Mircea Mustata 07-18-2005

Spaces of arcs and birational geometry

Idea: promoting a finite dimensional object to an infinite dimensional one can make certain properties more transparent.

Arcs: germs of formal curves in an algebraic variety

X – variety over $\mathbb{C}$
The space of arcs on X is the space

$$X_\infty = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X)$$

$\mathrm{Spec}(\mathbb{C}[[t]])$ has only two points
- a closed point o
- a generic point η

Example: $X = \mathbb{C}^n$, $\gamma \in X_\infty$ is an algebra homomorphism

$$\mathbb{C}[x_1, x_2, \ldots, x_n] \to \mathbb{C}[[t]]$$

or equivalently a collection of formal power series $f_1, \ldots, f_n$

Where
$$\begin{array}{ccc} x_i & \longrightarrow & f_i \\ \cap & & \cap \\ \mathbb{C}[x_1, \dots, x_n] & & \mathbb{C}[[t]] \end{array}$$

So
$$(\mathbb{C}^n)_\infty = (\mathbb{C}[[t]])^n$$

<u>Example</u>: If X is a smooth variety choose local coordinates $x_1, x_2, \dots, x_n$ on X near $p \in X$

Then
$$\gamma \in X_\infty \iff \gamma(0) = p \in X$$
together with a homomorphism
$$\mathcal{O}_{X,p} \longrightarrow \mathbb{C}[[t]]$$
of local $\mathbb{C}$-algebras

$$(x_i - x_i(p)) \to h_i \in t\,\mathbb{C}[[t]]$$

<u>Remarks</u>: • If X - smooth, then locally over X we have
$$X_\infty \cong X \times (t\,\mathbb{C}[[t]])^n$$

• So far we denoted the space of arcs as a set. However one can check that there is a scheme X_∞ so that what we had before was the set of $\mathbb{C}$-valued points of this scheme.

Why is X_∞ relevant to birational geometry?

(1) One reason is the valuative criterion for properness:

If $\begin{array}{c} X' \\ \downarrow f \\ X \end{array}$ is proper and birational,

then we get

$$\begin{array}{c} X'_\infty \\ \downarrow f_\infty \\ X_\infty \end{array}$$

and if f = isomorphism over $X - Z$

then

f_∞ = bijection over $X_\infty - Z_\infty$

(on sets of $\mathbb{C}$-points)

Remark: If $Z \underset{\neq}{\hookrightarrow} X \Rightarrow$

$Z_\infty \subseteq X_\infty$ is small
(given by ∞ many equations)

e.g.
$X = \mathbb{C}^n$, $Z = \mathbb{C}^{n-1}$

$$Z_\infty = \mathbb{C}[[t]]^{n-1}$$
$$\cap$$
$$X_\infty = \mathbb{C}[[t]]^n$$

Valuative criterion: An arc γ on X can be lifted to X' if $\gamma(\eta) \notin Z$.

Application: If X = singular variety
and
$$f: X' \to X$$
is a resolution of singularities
(i.e. f - proper birational
and X' - smooth)

we can use the valuative criterion to deduce properties of X_∞ from obvious properties of X'_∞.

For instance we can easily prove the following

Thm (Kolchin) If X is a variety over $\mathbb{C}$, then X_∞ is irreducible (in general not reduced).

Proof: We know this when X is smooth. For a singular X we will use induction on $\dim X$.

Suppose we know the statement for all varieties of $\dim \leq n-1$.

Suppose X is a variety of $\dim = n$ and let

$$X' \xrightarrow[f]{} X$$

be a resolution of singularities.

X'_∞ is irreducible $\Rightarrow$ $\overline{f_\infty(X'_\infty)}$ is irreducible

But $\overline{f_\infty(X'_\infty)} \supseteq X_\infty - Z_\infty$

$(Z \subset X$ s.t. f - iso over $X - Z)$

If $Z = Z_1 \cup \dots \cup Z_r$ - irred components

$\Rightarrow$ it suffices to show that

$$(Z_i)_\infty \subseteq \overline{f_\infty(X'_\infty)} \quad \forall i.$$

However we can find

$$\begin{array}{c} W_i \subseteq X' \\ \text{surj.} \downarrow \\ Z_i \end{array}$$

and by the generic smoothness property implies that W_i is smooth over some

$U_i \subset Z_i$ - open

and hence

$$(U_i)_\infty \subset \overline{f_\infty(X'_\infty)}$$

But $(Z_i)_\infty \supset (U_i)_\infty$ - open and by the inductive assumption

$(z_i)_\infty$ - irreducible. Thus

$$(z_i)_\infty \in \mathrm{Int}(f_\infty)$$

and we are done

$\square$

<u>Remark:</u>

$$\begin{tikzcd} X_\infty \arrow[d, "\pi"'] & \ni & \gamma \arrow[d] \\ X & \ni & \gamma(0) \end{tikzcd}$$

- natural map

We can look at $\pi^{-1}(X_{sing})$

<u>Conjecture (Nash)</u> The irreducible components of $\pi^{-1}(X_{sing})$ should be related with the exceptional divisors that appear in every resolution of singularities of X.

Nash showed that every irreducible component defines a divisor appearing in every resolution. He asked if the resulting injection is a bijection.

Ishii - Kollár showed that this is not the case in general.

Question: How to distinguish the divisors that come from irreducible components of $\pi^{-1}(X_{sing})$.

Modern development: The change of variables formula

let $X' \xrightarrow{f} X$ - proper birational

f_∞ - "almost bijective" on points but it is very far from being a scheme iso

Question: Understand how f_∞ changes the scheme structure on arc spaces.

In the algebraic category we can't expect to be able to stratify the base X so that f becomes locally trivial.

<u>However:</u> We can stratify X_∞ so that f_∞ is locally trivial on strata

<u>Preliminaries</u> Look at truncated arc spaces

$$X_\infty = \mathrm{Hom}(\mathrm{Spec}\,\mathbb{C}[[t]], X)$$

$$X_m = \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}[t]/t^{m+1}), X)$$

$$\vdots$$

$$X_1 = \mathrm{tot}(TX)$$

$$X_0 = X$$

For instance if X - smooth

$$X_\infty \underset{\substack{\text{locally} \\ \text{on } X}}{\cong} X \times (t\,\mathbb{C}[[t]])^n$$

$$X_m \underset{\substack{\text{locally} \\ \text{on } X}}{\cong} X \times (t\,\mathbb{C}[t]/t^{m+1})^n$$

$X_{m+1} \to X_m$ - locally trivial, fiber $\mathbb{A}^{\dim X}$

Subset to work with:

cylinders: $C = \pi_m^{-1}(S)$

for some $S \subset X_m$

For simplicity assume X - smooth
$\Rightarrow \operatorname{codim}(C) = \operatorname{codim}(S)$

<u>Examples:</u>

- If $Y \hookrightarrow X$ subscheme

consider

$$\operatorname{Cont}^m(Y) = \{ \gamma \in X_\infty \mid \operatorname{ord}_\gamma(I_Y) = m \}$$

Here if

$$\gamma : \operatorname{Spec}(\mathbb{C}[[t]]) \to X,$$

then $\operatorname{ord}_\gamma(I_Y) := \operatorname{ord}(\gamma^* I_Y)$

<u>Note</u>: $\operatorname{ord}_\gamma(I_Y) = \infty \Leftrightarrow \gamma \in Y_\infty$

The loci $\operatorname{Cont}^m(Y)$ turn out to be cylinders (i.e. come from a finite level).

Indeed if we look at

$$\begin{array}{ccc} \mathrm{Cont}^{\geq m}(Y) & \hookrightarrow & X_\infty \\ \| & & \\ \psi_{m-1}^{-1}(Y_{m-1}) & & \downarrow \psi_{m-1} \\ \downarrow & & \\ Y_{m-1} & \underset{\text{closed}}{\hookrightarrow} & X_{m-1} \end{array}$$

then

$$\mathrm{Cont}^{m}(Y) = \mathrm{Cont}^{\geq m}(Y) - \mathrm{Cont}^{\geq m-1}(Y)$$

($\Rightarrow$ is locally closed)

<u>Thm (Kontsevich)</u> If $X' \xrightarrow{f} X$ is proper birational with X, X' smooth then if we look at

$$K_{X'/X} \subset X'$$

the subscheme defined locally by det (Jac(f)), then we get a locally closed subscheme

$$C^m := \mathrm{Cont}^m(K_{X'/X}) \subset X'_\infty$$

and $\forall$ $p \geq 2m$ we get that

the space C^m is locally trivial with fiber $\mathbb{A}^n$ at any finite level p.

More precisely if we look at

$$\begin{tikzcd}
& \supset C_m \arrow[r] & (\text{image of } C_m) \arrow[dd] \\
X'_\infty \arrow[r] \arrow[d, "f_\infty"'] & X'_p \supset \arrow[d, "f_p"] & \\
X_\infty \arrow[r] & X_p \supset & f_p(\text{image of } C_m)
\end{tikzcd}$$

$\Rightarrow$ $(\text{image of } C_m)$

$$\downarrow$$

$f_p(\text{image of } C_m)$

is locally trivial with fiber $\mathbb{A}^n$.

Why is this useful?

Suppose X, Y are K-equivalent i.e. suppose $\exists$ X' - smooth with birational morphisms

$$\begin{tikzcd} & X' \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

so that $K_{X'/X} \cong K_{X'/Y}$

(e.g. X, Y ~ birational CYs)

Then we get

$$\begin{tikzcd} & X'_\infty \arrow[dl, "f_\infty"'] \arrow[dr, "g_\infty"] & \\ X_\infty & & Y_\infty \end{tikzcd}$$

and a decomposition of X'_∞ into locally closed strata C^m s.t.

$$\begin{tikzcd} & C^m \arrow[dl] \arrow[dr] & \\ f_\infty(C^m) & & g_\infty(C^m) \end{tikzcd}$$

are locally trivial fibrations with fibers $\mathbb{A}^m$.

Now if we can find an invariant α of varieties s.t.

- $$\alpha(X) = \alpha(Y) + \alpha(U)$$

if $U \subset X$ - open
$Y = X - U$.

- α is well behaved under products + we have a good counting procedure along the label m i.e. $\sum \alpha(C^m) \cdot (\text{weight}_m)$
(this is what is known as motivic integration)

Then
$$\alpha(X) = \alpha(Y).$$

Basic example of α: Hodge #'s

There is an extension of

$$X \to h^{p,q}(X)$$

to not necessarily smooth or compact varieties: these are the Deligne $i^{p,q}$ numbers

$$X \to i^{p,q}(X)$$

The Hodge polynomial $\sum i^{p,q}(X) u^p u^q$

satisfies the 2 properties above

In particular

$$\text{(Kontsevich)} \quad X \overset{K\text{-equiv}}{\cong} Y$$

implies

$$h^{p,q}(X) = h^{p,q}(Y)$$

Also: $K_{X'/X}$ appears when looking at singularities of subvarieties of X

Then we get a dictionary

$$\begin{pmatrix} \text{invariants} \\ \text{of singularities} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{asymptotic behavior} \\ \text{of codim of} \\ \text{suitable cylinders} \end{pmatrix}$$