Spaces of arcs and birat. geometry

Let X = variety / $\mathbb{C}$

X_∞ = space of formal arcs on X

= $\mathrm{Hom}(\mathrm{Spec}\, \mathbb{C}[[t]], X)$

Basic example: $X = \mathbb{A}^n$

$\gamma : \mathrm{Spec}\, \mathbb{C}[[t]] \longrightarrow X \iff \mathbb{C}[x_1, \ldots x_n] \to \mathbb{C}[[t]]$

$x_i \longrightarrow f_i$

$\Rightarrow X_\infty \simeq (\mathbb{C}[[t]])^n$

More gen: if X smooth and $x_1, \ldots x_n$ are coord in an open set, giving arc $\gamma \iff$

a point $x \in X$ ($x = \gamma(o)$ + closed pt of $\mathrm{Spec}\, \mathbb{C}[[t]]$)

+

morphism $\mathcal{O}_{X,x} \longrightarrow \mathbb{C}[[t]]$

$\wr\wr$

$x_i - x_i(x) \longrightarrow f_i \in t\,\mathbb{C}[[t]]$

$\Rightarrow$ X_∞ is locally isom to $X \times (t\,\mathbb{C}[[t]])^{\dim X}$

<u>Note</u>: X_∞ is not just a set, it has a <u>scheme structure</u>

s.t. $\mathrm{Hom}(\mathrm{Spec}\, A, X_\infty) \cong \mathrm{Hom}(\mathrm{Spec}\, A[[t]], X)$

What do spaces of arcs have to do with birational geometry?

① Val. crit for properness:

suppose that $X \xrightarrow{f} Y$ is a <u>proper</u>, <u>birational</u> morphism of ~~[illegible]~~ varieties

Let $Z \subsetneq Y$ closed s.t. f isom over $Y \setminus Z$

Then $f_\infty : X_\infty \longrightarrow Y_\infty$ is bijective outside $Z_\infty \subseteq Y_\infty$

Pf:
$$\begin{tikzcd} \mathrm{Spec}\,\mathbb{C}((t)) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,\mathbb{C}[[t]] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

exists since $\mathrm{Spec}\,\mathbb{C}((t)) \to X \to Y$, $\searrow Y \setminus Z$ ($\cup\!\mid$)

Apply valuative criterion!

Rmk: since $Z \underset{\neq}{\subseteq} Y$, $Z_\infty \subseteq Y_\infty$ is "very small"

e.g.: it has infinite codim

(think of $\mathbb{A}^{n-1} \subseteq \mathbb{A}^n \Rightarrow (\mathbb{C}[[t]])^{n-1} \subseteq (\mathbb{C}[[t]])^n$ is given by the vanishing of inf. many coordinates).

In all this story, such subsets can be ignored.

Application

In order to understand in general X_∞, take a resolution of sing of X:

X' $\xrightarrow{f}$ X proper, birat

X' smooth $\Rightarrow$ X'_∞ "easy to understand"

$X'_\infty \to X_\infty$ "almost bij"

Theorem 1 (Kolchin) X = variety $\Rightarrow X_\infty$ = irreducible

Pf. If $\begin{array}{c} X' \\ {\scriptstyle f}\downarrow \\ X \end{array}$ res of sing, isom over $X \setminus Z$

$\underset{\text{irred}}{X'_\infty} \Rightarrow X_\infty \setminus Z_\infty$ irred

Hence, enough to show (using induction on dim):

if $Z = Z_1 \cup \dots \cup Z_n$ irred dec,

$\forall i$, there is an open subset of $(Z_i)_\infty$ contained in $f_\infty(X'_\infty)$.

If $\begin{array}{c} W_i \\ \downarrow \\ Z_i \end{array}$ surj it is smooth over some open subset $U_i \subseteq Z_i$

$\Rightarrow (U_i)_\infty \subseteq \operatorname{Im}(f_\infty)$ □

Rmk: In general, $X_\infty \neq$ reduced. However, in this study one tends to ignore the info coming from non-reduced structure.

Rmk: we have

$$\begin{tikzcd} X_\infty \arrow[d, "\pi"'] & \ni \gamma \arrow[d] \\ X & \ni \gamma(0) \end{tikzcd}$$

Interesting to study: irred comp of $\pi^{-1}(X_{sing})$

Nash: if f is a resolution as above (whose except. locus is a div) every such irred comp corresp to a unique div on X' (these are div. that appear on any such res)

He asked whether this accounts for all div that appear on every res

Neg answer: Ishii - Kollár

(2) Change of var. formula

Let $\begin{array}{c} X \\ f\downarrow \\ Y \end{array}$ proper, birat

For simplicity: assume both X, Y smooth

- $f_\infty : X_\infty \to Y_\infty$ bij morphism, but far from isom; however, one can say more about its structure
- in the algebraic categ, can't stratify f s.t it becomes locally trivial on each piece.

However: this is possible for f_∞!

Some preliminaries: for arbitrary X, X_∞ is limit of spaces of m-truncated arcs (m-jets)

$$\begin{array}{l} X_\infty \\ \pi_m \downarrow \\ X_m = Hom(\,Spec\ \mathbb{C}[t]/(t^{m+1}), X) \\ \downarrow \\ X_{m-1} \\ \downarrow \\ X \end{array}$$

$X_0 = X$

E.g: X_1 = total tangent space of X

If X smooth, these truncation maps are surjective and locally trivial: locally over X,

$$X_m \simeq X \times \left(t\,\mathbb{C}[t]/(t^{m+1}) \right)^n$$

The sets one usually works with are <u>cylinders</u>:

$C = \pi_m^{-1}(S)$ for some locally closed $S \subseteq X_m$

<u>Typical example</u>: if $Y \subseteq X$ subscheme

$$\mathrm{Cont}^m(Y) = \left\{ \gamma \in X_\infty \;\middle|\; \mathrm{ord}\, \gamma^* I_Y = m \right\}$$

$\gamma : \mathrm{Spec}\, \mathbb{C}[[t]] \longrightarrow X$

This is det. by the corresp $(m-1)$-jets:

$$\mathrm{Cont}^{\geq m}(Y) = \pi_{m-1}^{-1}(Y_{m-1})$$

Theorem (Kontsevich)

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

proper, birat; X, Y smooth

$K_{X/Y}$ = eff. div on X locally def by $\det(\mathrm{Jac}(f))$

For every m, if $p \gg 0$

$$\begin{tikzcd}
X_\infty \arrow[dr] \arrow[dd] & \supseteq & \mathrm{Cont}^m(K_{X/Y}) \arrow[dd] \arrow[dr] & \\
& X_p \arrow[dd] & \supseteq & \text{Image of } \mathrm{Cont}^m(K_{X/Y}) \arrow[dd, "\substack{\text{locally trivial,}\\ \text{fiber } \mathbb{A}^m}"] \\
Y_\infty \arrow[dr] & \supseteq & f_\infty(\mathrm{Cont}^m(K_{X/Y})) \arrow[dr] & \\
& Y_p & \supseteq & \text{Image of } f_\infty(\mathrm{Cont}^m(K_{X/Y}))
\end{tikzcd}$$

Caveat: if $C \subseteq \mathrm{Cont}^m(K_{X/Y})$ cylinder

$\Rightarrow f_\infty(C)$ cyl

$$\mathrm{codim}(f_\infty(C)) = m + \mathrm{codim}(C)$$

(despite the fact that f_∞ bij)

Why is the appearance of $K_{X/Y}$ important:

(1) Suppose

$$\begin{array}{ccc} & X & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ Y_1 & & Y_2 \end{array}$$

and $K_{X/Y_1} = K_{X/Y_2}$

(i.e. "Y_1 and Y_2 are K-equiv")

$\Rightarrow$ Then the corresp stratif of X_∞ w.r.t $(f_1)_\infty$, $(f_2)_\infty$ are the same

If you have invar that are $\left\{\begin{array}{l}\text{additive w.r.t decomp} \\ \text{behave well w.r.t fibrations}\end{array}\right.$

+ good summing procedure w.r.t. infinite dec (motivic integration)

$\Rightarrow$ the corresp invar of Y_1, Y_2 are equal

E.g: Kontsevich: Y_1, Y_2 have equal Hodge, Betti #'s

(2) $K_{X/Y}$ appears in the study of sing of points of the form (X, D)

$\Rightarrow$ can use this framework to describe invariants of sing in terms of codim. of certain subsets of X_{∞}.