Mike Roth 07-20-05

## Hilbert schemes

Want to study moduli spaces, i.e. some space $\mathcal{M}$ so that

$$\mathrm{Hom}(B, \mathcal{M}) = \left\{ \begin{array}{l} \text{flat families of} \\ \text{some kind of objects}/B \end{array} \right\}$$

It is not clear how to go about constructing such $\mathcal{M}$ (fine or coarse)

General strategy: Add some extra data to your objects, and

(1) find a fine moduli space for (objects + extra data).

(2) "quotient" the answer from (1) by a suitable equivalence relation killing the extra data

The Hilbert scheme is a solution to (1). It is a fine moduli space and a projective scheme

We will study the moduli problem describing the deformations of a projective variety $X$.

Extra data: Instead of considering $X$ as an abstract variety look at $X$, along with an embedding

$$X \hookrightarrow \mathbb{P}^n$$

How does the presence of the extra data modify our moduli problem?

We want to look at flat families of objects in $\mathbb{P}^n$ that "look like $X$"

We need to be more precise about what "looks like $X$" means. To do this we use the Hilbert polynomial.

Two definitions of Hilbert polynomials

(I) (Hilbert) Let $A = k[z_0, \dots, z_n]$

be a polynomial ring, and let

$$I_X \lhd A$$

be the homogeneous ideal defining $X$.

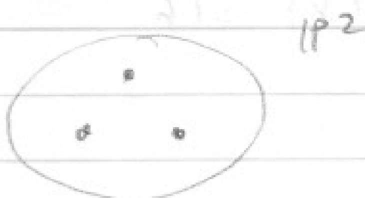
The Hilbert function of $X$ is the function

$$HF_X(d) := \dim_k \underbrace{(A/I_X)_d}_{\text{in degree } d}$$

<u>Thm</u>: For large $d$, $HF_X(d)$ is given by a polynomial in $d$.

We will write $P_X(d)$ for that polynomial.

<u>Examples</u>: (1) Suppose $X \subset \mathbb{P}^2$ is the collection of 3 general points



In this case

| $d$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $HF_X$ | 1 | 3 | 3 | 3 | 3 |
| $P_X$ | 3 | 3 | 3 | 3 | 3 |

$$P_X(d) = 3$$

(2) Suppose $X \subset \mathbb{P}^2$ consists of 3 collinear points. In this case

| $d$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $HF_X$ | 1 | 2 | 3 | 3 | 3 |
| $P_X$ | 3 | 3 | 3 | 3 | 3 |

$$P_X(d) = 3$$

(3) $X$ = intersection of two degree 4 hypersurfaces in $\mathbb{P}^3$

In the transversal case $X$ is a smooth curve

$\deg(X) = 16 \qquad g(X) = 33$

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $HF_X$ | 1 | 4 | 10 | 20 | 33 | 48 | 64 | 80 |
| $P_X$ | -32 | -16 | 0 | 16 | 32 | 48 | 64 | 80 |

$HF_X$ becomes a poly

$$P_X(t) = 16t - 32$$

Why is the Hilbert function a polynomial for large $d$?

Look at example (3). Here we can resolve $A/I$

$$0 \to A(-8) \to \begin{matrix} A(-4) \\ \oplus \\ A(-4) \end{matrix} \to A \to (A/I) \to 0$$

Here $A(-k)$ = shifted version of $A$

$$[A(-k)]_d := A_{d-k}$$

Now

$$\dim (A/I)_d = \dim A_d - 2 \dim A(-4)_d + \dim A(-8)_d$$

$$= \binom{d+3}{3} - d\binom{d-4+3}{3} + \binom{d-8+3}{3}$$

Here $\binom{m}{n} = \begin{cases} 0 & \text{if } m < n \\ \text{usual } \binom{m}{n} & \text{if } m \geq n \end{cases}$

Thus $HF_X$ is a poly for large $d$.

Note: This is the general proof that $HF_X$ is a poly for large $d$. To carry out the proof in general we need Hilbert's syzygy theorem – resolutions stop after finitely many steps.

(II) (Serre) Let $\mathcal{O}_X(d) = \mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)$

Define

$$\chi(\mathcal{O}_X(d)) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{O}_X(d))$$

Thm This is a polynomial function in $d$ $= P_X(d)$

Sketch of proof: To check that

$\chi(\mathcal{O}_X(d))$ is a polynomial take
$Y$ - general hyperplane section of
$X$ $\Rightarrow$ get an exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

Twist by $\mathcal{O}_X(d)$ $\Rightarrow$

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to \mathcal{O}_Y(d) \to 0$$

$\Rightarrow$

$$\underbrace{\chi(\mathcal{O}_Y(d))}_{\substack{\text{polynomial} \\ \text{by induction}}} = \underbrace{\chi(\mathcal{O}_X(d)) - \chi(\mathcal{O}_X(d-1))}_{\substack{\text{the first difference} \\ \text{of } \chi(\mathcal{O}_X(d))}}$$

But if the first difference of a function is polynomial, then the function must be polynomial

The fact that $\chi(\mathcal{O}_X(d))$ and $P_X(d)$ agree is a consequence of Serre vanishing.

Consider the short exact sequence

$$0 \to I_X(d) \to \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_X(d) \to 0$$

For large $d$ we have

$$\chi(I_X(d)) = \dim (I_X)_d$$

$$\chi(\mathcal{O}_{\mathbb{P}^n}(d)) = \dim (A)_d$$

$$\chi(\mathcal{O}_X(d)) = \dim (A/I_X)_d$$

by Serre vanishing

$\square$

Going back to the flat families of things that "look like $X$" in $\mathbb{P}^n$: What does the Hilbert polynomial tell us?

If $\dim X = m$, $\deg X = d$, then

$$P_X(t) = \frac{d}{m!} t^m + \text{lower terms} + \chi(\mathcal{O}_X)$$

<u>Example</u>: If $X$ is a curve of genus $g$ and degree $d$ in

in some $\mathbb{P}^n$, then

$$P_X(t) = dt + (1-g)$$

Similarly, if $X$ is a smooth surface in $\mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)_{|X}$

then

$$P_X(t) = \frac{c_1(L)^2}{2} t^2 - \frac{c_1(K)c_1(L)}{2} t + \chi(\mathcal{O}_X)$$

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If $B$ - connected scheme and

$$\mathcal{X} \subset B \times \mathbb{P}^n$$

a family of subschemes in $\mathbb{P}^n$ parameterized by $B$, i.e.

$$\begin{tikzcd} \mathcal{X} \arrow[dr] & \subset & B \times \mathbb{P}^n \arrow[dl, "pr_B"] \\ & B & \end{tikzcd}$$

then, for each $b \in B$ we can look at $X_b \subseteq \mathbb{P}^n$ and its Hilbert polynomial $P_{X_b}(t)$

<u>Thm (Serre)</u> If $X$ is flat over $B$, then $P_{X_b}(t)$ is independent of $b \in B$.

Conversely if $P_{X_b}(t)$ is independent of $b$ and $B$ is reduced then the family is flat.

Now this suggests the following setup. Given a polynomial $P(t)$, look for a fine moduli scheme $\mathrm{Hilb}_{P(t)}$ which parameterizes flat families of schemes in $\mathbb{P}^n$, with Hilbert polynomial $P(t)$.

<u>Example</u>: If $\Lambda \subset \mathbb{P}^n$ is a $k$-plane then the Hilbert polynomial of $\Lambda$ is

$$P_\Lambda(t) = \binom{t+k}{k}$$

The Hilbert scheme in this case is just the Grassmannian which we know how to construct.

What can we do in general?
Let us try and parameterize all ideals $I_X$. The essential difficulty is that $I_X$ is infinite dimensional over $k$ so we should make some reductions first.

Remarks: • If we know $I_{X,d}$ (i.e. the ideal in degree $d$), then we know $I_{X,d'}$ for all $d' \leq d$.

Indeed if $f \in I_{X,d-1}$, then $z_0 f, z_1 f, \ldots, z_n f \in I_{X,d}$ and the common zeros of these are the same zeros of $f$.

• If $I_X$ is generated in degree $\leq d$, then knowing $I_{X,d}$ lets us compute $I_{X,d'}$ for all $d'$.

<u>Idea</u>: look at the finite dimensional vector space $A_d$ and the subspace

$$I_{X,d} \subset A_d$$

The dimension $\dim I_{X,d} = ?$
For $d$ large enough we have

$$\dim (A/I_X)_d = P_X(d)$$

$\Rightarrow$ for large $d$ we have

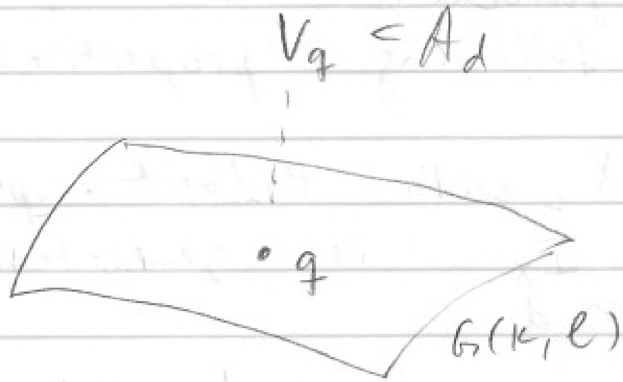
$$\dim I_{X,d} = \binom{n+d}{n} - P_X(d).$$

Now to get the construction going we need to show that we can choose $d$ (large enough) which works for all $X$'s with our fixed Hilbert polynomial.

Suppose we can find such a $d$. What more do we need to do to construct $\mathcal{H}ilb_{P(t)}$?

Fix $d$. Set $k = \dim I_{X,d}$
$\ell = \dim A_d$

Consider the Grassmanian $G(k, \ell)$
We need to restrict ourselves to points of $G(k,\ell)$ which generate ideals with our Hilbert polynomial




Require that

$$\dim \text{image} (V_q \cdot A_r) \subseteq A_{d+r}$$

is the dimension we expect i.e. is

$$= \binom{d+r+n}{n} - P(d+r)$$

There are two issues with this setup:

- the property that a linear map has a fixed rank is <u>not</u> a closed condition.

• When do we stop adding equations? (i.e. up to what degree do we need to check?)

<u>Theorem (Gotzmann)</u> Given a Hilbert polynomial $P(t)$, $\exists\, d$ with the following properties

• $\forall\, X$ with Hilbert poly $P(t) \Rightarrow I_X$ is generated in degree $\leq d$

• $\dim I_{X,d'} = \binom{n+d'}{n} - P(d')$

for all $d' \geq d$.

• for any subspace of dimension $\binom{n+d}{n} - P(d)$ in $A_d$, the image of multiplication by $z_0, z_1, \dots, z_n$ in degree $d+1$ is at least

$$\binom{d+1+n}{n} - P(d)$$

In particular, the equations for $d \to d+1$ are sufficient

In fact the theorem tells us how to compute $d$.

Given $P(t)$ there exists a unique sequence of integers

$$a_0 \geq a_1 \geq \cdots \geq a_r \geq 0$$

so that

$$P(t) = \binom{t+a_0}{a_0} + \binom{t+a_1-1}{a_1} + \cdots + \binom{t+a_r-r}{a_r}$$

Note: $$\binom{t+a_k-k}{a_k} = \frac{1}{a_k!} t^{a_k} + \cdots$$

$\Rightarrow$ $d = r+1$ in the theorem

Example: $P(t) = 3t+1$

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 0$$

$\Rightarrow$ $d = 4$.