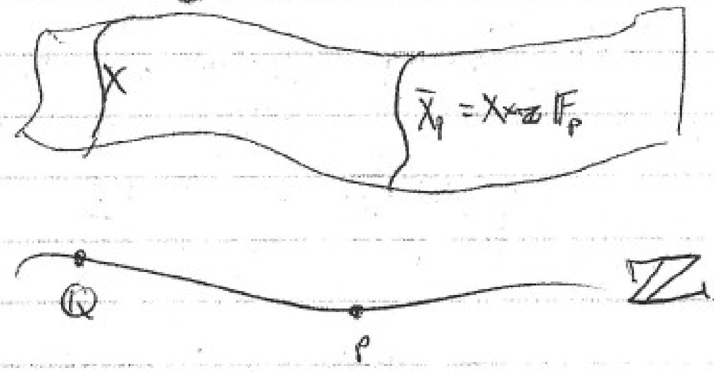
- How to use alg geometry to study this rep^{n}?

What does
$$H^{*}(X_{\mathbb{Q}_p} \times_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \overline{\mathbb{Q}}_\ell) \circlearrowleft G(\mathbb{A}) \times W_p$$
encode (as a variety)?



Get: • geometry of fiber $\bar{X}_p$
- • monodromy at p

need X proper, with $G(\mathbb{A})$ action
$\rightsquigarrow$ so hard to compactify Shimura varieties.
$\Rightarrow$ we want to study the geometry of moduli spaces of abelian varieties in char. p.

- Two more alg-geom tools to get info about this cohomology (as a rep^{n}).

$\to$ In general, if $G \circlearrowright V$, + you know $\mathrm{tr}(g)$ $\forall g \in G$, you basically know V.

In alg geom, have the trace formula.

Eg: Stratification. $X \supseteq U$ open, $Z = X \setminus U$, closed.

Get Long exact seq $\frac{1}{e}$
$$\cdots \longrightarrow H^i_c(U) \longrightarrow H^i(X) \longrightarrow H^i(Z) \longrightarrow H^{i+1}_c(U) \longrightarrow \cdots$$

If the strat. preserved by gp action, this is a long exact seq of repⁿs.

Set $H^\bullet(X) = \sum (-1)^i H^i(X)$
$= H^\bullet(Z) \oplus H^\bullet_c(U)$

This is very useful for breaking up repⁿs.

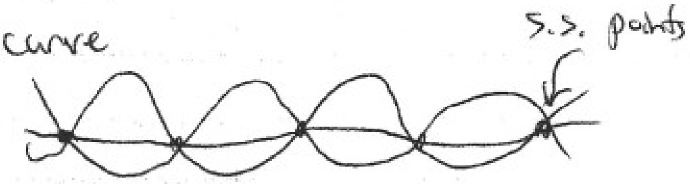
Eg: Künneth Formula: if $X = Z \times Y$, have a spectral sequence
$$H^p(Z) \otimes H^q(Y) \Longrightarrow H^{p+q}(X)$$
$$\rightsquigarrow H^\bullet(X) = H^\bullet(Z) \otimes H^\bullet(Y)$$
If $G(\mathbb{A})$ acts on the product, this is a tensor product of repⁿs. If $\mathbb{A}^p$ acts on one side, $\mathbb{Q}_p$ on the other, have
$$\underset{G(\mathbb{A})}{\overset{\circlearrowleft}{H^\bullet(X)}} = \underset{G(\mathbb{A}^p)}{\overset{\circlearrowleft}{H^\bullet(Z)}} \otimes \underset{G(\mathbb{Q}_p)}{\overset{\circlearrowleft}{H^\bullet(Y)}} \quad \Big\} \text{ geometric}$$

Similar techniques also work well

Thm (Iguza): let $\overline{X}_N$ = mod curves of level
$$\Gamma = \Gamma_N = \{A \equiv \mathbb{1}_2 (N)\}$$
in char p. Then

$p \nmid N \rightsquigarrow$ smooth curve

$p | N \rightsquigarrow$ not smooth:



Hecke operators preserve smooth components, except the one at p, which switches them.

Thm (Harris–Taylor): the picture is the same in higher dimensions.