



$\Rightarrow X_\infty$  is locally isom to  $X \times (\mathbb{A}^1 \setminus \{0\})^{\dim X}$

Note:  $X_\infty$  is not just a set, it has a scheme structure

$$\text{s.t. } \text{Hom}(\text{Spec } A, X_\infty) \cong \text{Hom}(\text{Spec } A[\mathbb{A}^1 \setminus \{0\}], X)$$

What do spaces of maps have to do with birational geometry?

① Val. crit for properness:

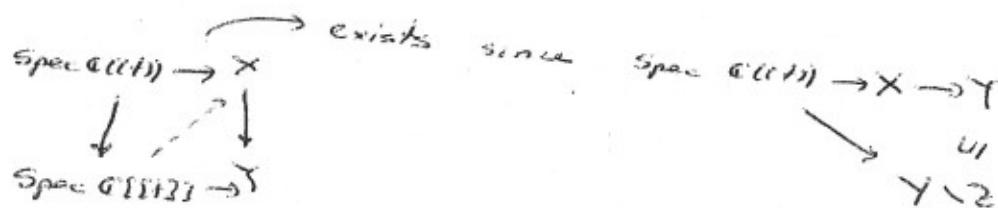
suppose that  $X \xrightarrow{f} Y$  is a proper, birational morphism of smooth varieties

Let  $Z \subset Y$  closed s.t.  $f$  isom over  $Y \setminus Z$

Then  $f_\infty : X_\infty \rightarrow Y_\infty$  is bijective outside  $Z_\infty \subset Y_\infty$

②

Pf:



Apply valuative criterion!

Remark:

since  $Z \subsetneq Y$ ,  $Z_\infty \subsetneq Y_\infty$  is "very small"

e.g: it has infinite codim

(think of  $\mathbb{A}^{n-1} \subset \mathbb{A}^n \Rightarrow (\mathbb{C}[[t]])^{n-1} \subset (\mathbb{C}[[t]])^n$   
 is given by the vanishing  
 of inf. many coordinates).

In all this stuff, such subsets can be ignored.

Application

In order to understand in general  $X_\infty$ , take  
 a resolution of sing of  $X$ :

$$\begin{array}{c}
 X' \\
 \downarrow f \\
 X
 \end{array}$$

proper, birat

$X'$  smooth

$\Rightarrow$   $X'$  is "easy to understand"  
 $\downarrow$  "almost bij"  
 $X_\infty$

Theorem 1 (Kobrin)  $X = \text{variety} \Rightarrow X_\infty = \text{irreducible}$

Pf. If  $\begin{array}{c} X' \\ \downarrow f \\ X \end{array}$  res of sing, isom over  $X \setminus Z$

$X'_\infty \Rightarrow X_\infty \setminus Z_\infty$  irred  
irred

Hence, enough to show (using induction on dim):  
if  $Z = Z_1 \cup \dots \cup Z_r$  irred dec,

$\forall i$ , there is an open subset of  $(Z_i)_\infty$   
contained in  $f_\infty(X'_\infty)$ .

If  $\begin{array}{c} W_i \\ \downarrow \text{sing} \\ Z_i \end{array}$

it is smooth over some  
open subset  $U_i \subseteq Z_i$

$\Rightarrow (U_i)_\infty \subseteq \text{Im}(f_\infty) \cap$

Remark:

In general,  $X_\infty \neq \text{reduced}$ . However, in  
this study one tends to ignore the info  
coming from non-reduced structure.

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Rmk:

we have

$$\begin{array}{ccc}
 X_{\text{res}} & \supset & Y \\
 \pi \downarrow & & \downarrow \\
 X & \supset & Y(0)
 \end{array}$$

Interesting to study: irred comp of  $\pi^{-1}(X_{\text{sing}})$

Nash: if  $f$  is a resolution as above

(whose except. locus is a div) every such

irred comp corresp to a unique div on  $X'$

(these are div. that appear on any such res)

He asked whether this accounts for all div that appear on every res

Neg answer: Ishii - Kollar

## ② Change of var. formula

Let  $X$  proper, birat

$$\begin{array}{c} X \\ f \downarrow \\ Y \end{array}$$

For simplicity: assume both  $X, Y$  smooth

- $f_\infty : X_\infty \rightarrow Y_\infty$  bij morphism, but  
 $f_\infty$  from isom; however, one can  
 say more about its structure
- in the algebraic categ, can't stratify  $f$  & it  
 becomes locally trivial on each piece.  
 However: this is possible for  $f_\infty$ !

Some preliminaries: for arbitrary  $X$ ,  $X_\infty$  is  
 limit of spaces of  $m$ -truncated arcs ( $m$ -jets)

$$\begin{array}{c} X_\infty \\ \pi_m \downarrow \\ X_m = \text{Hom}(\text{Spec } \mathbb{C}[t]/(t^{m+1}), X) \\ \downarrow \\ X_{m-1} \\ \downarrow \\ X \end{array}$$

$$X_0 = X$$

E.g:  $X_1$  = total tangent  
 space of  $X$

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If  $X$  smooth, these transition maps are surjective  
and locally trivial: locally over  $X$ ,

$$X_m \cong X \times (\mathbb{A}^1 \setminus \{0\} / (\mathbb{A}^1 \setminus \{0\})^n)$$

The sets one usually works with are cylinders:

$$C = \pi_m^{-1}(S) \quad \text{for some locally closed } S \subseteq X_m$$

Typical example: if  $Y \subseteq X$  subscheme

$$\text{Cont}^m(Y) = \{ \gamma \in X_m \mid \text{ord } \gamma^* I_Y = m \}$$

$$\gamma: \text{Spec } \mathbb{C}[[t]] \rightarrow X$$

This is det. by the corresp  $(m-1)$ -jets:

$$\text{Cont}^{\geq m}(Y) = \pi_{m-1}^{-1}(Y_{m-1})$$

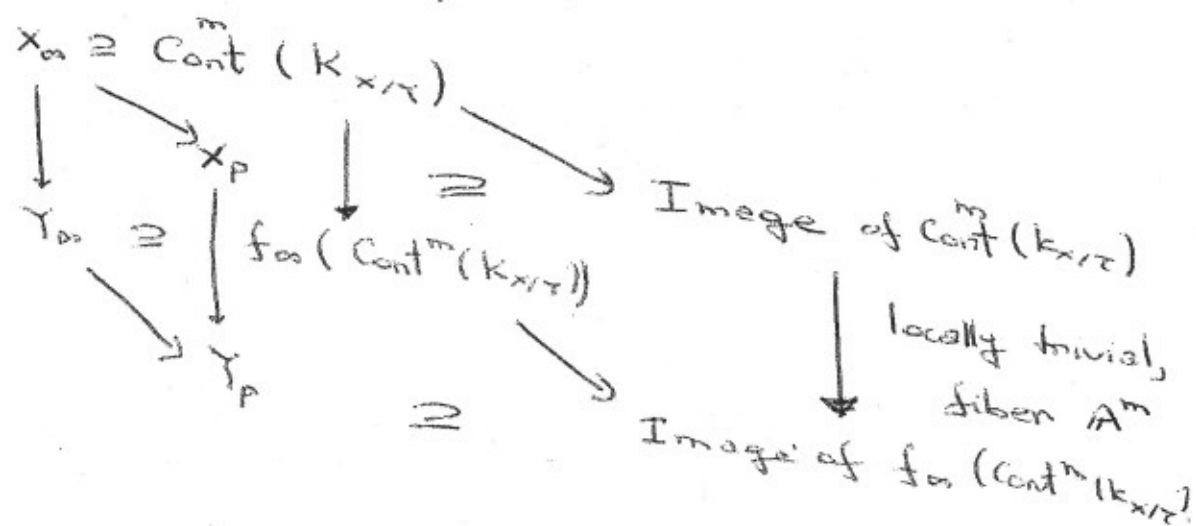
Theorem (Kontsevich)

$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

proper, birat;  $X, Y$  smooth

$K_{X/Y}$  = eff. div on  $X$  locally def by  $\det(J_{\text{loc}}(f))$

For every  $m$ , if  $p \gg 0$



Caveat: if  $C \subseteq \text{Cont}^m(K_{X/Y})$  cylinder  
 $\Rightarrow f_{\infty}(C)$  cyl

$$\text{codim}(f_{\infty}(C)) = m + \text{codim}(C)$$

(despite the fact that  $f_{\infty}$  big)

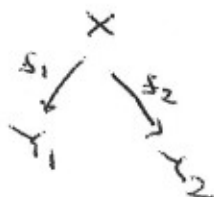


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Why is the appearance of  $K_{X/Y}$  important:

①

Suppose



and  $K_{X/Y_1} = K_{X/Y_2}$

(i.e. " $Y_1$  and  $Y_2$  are  $K$ -equiv")

$\Rightarrow$  Then the corresp stratif of  $X$  w.r.t  
 $(f_1)_*$ ,  $(f_2)_*$  are the same

If you have invar that are  $\left\{ \begin{array}{l} \text{additive w.r.t} \\ \text{decomp} \\ \text{behave well w.r.t} \\ \text{fibrations} \end{array} \right.$

+ good summing procedure w.r.t. infinite dec.  
( motivic integration)

$\Rightarrow$  the corresp invar of  $Y_1, Y_2$  are equal

E.g:

Kontsevich:  $Y_1, Y_2$  have equal  
Hodge, Betti #s

②  $K_{X/Y}$  appears in the study of sing of points of the form  $(X, D)$

$\Rightarrow$  can use this framework to describe invariants of sing in terms of codim. of certain subsets of  $X_{\text{sing}}$ .