

Mircea Mustata

07-18-2005

Spaces of arcs and birational geometry

Idea: promoting a finite dimensional object to an infinite dimensional one can make certain properties more transparent.

Arcs: germs of formal curves in an algebraic variety

X -variety over $\mathbb{C}$

The space of arcs on X is the space

$$X_{\infty} = \text{Hom}(\text{Spec } \mathbb{C}[[t]], X)$$

$\text{Spec}(\mathbb{C}[[t]])$ has only two points

- a closed point o
- a generic point η

Example: $X = \mathbb{C}^n$, $\gamma \in X_{\infty}$
is an algebra homomorphism

$$\mathbb{C}[x_1, x_2, \dots, x_n] \rightarrow \mathbb{C}[[t]]$$

or equivalently a collection
of formal power series $f_1, \dots, f_n$

where

$$\begin{array}{ccc} x_i & \rightarrow & f_i \\ \uparrow & & \uparrow \\ \mathbb{C}[x_1, \dots, x_n] & & \mathbb{C}[[t]] \end{array}$$

So

$$(\mathbb{C}^n)_\infty = (\mathbb{C}[[t]])^n$$

Example: If X is a smooth variety choose local coordinates

$x_1, x_2, \dots, x_n$ on X near $p \in X$

Then

$$\gamma \in X_\infty \iff \begin{array}{l} \gamma(0) = p \in X \\ \text{together with} \\ \text{a homomorphism} \\ \mathcal{O}_{X,p} \rightarrow \mathbb{C}[[t]] \\ \text{of local } \mathbb{C}\text{-algebras} \end{array}$$

$$(x_i - x_i(p)) \rightarrow h_i \in t\mathbb{C}[[t]]$$

Remarks: • If X is smooth, then locally over X we have

$$X_\infty \cong X \times (t\mathbb{C}[[t]])^n$$

• So far we denoted the space of arcs as a set. However one can check that there is a scheme X_∞ so that what we had before was the set of $\mathbb{C}$ -valued points of this scheme.

Why is X_∞ relevant to birational geometry?

(1) One reason is the valuative criterion for properness:

If X' is proper and birational,
 $\downarrow f$
 X
 then we get
 X'_∞
 $\downarrow f_\infty$
 X_∞

and if f = isomorphism over $X - Z$
 then

f_∞ = bijection over $X_\infty - Z_\infty$
 (on sets of $\mathbb{C}$ -points)

Remark: If $Z \subsetneq X \Rightarrow$

$$Z_\infty \subsetneq X_\infty \text{ is small}$$

(given by as many equations)

e.g.

$$X = \mathbb{C}^n, \quad Z = \mathbb{C}^{n-1}$$

$$Z_\infty = \mathbb{C}[[t]]^{n-1}$$

$$X_\infty = \mathbb{C}[[t]]^n$$

Valuative criterion: An arc γ on X can be lifted to X' if $\gamma(\eta) \notin Z$.

Application: If X = singular variety and

$$f: X' \rightarrow X$$

is a resolution of singularities (i.e. f - proper birational and X' - smooth)

we can use the valuative criterion to deduce properties

of X_∞ from obvious properties of X'_∞ .

For instance we can easily prove the following

Thm (Koldin) If X is a variety over $\mathbb{C}$, then X_∞ is irreducible (in general not reduced)

Proof: We know that when X is smooth. For a singular X we will use induction on $\dim X$.

Suppose we know the statement for all varieties of $\dim \leq n-1$.

Suppose X is a variety of $\dim = n$ and let

$$\begin{array}{ccc} X' & \rightarrow & X \\ f & & \end{array}$$

be a resolution of singularities.

X'_∞ is irreducible $\Rightarrow \overline{f_\infty(X'_\infty)}$ is irreducible

6.

But $\overline{f_\infty(X'_\infty)} \supseteq X_\infty - Z_\infty$

($Z \subset X$ s.t. f is iso over $X - Z$)

If $Z = Z_1 \cup \dots \cup Z_r$ - irred components
 $\Rightarrow$ it suffices to show that

$$(Z_i)_\infty \subseteq \overline{f_\infty(X'_\infty)} \quad \forall i.$$

However we can find

$$\begin{array}{c} W_i \subset X' \\ \text{surj.} \downarrow \\ Z_i \end{array}$$

and by the generic smoothness property implies that W_i is smooth over some

$$U_i \subset Z_i - \text{open}$$

and hence

$$(U_i)_\infty \subset \overline{f_\infty(X'_\infty)}$$

But $(Z_i)_\infty \supset (U_i)_\infty$ - open
 and by the inductive assumption

$(Z_i)_\infty$ - irreducible. Thus

$$(Z_i)_\infty \in \overline{I_{\text{irr}}(f_\infty)}$$

and we are done

□

Remark: $X_\infty \ni \delta$ - natural
 $\pi \downarrow \quad \downarrow$ map
 $X \ni \delta(0)$

We can look at $\pi^{-1}(X_{\text{sing}})$

Conjecture (Nash) The irreducible components of $\pi^{-1}(X_{\text{sing}})$ should be related with the exceptional divisors that appear in every resolution of singularities of X .

Nash showed that every irreducible component defines a divisor appearing in every resolution. He asked if the resulting injection is a bijection.

Ishii - Kollar showed that this is not the case in general.

Question: How to distinguish the divisors that come from irreducible components of $\pi^{-1}(X_{\text{sing}})$.

Modern development: The change of variables formula

let $X' \xrightarrow{f} X$ - proper
birational

f_* - "almost bijective" on points
but it is very far
from being a scheme iso

Question: Understand how f_* changes the scheme structure on arc spaces.

In the algebraic category we can't expect to be able to stratify the base X so that f becomes locally trivial.

However: We can stratify X_∞
so that f_∞ is locally
trivial on strata

Preliminaries Look at truncated
arc spaces

$$X_\infty = \text{Hom}(\text{Spec } \mathbb{C}[[t]], X)$$

$$X_m = \text{Hom}(\text{Spec}(\mathbb{C}[t]/t^{m+1}), X)$$

⋮

$$X_1 = \text{tot}(TX)$$

$$X_0 = X$$

For instance if X is smooth

$$X_\infty \underset{\substack{\text{locally} \\ \text{on } X}}{\cong} X \times (\mathbb{A}^n \mathbb{C}[[t]])^n$$

$$X_m \underset{\substack{\text{locally} \\ \text{on } X}}{\cong} X \times (\mathbb{A}^n \mathbb{C}[t]/t^{m+1})^n$$

$X_{m+1} \rightarrow X_m$ - locally trivial, fiber $\mathbb{A}^n$

Subst to work with:

cylinders: $C = \pi_m^{-1}(\mathcal{S})$

for some $\mathcal{S} \subset X_m$

For simplicity assume X smooth
 $\Rightarrow \text{codim}(C) = \text{codim}(\mathcal{S})$

Examples:

• If $Y \hookrightarrow X$ subscheme
 Consider

$$\text{Cont}^m(Y) = \{ \mathfrak{p} \in X_\infty \mid \text{ord}_{\mathfrak{p}}(I_Y) = m \}$$

Here if

$$\gamma: \text{Spec}(\mathbb{C}[[t]]) \rightarrow X,$$

then $\text{ord}_{\mathfrak{p}}(I_Y) := \text{ord}(\gamma^* I_Y)$

Note: $\text{ord}_{\mathfrak{p}}(I_Y) = \infty \Leftrightarrow \mathfrak{p} \in Y_\infty$

The loci $\text{Cont}^m(Y)$ turn out to be cylinders (i.e. come from a finite level).

Indeed if we look at

$$\begin{array}{ccc}
 \text{Cont}^{\geq m}(Y) & \hookrightarrow & X_{\infty} \\
 \downarrow \psi_{m-1}^{-1} & & \downarrow \psi_{m-1} \\
 Y_{m-1} & \hookrightarrow & X_{m-1} \\
 & & \text{closed}
 \end{array}$$

then

$$\begin{aligned}
 \text{Cont}^m(Y) &= \text{Cont}^{\geq m}(Y) - \text{Cont}^{\geq m-1}(Y) \\
 (\Rightarrow \text{ is locally closed})
 \end{aligned}$$

Thm (Kontsevich) If $X' \xrightarrow{f} X$ is proper birational with X, X' smooth then if we look at

$$K_{X'/X} \subset X'$$

the subscheme defined locally by $\det(\text{Jac}(f))$, then we get a locally closed subscheme

$$C^m := \text{Cont}^m(K_{X'/X}) \subset X'_{\infty}$$

and $\forall p \geq 2m$ we get that

the space C^n is locally trivial
with fiber A^n at any
finite level p .

More precisely if we
look at $C^n \rightarrow (\text{image of } C^n)$

$$\begin{array}{ccc} X'_\infty & \rightarrow & X'_p \\ \downarrow f_\infty & & \downarrow f_p \\ X_\infty & \rightarrow & X_p \supset f_p(\text{image of } C^n) \end{array}$$

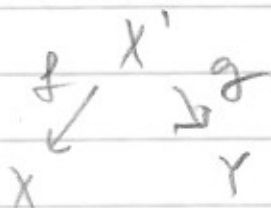
$\Rightarrow (\text{image of } C^n)$

$\downarrow$
 $f_p(\text{image of } C^n)$

is locally trivial with fiber A^n .

Why is this useful?

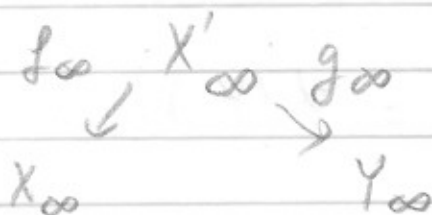
Suppose X, Y are K -equivalent
i.e. suppose $\exists X'$ - smooth
with birational morphisms



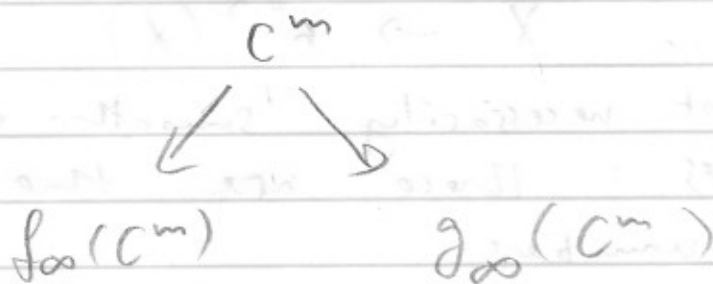
So that $K_{X'/X} \cong K_{X'/Y}$

(e.g. X, Y ~ birational $\mathbb{C}P^2$)

Then we get



and a decomposition of X'_∞
into locally closed strata C^m
s.t.



are locally trivial fibrations with
fibers $\mathbb{A}^m$.

Now if we can find an invariant
 α of varieties

s.t.

$$\bullet \quad \chi(X) = \chi(Y) + \chi(U)$$

if

$$U \subset X - \text{open}$$

$$Y = X - U.$$

$\bullet$ χ is well behaved under products + we have a good counting procedure along the label m i.e. $\sum \chi(C^m) \cdot (\text{weight}_m)$
 (this is what is known as motivic integration)

Then

$$\chi(X) = \chi(Y).$$

Basic example of χ : Hodge #'s

There is an extension of

$$X \rightarrow h^{p,q}(X)$$

to not necessarily smooth or compact varieties: these are the Deligne $i^{p,q}$ numbers

$$X \rightarrow i^{p,q}(X)$$

The Hodge polynomial $\sum i^{p,q}(X) u^p u^q$

Satisfies the 2 properties above
In particular

$$(Kontsevich) \quad X \stackrel{K\text{-equiv}}{\cong} Y$$

implies

$$h^{p,q}(X) = h^{p,q}(Y)$$

Also: $K_{X'/X}$ appears when looking
at singularities of
subvarieties of X

Then we get a dictionary

$$\left(\begin{array}{l} \text{invariants} \\ \text{of singularities} \end{array} \right) \longleftrightarrow \left(\begin{array}{l} \text{asymptotic behavior} \\ \text{of codim of} \\ \text{suitable cylinders} \end{array} \right)$$