Barotropic Instability of Zonal Flows

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ABSTRACT

The problem of barotropic instability of zonal flows to infinitesimal normal-mode perturbations is considered. The zonal flow is assumed to be continuous, but is allowed to be monotonic or nonmonotonic, and can have one or more inflection-points (which are the zeroes of the mean vorticity gradient; the zeroes are allowed to be of any order). A sufficient condition for instability is derived for this general flow profile. The present result complements the condition for stability found by Arnol'd (1965).

1. Introduction

It was argued in Lindzen and Tung (1978) that instability of shear flows is caused by wave overreflection from the critical level where the wave speed equals the basic flow speed, and so it seems that the condition under which a wave is overreflected should be a necessary condition for instability. It was further speculated that in order to be *sufficient* for instability, the wave not only has to be overreflected, but it also must be able to fit into the waveguide when boundaries are present. It was the purpose of the present paper to investigate the last requirement, the so-called quantization condition, in the hope that this would supply the missing condition sufficient for instability. For the problem of barotropic instability,[1] it is found that the quantization condition alone is sufficient for instability, the requirement for wave overreflection being contained in it. It turns out to be a stronger condition than the necessary conditions of Rayleigh-Kuo and Fjørtoft, especially when rigid boundaries are placed at finite distances apart. For many profiles in the infinite domain, where the effect of the boundaries are absent, the condition reduces to that of Fjørtoft, and so in these cases, Fjørtoft's condition is both necessary and sufficient for instability.

This paper concerns itself mainly with the nature of instability for a wide class of zonal profiles, and not with the solution of the stability problem for a particular profile. For the latter purpose a numerical eigenvalue solver may be more appropriately applied to obtain explicit solutions. The present result can more profitably be used in conjunction with the concept of overreflection, which is a local analysis, to give a more detailed interpretation of the mechanism of instability than what can be inferred from the global integral stability conditions alone. The concept of overreflection can be used to deduce which wave mode is capable of extracting energy from the mean state. The present result on quantization will determine which of the overreflected wave modes can actually lead to instability.

A majority of the methodology used here is not new, but can be found scattered in the literature on hydrodynamic instability. Some of the classical arguments can be carried over directly to the barotropic instability problem, while the application of others are far from being straightforward. An example is the method used by Rosenbluth and Simon (1964), who derived the necessary and sufficient condition for the instability of a class of profiles in the nonrotating case; the assumption of $\beta = 0$ was essential and to extend to the present case, a very different approach has to be adopted.

In Section 2, the barotropic instability problem is defined and some of the existing necessary conditions for instability relevant to the present problem are reviewed. One such condition is the famous inflection-point theorem of Rayleigh (1880), which in a form modified by Kuo (1949) to include rotation, states that *all* unstable solutions must have an "inflection point" in the domain, i.e., the mean-vorticity gradient $\beta - U_{yy}$ must vanish somewhere in the flow field. In Section 3 the existence of such an "inflection point" is assumed and the additional condition under which the existence of an unstable solution is guaranteed is sought. As a first step, the

[1] The present paper deals mainly with the simpler problem of barotropic instability, where a disturbance draws its energy only from the kinetic energy of the basic flow. With slight modifications, the baroclinic problem can be cast into a mathematically equivalent form (see Lindzen and Tung, 1978), although the physical interpretation is quite different, and many of the results to be presented here, especially those of the neutral modes, need to be modified.

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condition for the existence of a *neutral* solution whose critical level is situated at the "inflection point" is derived. We then proceed to establish that the existence of such a neutral solution necessarily implies the existence of a neighboring unstable solution. Thus, it is proved that a *sufficient* condition for instability is the condition for the existence of this neutral solution. It is a sufficient, instead of both necessary and sufficient, condition because there may possibly exist unstable solutions not contiguous to such a particular neutral mode. Indeed, as is shown in Section 4, an additional form of neutral solution exists for the rotating ($\beta \neq 0$) case which also may have neighboring instabilities. The neutral solution is characterized by having a critical level at one of the boundaries ($c = U_{\min}$), and is absent when $\beta \equiv 0$ [by the semicircle theorem (Howard, 1961)]. This second neutral mode exists under a different condition than the "inflection-point" mode and, moreover, the existence of "inflection points" is not required. However, *in order to have unstable neighboring solutions*, it is found that an "inflection point" must exist in the interior and the same condition as for the inflection mode instability is satisfied. Since it can be shown that for monotonic profiles there are no other neutral modes with neighboring instabilities, the derived condition becomes both necessary and sufficient for contiguous instability in that case. Non-contiguous instabilities (i.e., unstable solutions which are not connected in the parameter space to a neutral solution) do not arise in the present problem where no-flux boundary conditions are imposed. The no-flux boundary condition is the correct one for the relevant geophysical problem which originates from flow on a sphere. On a sphere, the boundedness condition at the poles demands that the solutions behave like $e^{-s|y|}$, where y is the Mercator coordinate and s is the zonal wavenumber, as $y \to \pm\infty$ near the poles. Thus, it is seen that the radiating mode found by Howard and Drazin (1964) and by Dickinson and Clare (1973), though a legitimate solution on an infinite β-plane, does not have a counterpart on the sphere.

In Section 5, it is mentioned that our derived condition for *instability* is in a very similar form as Arnol'd's condition for stability, which is also expressed in a variational form. Since Arnol'd's approach predicts stability but gives no information on instability, while the present result predicts instability but not sufficient in predicting stability when non-normal mode nonlinear disturbances are allowed, the results from the two approaches compliment each other.

2. Necessary conditions for instability

We consider the stability to infinitesimal perturbations of a shear flow $U(y)$ in a rotating frame of reference whose vorticity is given by the Coriolis parameter $f(y)$. The governing equation for small disturbances, say, in the northward perturbation velocity, of the form

$$\sim v(y)e^{i\alpha(x-ct)},$$

with $c = c_r + ic_i$ being the phase speed and $\alpha > 0$ and real being the wavenumber is

$$\frac{d^2}{dy^2}v + \left(\frac{\beta - U_{yy}}{U - c} - \alpha^2\right)v = 0, \qquad (2.1)$$

where $\beta \equiv df/dy$. In the present analysis, $U(y)$ is assumed to be a continuous function of y with continuous derivatives.[2] The boundary conditions to be used here are

$$v = 0 \quad \text{at} \quad y = y_1 \text{ and } y_2, \qquad (2.2)$$

where $y_2 > y_1$.

The set (2.1) and (2.2) forms an eigenvalue problem, with c as the eigenvalue, and traditionally constitutes the problem of barotropic instability (Kuo, 1949) containing the inviscid hydrodynamic instability problem as a subset. The aim of the study is to determine the condition(s) under which $c_i > 0$, which presumably leads to "self-excited instabilities" (Lin, 1955).

For the neutral solution $c_i = 0$. There is a singularity in Eq. (2.1) at the critical level where $U - c = 0$. The continuation across this singularity from one side to the other is not determinable from Eq. (2.1), which fails to describe the physical situation in the vicinity of that point. The problem was traditionally overcome by the introduction in the vicinity of the critical level of a thin inner friction layer inside which the full (linearized) Orr-Sommerfeld equation holds. The desired continuation was then obtained by matching the inner viscous solution asymptotically to the outer inviscid solutions of (2.1) (Wasow, 1950; and Lin, 1957; among others). It is

$$y - y_c \to -(y_c - y)e^{-i\pi} \qquad (2.3)$$

for $y - y_c < 0$, where y_c is the location of the critical level. That is, there is a $-\pi$ phase-shift as the critical level is crossed. This will be the approach adopted in the present study, mainly to conform to the traditional definition of the problem. The $-\pi$ value for the phase-shifted is diminished in magnitude if nonlinearity is introduced in the inner layer (Haberman, 1972), and approaches zero if viscosity is vanishingly small compared to nonlinearity (Benney and Bergeron, 1969; Davis, 1969). As far as the *existence* of unstable solutions is concerned, our results can be shown to be also valid[3] for values of phase-shift

[2] Discontinuous profiles are treated here as limits of continuous profiles. If no such limits exist, or the results for the discontinuous profiles differ from those obtained by taking continuous limits, we regard the problem with discontinuous profiles as ill-posed.

[3] Though the growth rates will be reduced when the magnitude of the phase-shift is decreased.

in $(-\pi,0)$, which covers the practical range; no instability of the kind governed by (2.1) exists when the phase shift is zero.

For complex c solutions, the classical results of Wasow and Lin[4] imply that the correct continuation is automatically satisfied by analytic continuation for the branch of the solution with $c_i > 0$ [i.e., below the singularity in the complex y plane (see Lin, 1955)]. The branch with $c_i < 0$, if also analytically continued, does not have any correspondence with the solutions of the viscous problem. However, it can be made to correspond to some of the damped disturbances of the viscous problem if the solution is continued *in the same way as the* $c_i > 0$ branch,[5] i.e., also, below the singular point (see Lin, 1955). This completes the problem definition.

One of the best known necessary conditions for instability is the inflection-point theorem of Rayleigh-Kuo (Rayleigh, 1880; Kuo, 1949). It is obtained by first multiplying Eq. (2.1) by v^*, the complex conjugate of v and then integrating the product from y_1 to y_2, to yield

$$\int_{y_1}^{y_2}\left(\frac{\beta - U_{yy}}{U - c} - \alpha^2\right)|v|^2 dy = \int_{y_1}^{y_2}|v_y|^2 dy, \quad (2.4)$$

after making use of the boundary conditions (2.2). The imaginary part of (2.4) is

$$c_i\int_{y_1}^{y_2}(\beta - U_{yy})\left|\frac{v}{U - c}\right|^2 dy = 0. \quad (2.5)$$

It then follows that a *necessary* condition for instability, $c_i > 0$, is that the mean vorticity gradient $\beta - U_{yy}$ must change sign in the domain. For convenience, we shall refer to the location $y = y_s$ where $\beta - U_{yy}$ vanishes as the inflection point. We allow the possibility of multiple inflection points, i.e., $\beta - U_{yy}$ can vanish at more than one location in $[y_1, y_2]$.[6] We also allow the possibility of double or multiple roots at the inflection point, i.e.,

$$\beta - U_{yy} \approx b(y - y_s)^m \quad \text{near} \quad y = y_s, \quad (2.6)$$

where m is an odd positive integer, since $\beta - U_{yy}$ has to change sign at y_s by definition.

The real part of Eq. (2.4) yields an inequality

$$\int_{y_1}^{y_2}(\beta - U_{yy})(U - c_r)\left|\frac{v}{U - c}\right|^2 dy > 0. \quad (2.7)$$

This is a form of the Fjørtoft theorem (Fjørtoft, 1950). We wish to point out that this necessary condition is to be satisfied by all solutions of Eqs. (2.1) and (2.2) and not just by the unstable solutions alone. For a neutral solution whose critical level is also at y_s, the inflection point, Eq. (2.7) reduces to

$$\int_{y_1}^{y_2} K(y)v^2 dy > 0, \quad (2.8)$$

where

$$K(y) = K[y, U(y_s)] \equiv \frac{\beta - U_{yy}}{U - U(y_s)}.$$

For an unstable wave ($c_i \neq 0$), one can add to Eq. (2.7) an identity [by virtue of Eq. (2.5)]:

$$[c_r - U(y_s)]\int_{y_1}^{y_2}(\beta - U_{yy})\left|\frac{v}{U - c}\right|^2 dy = 0$$

to get

$$\int_{y_1}^{y_2} K(y)[U - U(y_s)]^2\left|\frac{v}{U - c}\right|^2 dy > 0. \quad (2.9)$$

In the general case, no conclusion concerning the sign of $K(y)$ can be drawn directly from the inequalities (2.8) or (2.9). However, for the special case of a monotonic profile possessing only one inflection point, it is clear that $K(y)$ should be of one sign for $y \neq y_s$, so (2.8) or (2.9) suggests that

$$K(y) > 0 \quad (2.10)$$

throughout the domain, except if m, the multiplicity of $\beta - U_{yy}$ in (2.6), is greater than 1, in which case equality holds at (and only at) $y = y_s$.

3. A sufficient condition for instability

Since the Rayleigh-Kuo theorem implies that *all* unstable solutions have at least one inflection point we shall in this section restrict ourselves to the consideration of the class of flows that possess such a point or points in the interior of the domain. It is known that this class still contains solutions that are *not* unstable. A subset in this class will be shown in this section to consist of unstable solutions.

Let $y = y_s$ be the location where $\beta - U_{yy}$ vanishes. We seek the condition for the existence of a *neutral* solution that has its critical level at the inflection point, i.e., with $c = U(y_s)$. For this neutral mode, Eq. (2.1) reduces to the following Sturm-Liouville equation

$$\frac{d^2}{dy^2}v + [K(y) + \lambda]v = 0, \quad (3.1)$$

where

$$K(y) \equiv \frac{\beta - U_{yy}}{U - U(y_s)},$$

and $\lambda \equiv -\alpha^2$ is treated as the eigenvalue. Eq. (3.1) is to be solved subject to the boundary conditions

[4] In the complex y plane, the argument of $y - y_c$ must be $-7\pi/6 < \arg(y - y_c) < \pi/6$.

[5] The symmetry between the $c_i > 0$ and $c_i < 0$ solutions is thus destroyed. Despite frequent claims in literature complex c solutions do not all come in pairs and the existence of a $c_i < 0$ solution does not necessarily imply the existence of a $c_i > 0$ solution or vice versa.

[6] We shall use y_s to denote *the* "inflection point" when there is only one, and if there are more than one such location, y_s will be used to denote *one* of the "inflection points," the one under consideration.

(2.2). It is known that this system has an infinite number of eigenvalues, which can be ordered as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots ,$$

with the only limiting point at $\lambda \to \infty$. We wish to find out whether there exists at least one negative eigenvalue, i.e., $\lambda_0 < 0$, so that there exists a quantized mode with α real.

It can be shown, using variational principles applied to ordinary differential equations [see Morse and Feshbach (1953), or Howard (1964)], that the lowest eigenvalue λ_0 is given by the minimum value of the following so called Rayleigh's quotient:

$$\Omega(v) \equiv \int_{y_1}^{y_2} \left[\left(\frac{d}{dy} v\right)^2 - K(y)v^2\right]dy \times \left(\int_{y_1}^{y_2} v^2 dy\right)^{-1}. \quad (3.2)$$

In (3.2), v is *any* square-integrable function, with square-integrable derivative, satisfying only the boundary conditions (2.2). Suppose v_0 is the eigenfunction corresponding to the lowest value of Ω, then

$$\Omega(v_0) = \left\{\left[v_0 \frac{d}{dy} v_0\right]_{y_1}^{y_2} - \int_{y_1}^{y_2} v_0\left[\frac{d^2}{dy^2} v_0 + Kv_0\right]dy\right\}\left(\int_{y_1}^{y_2} v_0^2 dy\right)^{-1} = \lambda_0.$$

The next lowest eigenvalue λ_1 is obtained by minimizing (3.2) subject to the additional constraint that v_1 be orthogonal to v_0, and so on for other eigenvalues.

For nonmonotonic profiles, the function

$$K(y) = (\beta - U_{yy})[U(y) - U(y_s)]^{-1}$$

may be singular at a number of points. These points are the locations of those critical levels which do not coincide with the zeroes of $(\beta - U_{yy})$; these points will be denoted by y_c^i, $i = 1, 2, \ldots$. In general, the requirement that the eigenvalue λ be real places a severe restriction on the class of nonmonotonic profiles which can yield a neutral solution to Eq. (3.1). Viewing $K(y)$ as the limit, as $\epsilon \to 0$, of

$$(\beta - U_{yy})/(U - c), \quad \text{with} \quad c = U(y_s) + i\epsilon,$$

as prescribed by the method of continuation discussed in Section 2, then one can show (see later in this section) that the integral $\int Kv^2 dy$ appearing in (3.2) is, in general, complex, since

$$\int_{y_1}^{y_2} Kv^2 dy = ⨍_{y_1}^{y_2} Kv^2 dy + i\pi \sum_i [(\beta - U_{yy})/U_y \cdot v^2]\big|_{y_c^i},$$

where a bar on the integral sign means that the principal value is to be taken. For those nonmonotonic profiles which yield a nonzero imaginary term[7] in the above expression, no neutral solution whose inflection-point is located at one of the critical levels can exist. Incidentally, profiles that have an extremum at the inflection-point, i.e., $d/dy(U) = 0$ at $y = y_s$, are also excluded if they give an imaginary part to the above integral.

It is seen from (3.2) that if $K(y)$ is non-positive in the domain, then $\Omega(v) > 0$ for any v, and thus no negative eigenvalues can be found. In order for Ω to be negative for some v, it is *necessary* (but not sufficient) that

$$\int_{y_1}^{y_2} K(y)v^2 dy > 0. \quad (3.3)$$

Eq. (3.3) is recognized as the Fjørtoft theorem specialized to the neutral solution [cf. Eq. (2–8)].

In (3.2), $v(y)$ appears only as a dummy variational variable and hence can be replaced by *any* square-integrable (with square-integrable derivative) real function $g(y)$ satisfying

$$g(y) = 0 \quad \text{at} \quad y = y_1 \text{ and } y_2. \quad (3.4)$$

And, since λ_0 is the minimum value of Ω, we have

$$\lambda_0 \leq \Omega(g). \quad (3.5)$$

It follows that if one can find any g that gives $\Omega(g) < 0$ one has then shown that

$$\lambda_0 < 0.$$

So we have the following theorem:

THEOREM I. A neutral solution to Eqs. (2.1) and (2.2) with $c = U(y_s)$ and α real exists if and only if there exists a square-integrable real function $g(y)$ with square-integrable derivative, satisfying (3.4), such that

$$I[g, U(y_s)] \equiv \int_{y_1}^{y_2} \left\{\left(\frac{d}{dy} g\right)^2 - K[y, U(y_s)]g^2\right\}dy < 0 \quad \text{and real}, \quad (3.6)$$

where

$$K[y, U(y_s)] \equiv (\beta - U_{yy})/[U - U(y_s)] = K(y).$$

Condition (3.6) is a function of $K(y)$ only and so (3.6) is a condition on $U(y)$ and β only, as $g(y)$ is merely a dummy variable.

It turns out that (3.6) is actually a sufficient condition for *instability*, for it can be shown, as will be done below, that unstable solutions exist neigh-

[7] The imaginary part turns out to be the sum of jumps in momentum fluxes (or called Reynolds stresses in the hydrodynamics problem) across critical levels in the domain (see Lindzen and Tung, 1978). Since for a neutral wave the momentum flux is constant away from critical levels, and has to vanish at both boundaries in order to satisfy (2.2), the jumps in the momentum flux either have to be zero or cancel each other out to give a zero sum.

boring the neutral solution.[8] The procedure to be used here is a generalized version of that used by Lin (1945).

We let the neutral solution be denoted by

$$\left.\begin{aligned} v(y) &= v_s(y) \\ c &= c_s = U(y_s) \\ \alpha^2 &= \alpha_s^2 = -\lambda_s \end{aligned}\right\} .$$

It is desired to establish the existence of a contiguous unstable solution. Let Eq. (2.1) be written as

$$L(v) \equiv v'' + \lambda v + \frac{q'}{U - c} v = 0, \quad (3.7)$$

where $q \equiv f - U_y$, and primes denote differentiation with respect to y. Differentiating (3.7) with respect to λ gives

$$L(v_\lambda) \equiv v_\lambda'' + \lambda v_\lambda + \frac{q'}{U - c} v_\lambda = \left[-1 - \frac{d}{d\lambda}\left(\frac{q'}{U - c}\right)\right] v, \quad (3.8)$$

where

$$v_\lambda \equiv \frac{\partial v}{\partial \lambda} + \frac{\partial v}{\partial c}\frac{dc}{d\lambda} .$$

We multiply v by Eq. (3.8), v_λ by (3.7), and subtract to yield

$$vL(v_\lambda) - v_\lambda L(v) = \frac{d}{dy}(vv_\lambda' - v_\lambda v') = \left[-1 - \frac{d}{d\lambda}\left(\frac{q'}{U - c}\right)\right] v^2. \quad (3.9)$$

Integrating between y_1 and y_2, evaluating (3.9) at $\lambda \to \lambda_s$, $c \to c_s$, and noting that v_{λ_s} satisfies the same boundary conditions as v_s, one obtains

$$\int_{y_1}^{y_2} v_s^2 dy + \int_{y_1}^{y_2} \frac{d}{d\lambda}\left(\frac{q'}{U - c}\right)_s v_s^2 dy = 0$$

or

$$\left(\frac{dc}{d\lambda}\right)_s = -\int_{y_1}^{y_2} v_s^2 dy \left(\int_{y_1}^{y_2} \frac{q'}{(U - c_s)^2} v_s^2 dy\right)^{-1} . \quad (3.10)$$

[8] The existence of neighboring solutions, unstable or otherwise, must first be established before one can justifiably apply the Tollmien-Lin perturbation scheme used in the following. For monotonic profiles, this is guaranteed by the Implicit Function Theorem and the fact that both the solution with $c_i \neq 0$ and the neutral solution of Theorem I are analytic in y. For nonmonotonic profiles there is the possibility that the neutral solution of Theorem I is log-singular. To circumvent this technical difficulty, we have adopted the approach, as stated in Section 2, that the neutral solution is to be viewed as the limit when c_i approaches zero from above. As a consequence, the neutral solution is not disconnected from its unstable neighbor, though it may be disconnected from solutions with c_i less than zero. It is believed that this also is the more physical approach, as viscosity, no matter how small, is never zero in the real atmosphere, and a small positive c_i simulates this effect.

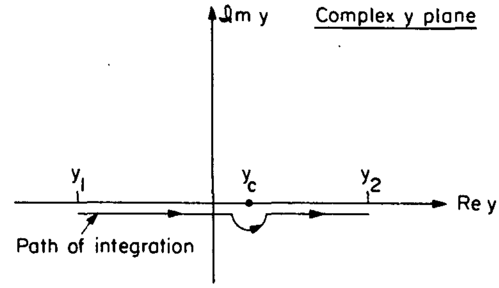

FIG. 1. Path of integration in the complex y plane.

The singular integral in the denominator is to be evaluated along a path dented below the real axis near y_s, in accordance with the prescription in Section 2. We take the path to be along the real axis from y_1 to $y_s - \epsilon$, and $y_s + \epsilon$ to y_2 and along a semicircle below the real axis of radius ϵ centered at y_s (see Fig. 1). In the limit $\epsilon \to 0$, the integral along the segments on the real axis gives the principle value integral

$$\fint_{y_1}^{y_2} \frac{K(y)}{(U - c_s)} v_s(y)^2 dy,$$

while the integration along the semicircle yields $i\pi K(y_s)v_s^2(y_c)/U'(y_s)$. Therefore,

$$\left(\frac{dc}{d\alpha^2}\right)_s = \frac{A - iB}{A^2 + B^2} , \quad (3.11a)$$

where

$$A \equiv \fint_{y_1}^{y_2} \frac{K(y)}{U - U(y_s)} v_s^2 dy \left(\int_{y_1}^{y_2} v_s^2 dy\right)^{-1}, \quad (3.11b)$$

$$B \equiv \pi K(y_s) v_s^2(y_c) \left(U'(y_s) \int_{y_1}^{y_2} v_s^2 dy\right)^{-1} . \quad (3.11c)$$

Let us for a moment assume that $v_s(y_c) \neq 0$; we shall come back to this point later in this section. If y_s is a simple root of $\beta - U_{yy}$, i.e., $m = 1$, $K(y_s)$ will be different from zero. Therefore, complex eigenvalues with $c_i \neq 0$ exist for α^2 slightly removed from α_s^2. The neutral curve $c = c_s(\alpha_s^2)$ is actually a *stability boundary*, because for this case the sign of c_i is different on different sides of the neutral curve. As an example, take the case of a monotonic profile $U_y > 0$ with a single inflection-point. Since $K(y_s) > 0$ from (2.10), one has $B > 0$. Thus (3.11a) implies that an unstable solution with $c_i > 0$ exists for α slightly less than α_s, and a stable solution with $c_i < 0$ exists for α slightly greater than α_s.

The sign of the real part of (3.11a) is also of some physical interest. An unstable solution will have its critical level to the south of the inflection point, i.e., $y_c < y_s$ if $A > 0$. The case of a tanh-profile studied by Dickinson and Clare (1973) has $A > 0$ for positive β, and so the critical level is in the region of negative mean vorticity gradient. The instability can be interpreted as arising from the Rossby wave in the westerly region in the north incident on the critical level in the south and over-reflected by it. The Rossby wave in the southern wave-guide with easterly wind and negative mean-vorticity gradient also can be overreflected by the critical level if the critical level is situated to the *north* of the inflection-point. Such a wave, though capable of extracting energy from the mean flow, does not lead to instability because it cannot be quantized. This fact is evident from (3.11a), which does not permit an unstable wave to have its critical level to the north of the inflection-point. The results of Michalke (1964) for the nonrotating case can also be interpreted in the above manner, only that as β decreases, the critical level of a quantized wave is made to be located closer and closer to the inflection-point, and, finally, when β becomes zero the two locations coincide, as is found by Michalke.

When y_s is a higher root of $\beta - U_{yy}$ (i.e., $m > 1$), $K(y)$ vanishes at $y = y_s$, and one then has $B = 0$. Nevertheless, one cannot conclude merely from Eq. (3.11) that no neighboring unstable solution exists. Derivatives higher than $(dc/d\lambda)_s$ must be considered, as was done by Lin (1945) for the case $\beta \equiv 0$. The rather tedious calculation is relegated to Appendix A. It is shown there that if near $\lambda = \lambda_s$, c is expanded in a power series

$$c - c_s = \left(\frac{dc}{d\lambda}\right)_s (\lambda - \lambda_s) + \frac{1}{2!}\left(\frac{d^2c}{d\lambda^2}\right)_s (\lambda - \lambda_s)^2 + \cdots, \quad (3.12)$$

with λ restricted to real negative values, then

$$\mathrm{Im}\left(\frac{d^m}{d\lambda^m} c\right)_s \neq 0, \quad (3.13)$$

while all the lower derivatives are zero. Eq. (3.13) then implies that

$$\mathrm{Im}\{c - c_s\} = \frac{1}{m!}\mathrm{Im}\left(\frac{d^m}{d\lambda^m} c\right)_s \times (\lambda - \lambda_s)^m + \mathrm{O}[(\lambda - \lambda_s)^{m+1}]. \quad (3.14)$$

Since $(\lambda - \lambda_s)^m$ is an odd function (recalling that m must be odd for y_s to be the inflection point), the curve defined by $c = c_s(\lambda_s)$ is seen to be a stability boundary, i.e., the sign of c_i is different on the $\lambda - \lambda_s > 0$ side than on the $\lambda - \lambda_s < 0$ side.

The validity of the arguments in Appendix A leading to Eq. (3.14) depends on the assumption in (3.11) and (A7), that $v_s(y)$ does not have a node at the critical level, i.e.,

$$v_s(y_c) \neq 0. \quad (3.15)$$

It is not clear that such is always the case. However, Tollmien (1935) managed to show that for a monotonic profile (or symmetric profile divisible into monotonic halves) in the absence of β, (3.15) does hold *in the interior of the domain*. His proof can be extended to the rotating case with $\beta > 0$, and this is relegated to Appendix B. The results are important for the baroclinic instability problem. The proof fails for non-monotonic profiles and the possible existence of an eigenfunction with a node at one of the critical levels has to be allowed for in that case. For that case, it can be shown that, if $v_s(y_c) = 0$, the appearance of a complex $[(d^n/d\lambda^n)c]_s$ is merely delayed by two orders, so that (3.14) and (A8) is replaced by

$$\mathrm{Im}\{c - c_s\} = -\frac{2\pi}{m!}[v_s'(y_c)]^2\left(\frac{d^m}{dy^m} q'\right)_s\left(\frac{dc}{d\lambda}\right)_s^{m+2} \times (\lambda - \lambda_s)^{m+2}\left[U'(y_s)^{m+3}\int_{y_1}^{y_2}\frac{q'}{(U - c_s)^2}v_s^2 dy\right]^{-1} + \mathrm{O}[(\lambda - \lambda_s)^{m+3}]. \quad (3.16)$$

The neutral curve is seen to be still a stability boundary.

THEOREM II. A stability boundary for the barotropic instability problem defined by (2.1) and (2.2) is given by the neutral solution whose critical level is located at the "inflection point." Consequently,

THEOREM III. A sufficient condition for the existence of unstable solutions to Eq. (2.1) and boundary condition (2.2) is that there exists a square-integrable real function $g(y)$, with square-integrable derivative, satisfying $g(y) = 0$ at $y = y_1$ and y_2, such that

$$I[g,U(y_s)] \equiv \int_{y_1}^{y_2}\left\{\left(\frac{d}{dy}g\right)^2 - K[y,U(y_s)]g^2\right\}dy < 0, \quad \text{and real}, \quad (3.17)$$

where

$$K[y,U(y_s)] \equiv (\beta - U_{yy})/[U - U(y_s)],$$

and y_s in (y_1,y_2).

This result applies to both monotonic and nonmonotonic profiles. From the way in which (III) is derived, it is seen that this condition for instability is simply the requirement for the quantization of the over-reflected wave.[9] Note also that this condition actually predicts instability, as compared to the necessary conditions mentioned in Section 2, which suggest the possibility of instability, but do not guarantee it. It should be pointed out that we have not shown that the above-stated condition is met by all unstable solutions; hence its violation does not

[9] Since the condition in (III) is the same as that in (I), it is strictly speaking the quantization condition for the neutral wave. However, (II) implies that if this neutral wave can fit between the boundaries, so can the unstable (or the overreflected) wave.

guarantee stability. No attempt at deriving a "sufficient condition for stability" is given here, since such a condition will not be physically meaningful without a discussion of other forms of instability, e.g., non-normal mode, non-infinitesimal disturbances. In the remaining part of this section more special conditions for instability (which are previously known) are reviewed.

a. Tollmien's (1935) problem

This problem is defined by system (2.1) and (2.2) with the following additional restrictions: (i) $\beta \equiv 0$; (ii) $U(y)$ symmetric with monotonic halves; (iii) only a single inflection point in each half; (iv) $U = 0$ at y_1 and y_2; and (v) $U \neq 0$ in (y_1,y_2). A suitable choice for the function $g(y)$ in (3.17) is (see Yih, 1969) $g(y) = U(y)$, and so the quantity $I[g,U(y_s)]$ is given by

$$I[U,U(y_s)] = \int_{y_1}^{y_2}\left(\frac{d}{dy}U\right)^2 dy + \int_{y_1}^{y_2}\frac{U_{yy}}{U - U(y_s)}U^2 dy$$
$$= UU_y\Big|_{y_1}^{y_2} - \int_{y_1}^{y_2}\frac{U_{yy}}{U - U(y_s)} \times \{U[U - U(y_s)] - U^2\}dy$$
$$= -\int_{y_1}^{y_2} K(y)UU(y_s)dy.$$

Since $U(y)U(y_s) > 0$ for all y in (y_1,y_2), it is clear that I is negative if Fjørtoft's *necessary* condition for instability (2.10) is satisfied. *The sufficient condition for instability (3.17) is met automatically by all solutions satisfying Fjørtoft's necessary condition.* It follows that (3.17), or Fjørtoft's condition (2.10), is both necessary and sufficient for instability in this restricted problem.

b. Howard's (1964) problem in an infinite domain

This problem is defined by[10] $K(y) = (\beta - U_{yy})/[U - U(y_s)]$ finite and of one sign. If Fjørtoft's necessary condition for instability is met, then $K(y) > 0$ except perhaps at a finite number of points where it can be zero. This class of profiles, called class-$\mathscr{K}$ by Howard (1964), includes, but is not restricted to, monotonic profiles which has one "inflection point." In the infinite domain $(y_1,y_2) = (-\infty,\infty)$, $g(y) = \exp\{-\epsilon|y|\}$ $\epsilon > 0$ is an admissible trial function (Howard, 1964). Substituting $g(y)$ into (3.17) and taking the limit $\epsilon \to 0^+$ then yields

$$I(g,U(y_s)) = \int_{y_1}^{y_2}[\epsilon^2 e^{-2\epsilon|y|} - K(y)e^{-2\epsilon|y|}]dy$$
$$= -\int_{y_1}^{y_2} K(y)e^{-2\epsilon|y|}dy,$$

which clearly is negative for the class-$\mathscr{K}$ problem at hand if Fjørtoft's necessary condition for instability is satisfied. Therefore, Fjørtoft's condition is both necessary and sufficient for instability in this case. Unstable solutions have already been found by Drazin and Howard (1962) and Howard and Drazin (1964) for sufficiently small wavenumber.

c. Finite domain

The problem is considerably more complicated when boundaries are present at a finite distance apart. The integral $\int_{y_1}^{y_2}[(dg/dy)^2]dy$ in general, cannot be made to vanish for any permissible choice of $g(y)$. Therefore it is expected that Fjørtoft's condition cannot be both necessary and sufficient for instability, not even for the class-$\mathscr{K}$ problem. The only known necessary and sufficient condition seems to be that of Rosenbluth and Simon (1964) for the more special problem defined by (i) $\beta \equiv 0$; (ii) $U_y > 0$ in the domain (can be relaxed to $U_y \neq 0$); (iii) U_{yy} has only a single zero; and (iv) $U_{yyy} \neq 0$ at the inflection point. The condition for instability was obtained in the form of the inequality

$$\frac{1}{U_y[U(y_s) - U(y)]}\Bigg|_{y_1}^{y_2} - \int_{y_1}^{y_2}\frac{U_{yy}}{U_y^3[U - U(y_s)]}dy > 0. \quad (3.18)$$

Because of the special way with which (3.18) was derived by Rosenbluth and Simon, restriction (i) is indispensible and cannot be relaxed. As a result, when $\beta \not\equiv 0$, we have not been able to reduce (3.17) to a more explicit form similar to (3.18).

4. Monotonic profiles

The condition (3.17) has not been shown to be a necessary condition for instability because for the general case there conceivably can exist (i) other neutral modes with neighboring instabilities and (ii) unstable solutions not contiguous to neutral modes. To examine the first possibility, we observe that for a neutral solution the Wronskian of Eq. (2.1) is an invariant,[11] except at the critical level, i.e.,

$$\mathrm{Im}\left\{v\frac{d}{dy}v^*\right\} = \text{constant} \quad \text{for} \quad y \neq y_c. \quad (4.1)$$

Near the critical level y_c, a Frobenius solution can be written as

$$v(y) = Av_1(y) + Bv_2(y), \quad (4.2a)$$

where

$$v_1(y) = (y - y_c) + \sum_{n=2}^{\infty} a_n(y - y_c)^n, \quad (4.2b)$$

[10] The problem of Howard (1964) is actually for the $\beta \equiv 0$ case but can be extended to include a nonzero β.

[11] It is sometimes referred to as the Eliassen and Palm theorem, for the momentum flux is proportional to the Wronskian.

$$v_2(y) = 1 + \sum_{n=2}^{\infty} b_n(y - y_c)^n - \frac{q'(y_c)}{U'(y_c)} \ln(y - y_c)v_1(y), \quad (4.2c)$$

with a_n and b_n being real.

For $(y - y_c) > 0$, we have

$$\mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_+ = \mathrm{Im}A^*B \quad (4.3)$$

and for $(y - y_c) < 0$,

$$\mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_- = \mathrm{Im}\{A^*B\} - \pi \frac{q'(y_c)}{U'(y_c)} |B|^2. \quad (4.4)$$

In arriving at (4.4), the branch (2.3) has been taken. There is therefore a jump in the values of the Wronskian across the critical level given by

$$\mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_+ - \mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_- = \pi \frac{q'(y_c)}{U'(y_c)} |B|^2. \quad (4.5)$$

However, the boundary conditions (2.2) for monotonic profiles (with only one critical level), imply that

$$\mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_+ = 0, \quad \mathrm{Im}\left\{v \frac{d}{dy} v^*\right\}_- = 0, \quad (4.6)$$

which contradicts Eq. (4.5) (and hence no neutral solutions exist), unless one of the following[12] occurs:

$$\left.\begin{array}{lll} \text{(i)} & q'(y_c) = 0, & \text{i.e., } y_c = y_s \\ \text{(ii)} & B = 0, & \text{i.e., } v(y_c) = 0 \\ \text{(iii)} & U \neq c, & \text{for } y \text{ in } (y_1,y_2) \end{array}\right\} .$$

Possibility (i) has been considered in Section 3. It has also been shown in Appendix B that possibility (ii) does not happen for monotonic profiles and c in the interior of the flow. The case of the critical level located on one of the boundaries will be considered later. The third possibility will be considered now. The arguments do not require the assumption of monotonicity of $U(y)$.

It is known, from a form of the semicircle theorem for nonrotating flows, that c must lie inside the range of $U(y)$. It is equally well known that for rotating flows with $\beta > 0$, the system (2.1) and (2.2) permits free Rossby waves whose phase speeds are less than the minimum of U. This has been proved by Kuo (1949) using the oscillation theorem for the Sturm-Liouville system. No neutral waves can exist, however, for c greater than or equal to the maximum of U. These results can be seen by observing that Eq. (2.1) can be written as

$$\frac{d}{dy}(U - c)^2 \frac{d}{dy} F + [-\alpha^2(U - c)^2 + \beta(U - c)]F = 0, \quad (4.7)$$

with $F \equiv v/(U - c)$. Multiplying Eq. (4.7) by F^* and integrating from $y = y_1$ to y_2 and after utilizing the boundary conditions (2.2)[13] to eliminate the term

$$F^* \frac{d}{dy} F(U - c)^2 \Big|_{y_1}^{y_2},$$

one obtains

$$\int_{y_1}^{y_2} (U - c)^2[|F_y|^2 + \alpha^2|F|^2]dy = \int_{y_1}^{y_2} \beta(U - c)|F|^2dy. \quad (4.8)$$

For c real and lying outside the range of $U(y)$, the left-hand side of (4.8) is positive, while the right-hand side is (i) zero if $\beta \equiv 0$, thus implying the nonexistence of neutral solutions with c outside the range of U, or (ii) negative if $\beta > 0$ and $c - U_{\max} \geq 0$, thus implying the nonexistence of neutral solutions for $c \geq U_{\max}$. As a matter of fact, no solution, neutral or not, exists for $c_r \geq U_{\max}$, for if $c_i \neq 0$, the imaginary part of (4.8) will yield

$$2\int_{y_1}^{y_2} (U - c_r)[|F_y|^2 + \alpha^2|F|^2] = \int_{y_1}^{y_2} \beta|F|^2dy, \quad (4.9)$$

which again implies that no solution exists for $c_r \geq U_{\max}$. This result has been obtained previously by Kuo (1949), Pedlosky (1964) and Miles (1964).

The condition for the existence of neutral solutions with $c < U_{\min}$ is still given by Eq. (3.6), except with $K(y)$ now replaced by

$$K(y,c) \equiv \frac{\beta - U_{yy}}{U - c}. \quad (4.10)$$

The question we want to address now is whether these neutral solutions have neighboring instabilities. The question seems to be a trivial one, as the answer can be expected to be negative without any further analysis. But in light of the fact that the semicircle theorem of, say, Pedlosky (1964), permits unstable solutions whose phase speed is $<U_{\min}$, it is important to give a proof to dispel the notion that such instabilities, if they exist, can be contiguous to the neutral modes whose phase speeds are $<U_{\min}$. The proof follows the same line as given in Section 3 in the investigation of the neighboring instabilities for the neutral mode, with the important difference that now since $U - c$ never vanishes, the integrands

[12] There is a fourth possibility which will not be considered here, the case of a nonlinearity-dominated critical layer with a zero phase-shift.

[13] Or, incidentally, the baroclinic boundary condition $dF/dy = 0$, so this result applies also to the baroclinic problem.

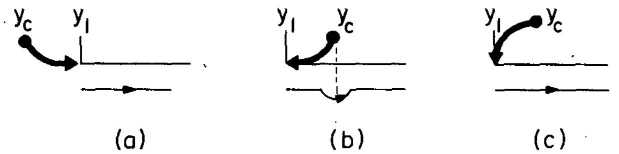

FIG. 2. Enlarged portion of the complex y plane of Fig. 1 near the end point y_1: (a) the case where the neighboring solution has its critical level to the left of y_1, (b) the case where the neighboring solution has its critical level to the right of y_1, and as the neutral solution is approached, c_i vanishes before $c_r - U(y_1)$, and (c) the case where $c_r - U(y_1)$ vanishes before c_i does as the neutral solution is approached.

in equations such as (A5) do not have singularities and, therefore, in the limit as the neutral solution is approached no imaginary parts to the integrals exist. Thus $\text{Im}[(d^n/d\lambda^n)c] = 0$ to all orders in n when the neutral solution is approached and one can conclude that no neighboring instabilities exist for the neutral modes whose phase speeds are less than the minimum of U.

THEOREM IV. The unstable solution, if one exists, whose phase speed has a real part falling outside the range of $U(y)$ can be contiguous only to a neutral mode whose phase speed lies in the range of $U(y)$.

Note that the above result applies to both the monotonic and nonmonotonic cases, and to both the barotropic and baroclinic instability problems. It is important for the wave overreflection interpretation of instability, since overreflection occurs only if a critical level is located in the domain.

Now it seems that the only possibility left for a neutral mode to exist, besides the ones considered in the preceding paragraphs, is to have the critical level situated at one of the boundaries.[14] For the monotonic profile under consideration, this can occur only at the boundary where $U = U_{\min}$, the $c = U_{\max}$ case having already been eliminated. Whether or not such a neutral solution exists will depend on whether a particular profile does or does not satisfy condition (3.6) with $K[y, U(y_s)]$ replaced by

$$K(y, U_{\min}) = \frac{\beta - U_{yy}}{U - U_{\min}}.$$

THEOREM V. A neutral solution to Eq. (2.1) and boundary condition (2.2) with α real and $c = U_{\min}$ exists if and only if there exists a square-integrable real function $g(y)$ with square-integrable derivative, satisfying $g(y) = 0$ at $y = y_1$ and y_2, such that

$$I(g, U_{\min}) \int_{y_1}^{y_2} \left\{ \left(\frac{d}{dy} g \right)^2 - K(y, U_{\min}) g^2 \right\} dy < 0, \quad \text{and real.} \quad (4.11)$$

For nonmonotonic profiles, the U at the boundary is not necessarily $U_{\min}$. The condition for the existence of the boundary mode is still given by (4.11), except with $U(y_1)$ replacing $U_{\min}$.

Note that the existence of the *neutral* solution does not require the occurrence of an inflection point in the interior.[15] However, the existence of the *unstable* neighboring solution does, as will be shown later. Note also that despite the singularity of $K(y, U_{\min})$ at where $U = U_{\min}$, the integral is regular and so is real because $g^2(y)$ is zero there, forced by the boundary condition.

Assuming that the condition (4.11) is satisfied, so a neutral solution exists for the monotonic profile whose critical level is located at one of the boundaries, say, y_1. It is clear that such a neutral solution, which shall be denoted by subscript p, is a regular function of y. We wish to investigate whether neighboring instabilities exist, and this turns out to be a more complicated job than previously encountered in Section 3. Due to the fact that the critical level is now located at the boundary, there may not be a need to "go around" it as has been the case in Section 3, where the imaginary parts to the derivatives $[(d^n/d\lambda^n)c]_s$ are found by following a path that goes under the singularity in the complex plane. For the present case, different results concerning the existence of neighboring instabilities are expected to arise depending on the different manner in which $c(\lambda)$ approaches $c_p(\lambda_p) = U(y_1)$. If c approaches c_p from the left (see Fig. 2a), i.e.,

$$c(\lambda) \rightarrow c_p^-, \quad y_c \rightarrow y_1^-, \quad (4.12a)$$

then there is no singularity in $[y_1, y_2]$, and the path of integration such as those in Eq. (A5) can be entirely on the real axis. In this case we find $(d^n c/d\lambda^n)_p$ has no imaginary part to all orders in n. On the other hand, if c approaches c_p from the right on the real axis (see Fig. 2b), i.e.,

$$c(\lambda) \rightarrow c_p^+, \quad y_c \rightarrow y_1^+, \quad (4.12b)$$

then the path of integration has to be dented as described in Section 3, and imaginary parts to $(d^n c/d\lambda^n)_p$ may result. The last possibility, that of c approaching c_p from the imaginary axis (see Fig. 2c), i.e.,

$$c \rightarrow c_p + i0^+, \quad (4.12c)$$

[14] There appear to be some unresolved difficulties in the slightly viscous problem when a critical layer approaches a boundary layer (see Lin, 1955). Therefore, results obtained here for the inviscid problem should be interpreted with caution in a physical context.

[15] It can be shown that (4.11) can be satisfied by a linear profile, e.g., Charney's problem, which does not have an "inflection point" in the interior of the domain (Charney, 1947).

is seen to leave the real axis free from the singularity and hence no imaginary part to $(d^n c/d\lambda^n)_p$ will result from the integrations along the real axis. This possibility is eliminated by calculating $(dc/d\lambda)_p$, which is [from Eq. (3.10)],

$$\left(\frac{dc}{d\lambda}\right)_p = -\int_{y_1}^{y_2} v_p{}^2 dy \times \left\{\int_{y_1}^{y_2} \frac{q'}{[U - U(y_1)]^2} v_p{}^2 dy\right\}^{-1}. \quad (4.13)$$

It is seen to be real and nonzero, as the integrand in the denominator is nonsingular [remembering that $v_p(y_1) = 0$]. Thus the approach to y_1 will be along the real axis in the close vicinity of y_1; whether it is from the left or the right depends on the sign of $(dc/d\lambda)_p(\lambda - \lambda_p)$. Suppose $(dc/d\lambda)_p$ is negative, then $c \to c_p^- + O[(\lambda - \lambda_p)^2]$ if $\lambda \to \lambda_p^+$, giving no neighboring instability, and if $\lambda \to \lambda_p^-$, one has $c \to c_p^+ + O[(\lambda - \lambda_p)^2]$, suggesting possible neighboring instabilities. At the second order, one finds [cf. (A5)] that

$$\operatorname{Im}\left(\frac{d^2}{d\lambda^2} c\right)_p = -2\left(\frac{dc}{d\lambda}\right)_p^2 \operatorname{Im}\left\{\lim_{c\to c_p} \int_{y_1}^{y_2} \frac{q'}{(U-c)^3} v_p{}^2 dy\right\}\left\{\int_{y_1}^{y_2} \frac{q'}{(U-c_p)^2} v_p{}^2 dy\right\}^{-1}. \quad (4.14)$$

Therefore, depending on the sign of $(\lambda - \lambda_p)$, one finds

$$\operatorname{Im}\left(\frac{d^2}{d\lambda^2} c\right) = \begin{cases} 0, & \text{if } \left(\dfrac{d}{d\lambda} c\right)_p (\lambda - \lambda_p) \to 0^- \quad (4.15\text{a}) \\[2ex] \dfrac{-2\pi\left(\dfrac{dc}{d\lambda}\right)_p^2 v_p'(y_1)^2 q'(y_1)}{\left\{U'(y_1)^3 \displaystyle\int_{y_1}^{y_2} \frac{q'}{(U-c_p)^2} v_p{}^2 dy\right\}}, & \text{if } \left(\dfrac{d}{d\lambda} c\right)_p (\lambda - \lambda_p) \to 0^+. \quad (4.15\text{b}) \end{cases}$$

If (4.15b) is positive, instability exists on one side of the neutral curve.

Note that no instability of the kind discussed above exists unless an inflection point also occurs in the interior. To show this, let us first assume that the profile is free of inflection points, i.e., $q'(y) \neq 0$. Then (4.11) necessarily implies that $q'(y) > 0$ in order to make $K(y, U_{\min})$ positive. This in turn implies that the integral in the denominator of (4.15b) is positive. No instability could arise unless $U'(y_1) < 0$, but $U'(y_1) < 0$ is contradictory to the fact that $U(y_1)$ is a minimum of $U(y)$. Therefore, $q'(y)$ must have zero(es) in the interior. The Rayleigh-Kuo necessary condition for instability is thus shown to be not contradicted.

One last detail, the possibility that $q'(y_1)$ vanishes, is now considered. Let $q'(y)$ have an order m zero at $y = y_1$, i.e.,

$$q'(y) = O[(y - y_1)^m], \quad m = 0, 1, 2, 3, \ldots.$$

(m is no longer restricted to odd integers). The first derivative in (3.12) to have a nonzero imaginary part is $(d^{m+2}c/d\lambda^{m+2})_p$. We find [cf. Eq. (A5)]:

$$\operatorname{Im}\left(\frac{d^{m+2}}{d\lambda^{m+2}} c\right)_p = \frac{-(m+2)!\left(\dfrac{dc}{d\lambda}\right)^{m+2} \operatorname{Im}\left\{\lim_{c\to c_p} \displaystyle\int_{y_1}^{y_2} \frac{q'}{(U-c)^{m+3}} v_p{}^2 dy\right\}}{\displaystyle\int_{y_1}^{y_2} \frac{q'}{(U-c_p)^2} v_p{}^2 dy},$$

so

$$\operatorname{Im}\left(\frac{d^{m+2}}{d\lambda^{m+2}} c\right)_p = \begin{cases} 0, & \text{if } \left(\dfrac{d}{d\lambda} c\right)_p (\lambda - \lambda_p) \to 0^- \quad (4.16\text{a}) \\[2ex] \dfrac{-\pi(m+2)(m+1)\left(\dfrac{dc}{d\lambda}\right)_p^{m+2} v_p'(y_1)^2 \dfrac{d^m}{dy^m} q'(y_1)}{\left\{U'(y_1)^{m+3} \displaystyle\int_{y_1}^{y_2} \frac{q'}{(U-c_p)^2} v_p{}^2 dy\right\}}, & \text{if } \left(\dfrac{d}{d\lambda} c\right)_p (\lambda - \lambda_p) \to 0^+. \quad (4.16\text{b}) \end{cases}$$

When m is even, neighboring instability occurs if (4.16b) is positive. When m is odd, neighboring instability occurs only if $\mathrm{Im}(d^{m+2}c/d\lambda^{m+2})_p(\lambda - \lambda_p)^{m+2}$ is positive on the side of $\lambda - \lambda_p$ which gives positive $[(d/d\lambda)c]_p(\lambda - \lambda_p)$. In other words, we have instability only when

$$\mathrm{Im}\left(\frac{d^{m+2}}{d\lambda^{m+2}}c\right)_p \Big/ \left(\frac{dc}{d\lambda}\right)_p > 0.$$

Thus, for either case, the condition for instability is

$$-\frac{d^m}{dy^m}q'(y_1) \Big/ \left\{U'(y_1)^{m+1}\int_{y_1}^{y_2}\frac{q'}{[U - U(y_1)]^2}v_p{}^2dy\right\} > 0 \quad (4.17)$$

$m = 0, 1, 2, \ldots$ being the order of the zero of $q'(y)$ at y_1. We have thus obtained another sufficient condition for instability.

Using an argument similar to the one just used for the $m = 0$ case, it can be shown that (4.17), the condition for neighboring solutions to have $c_i > 0$ for a general m, implies the necessity for an interior "inflection point" to occur.

THEOREM VI. The instability (if it exists) contiguous to the neutral solution whose critical level is situated at the boundary has to be accompanied by at least one "inflection point" in the interior of the domain.

We define

$$(g(y)) \equiv \int_{y_1}^{y_2}\{K[y,U(y_s)] - K(y,U_{\min})\}g(y)^2dy \quad (4.18)$$

and enquire about its sign. Since

$$\Delta = \int_{y_1}^{y_2}\left[\frac{\beta - U_{yy}}{U(y) - U(y_s)}\right]\left[\frac{U(y_s) - U_{\min}}{U(y) - U_{\min}}\right]g^2dy,$$

we have

$$\Delta(g) > \left[\frac{U(y_s) - U_{\min}}{U_{\max} - U_{\min}}\right]\int_{y_1}^{y_2}K(y)g^2dy > 0; \quad (4.19)$$

the integral in (4.19) is positive by Fjørtoft's theorem [Eq. (2.8)]. We next consider the sign of the difference

$$\delta\lambda_{00} \equiv (\lambda_0)_s - (\lambda_0)_p, \quad (4.20)$$

where subscripts s and p refer to the eigensolutions described in Theorems I and V, respectively. The index 0 in λ denotes the lowest eigenvalue of λ. $\delta\lambda_{00}$ can be expressed alternatively as

$$\delta\lambda_{00} = I[(v_0),U(y_s)] - I[(v_0)_p,U_{\min}], \quad (4.21)$$

assuming that the eigenfunctions are normalized so that

$$\int_{y_1}^{y_2}(v_0)_s{}^2dy = 1 = \int_{y_1}^{y_2}(v_0)_p{}^2dy.$$

Since $(v_0)_s$ is the function that yields the minimum value for $I[g,U(y_s)]$, we have

$$I[(v_0)_s,U(y_s)] \leq I[(v_0)_p,U(y_s)]. \quad (4.22)$$

Therefore,

$$\begin{aligned}\delta\lambda_{00} &\leq I[(v_0)_p,U(y_s)] - I[(v_0)_p,U_{\min}] \\ &= -\int_{y_1}^{y_2}\{K[y,U(y_s)] - K(y,U_{\min})\}(v_0)_p{}^2dy \\ &= -\Delta(v_{0p}). \end{aligned} \quad (4.23)$$

Using (4.19) in (4.23), we have, $\delta\lambda_{00} < 0$, i.e.,

$$(\lambda_0)_s < (\lambda_0)_p. \quad (4.24)$$

(Equality holds only if $y_s \rightarrow y_1$.) Thus it is seen that, for the same and $U(y)$, the condition (4.11), which is required for the existence of an unstable boundary mode, is more difficult to satisfy than the one in Theorem (III). Furthermore, since

$$(\lambda_0)_p \leq 0 \Rightarrow (\lambda_0)_s < 0$$

and

$$(\lambda_0)_s \geq 0 \Rightarrow (\lambda_0)_p > 0,$$

we have the following theorems:

THEOREM VII. The sufficient condition for the existence of unstable solution contiguous to the neutral mode whose critical level is located at the boundary is also sufficient for the existence of unstable solution contiguous to the neutral mode whose critical level is located at the inflection point.

THEOREM VIII. The sufficient condition for the existence of unstable solution contiguous to the neutral mode whose critical level is located at the inflection point is also necessary for the existence of unstable solution contiguous to the neutral mode whose critical level is located at the boundary.

In other words, if the unstable solution of Theorem III does not exist, there is also no unstable solution of the kind discussed in this section. Combining the two theorems, one sees that as far as the existence of *any* unstable contiguous is concerned, Theorem III is both necessary and sufficient. So

THEOREM IX. Condition (3.17) in Theorem (III) is both necessary and sufficient for the existence of an unstable solution contiguous to a neutral curve, when the profile is monotonic.

The last step in showing that (3.17) is the necessary and sufficient condition for instability is to prove that noncontiguous instabilities do not occur.

The proof seems to be difficult using the present approach and so will not be attempted here. It can be mentioned that no contiguous instability has ever been reported for the system (2.1) and (2.2).

5. Concluding remarks

We have studied in this paper the barotropic instability of continuous (analytic) zonal flows $U(y)$ and a condition for instability is derived by extending the classical results of Tollmien, Fredericks and Lin. Since only infinitesimal normal-mode perturbations are treated, it has not been appropriate to make a statement concerning the *stability* of flows using the present approach. For that purpose a more powerful method using concepts from Liapunov stability theory is available which does not involve the restriction to linear normal-mode perturbations. Arnol'd (1965) has successfully applied it to the problem of hydrodynamic stability, and the result is later extended by Dikii (1965) and Blumen (1968) to the problem of barotropic and baroclinic stability. The derived condition for stability also is stated in a variational formula, and the *form* of that condition is remarkably similar to our formula for instability [*viz.*, (3.17)]. For the present problem on one-dimensional zonal flows, Arnol'd's theorem can be stated in the following form:

The flow U(y) *is stable, in the sense of Liapunov, if the quadratic form*

$$J(g) \equiv \int_{y_1}^{y_2} \left[g^2 - \frac{1}{K(y)} \left(\frac{d}{dy} g \right)^2 \right] dy \qquad (5.1)$$

is definite,

i.e., of one sign for all variations $g(y)$ which vanish on the boundaries. Condition (5.1) should be compared with our (3.17), which states:

The flow U(y) *is unstable if the following quadratic form:*

$$I(g) \equiv \int_{y_1}^{y_2} \left[\left(\frac{d}{dy} g \right)^2 - K(y) g^2 \right] dy \qquad (5.2)$$

is non-definite,

i.e., positive for some $g(y)$ and negative for some other $g(y)$, $g(y)$ vanishing on the boundaries.

Because of the fact that Arnol'd's result predicts stability but not instability, and our result predicts instability but not stability, the two approaches complement each other and further study in these areas is likely to be fruitful.

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APPENDIX A

Stability Boundary

Near $\lambda = \lambda_s$, one has, in general,

$$\mathrm{Im}\{c - c_s\} = \sum_{n=1}^{\infty} \frac{1}{n!} \mathrm{Im} \left(\frac{d^n c}{d\lambda^n} \right)_s (\lambda - \lambda_s)^n, \qquad (A1)$$

with λ restricted to real negative values.

When y_s is a higher root of $\beta - U_{yy}$ (i.e., $m > 1$), $(dc/d\lambda)_s$ has no imaginary part, as seen in Eq. (3.11). To determine whether neighboring unstable solution can exist, derivatives higher than $(dc/d\lambda)_s$ have to be considered. We want to show that not all of the derivatives in (A1), $\mathrm{Im}[(d^n c/d\lambda^n)_s]$, are zero, so that neighboring unstable solution does exist. Furthermore, it will be shown that the first derivative in (A1) for which the imaginary part does not vanish is of the odd order. Therefore, $c = c_s(\alpha_s^2)$ is a *stability boundary*.

When Eq. (3.7) is differentiated n times with respect to λ, it can be deduced from the form of Eq. (3.8) that the following is true for a general n:

$$L[v_{\lambda^{(n)}}] = -n v_{\lambda^{(n-1)}} - \sum_{r=1}^{n} C_r^n v_{\lambda^{(n-r)}} \frac{d^r}{d\lambda^r} \left(\frac{q'}{U - c} \right), \quad (A2)$$

where $C_r^n \equiv n!/(n-r)!r!$ are the binomial coefficients. In the limit $\lambda \to \lambda_s$ and $c \to c_s$, the solution v_s of Eq. (3.7) is real. In the same limit, the solution v_{λ_s} of (3.8) is real if $\mathrm{Im}\{(dc/d\lambda)_s\} = 0$, which is the case when $K(y_s) = 0$. In general, if $y = y_s$ is a root of $q' = 0$ of multiplicity m, i.e.,

$$\left. \frac{q'}{(U - c_s)^r} \right|_{y=y_s} = 0 \quad \text{for} \quad r = 1, 2, \ldots, m-1,$$

but

$$\left. \frac{q'}{(U - c_s)^m} \right|_{y=y_s} \neq 0,$$

then it can be shown from Eq. (A2) for $n = 1, 2, \ldots, m-1$ that

$$\left. \begin{array}{l} (v_\lambda)_s, [v_{\lambda^{(2)}}]_s, \cdots, [v_{\lambda^{(m-1)}}]_s \\ \left(\dfrac{d}{d\lambda} c \right)_s, \left(\dfrac{d^2}{d\lambda^2} c \right)_s, \cdots, \left(\dfrac{d^{m-1}}{d\lambda^{m-1}} c \right)_s \end{array} \right\} \qquad (A3)$$

are all real. We now want to show that

$$\mathrm{Im} \left(\frac{d^m}{d\lambda^m} c \right)_s \neq 0.$$

By taking

$$vL[v_{\lambda^{(n)}}] - v_{\lambda^{(n)}} L(v)$$

it follows that

$$\frac{d}{dy} \left[v \frac{d}{dy} v_{\lambda^{(n)}} - v_{\lambda^{(n)}} \frac{d}{dy} v \right] = -n v_{\lambda^{(n-1)}} v - \sum_{r=1}^{n} C_r^n v_{\lambda^{(n-r)}} v \frac{d^r}{d\lambda^r} \left(\frac{q'}{U - c} \right). \quad (A4)$$

We shall consider Eq. (A4) for $n = m$ and $\lambda \to \lambda_s$, $c \to c_s$. The left-hand side of the equation is not real in general, but its integral from $y = y_1$ to $y = y_2$ vanishes. Therefore, making use of (A3), we have

$$m!\left(\frac{dc}{d\lambda}\right)_s^m \lim_{c\to c_s}\int_{y_1}^{y_2} v_s^2 \frac{q'}{(U-c)^{m+1}}dy + \left(\frac{d^m}{d\lambda^m}c\right)_s \int_{y_1}^{y_2}\frac{q'}{(U-c_s)^2}v_s^2 dy = \text{real}, \quad m > 1. \quad \text{(A5)}$$

[If $m = 1$, the second integral on the left-hand side of Eq. (A5) is to be replaced by $\int_{y_1}^{y_2} v_s^2 dy$.] Taking the same path of integration as previously described, the singular integral

$$\lim_{c\to c_s}\int_{y_1}^{y_2} v_s^2\frac{q'}{(U-c)^{m+1}}dy$$

can be shown to have the imaginary part

$$\pi v_s^2(y_c)\left(\frac{d^m}{dy^m}q'\right)_s [m!U'(y_s)^{m+1}]^{-1}. \quad \text{(A6)}$$

Since both

$$\left(\frac{dc}{d\lambda}\right)_s \quad \text{and} \quad \int_{y_1}^{y_2}\frac{q'}{(U-c_s)^2}v_s^2 dy$$

are real, Eq. (A5) implies

$$\mathrm{Im}\left(\frac{d^m}{d\lambda^m}c\right)_s = \frac{-\pi v_s^2(y_c)\left(\dfrac{d^m}{dy^m}q'\right)_s\left(\dfrac{dc}{d\lambda}\right)_s^m [U'(y_s)]^{-(m+1)}}{\displaystyle\int_{y_1}^{y_2}\frac{q'}{(U-c_s)^2}v_s^2 dy} \quad \text{(A7)}$$

and from (A1)

$$\mathrm{Im}\{c - c_s\} = \frac{1}{m!}\mathrm{Im}\left(\frac{d^m}{d\lambda^m}c\right)_s(\lambda-\lambda_s)^m + O[(\lambda-\lambda_s)^{m+1}]. \quad \text{(A8)}$$

Since $(\lambda - \lambda_s)^m$ is an odd function (recalling that m must be odd), $c = c_s(\lambda_s)$ is seen to be a stability boundary, that is, the sign of c_i is different depending on whether λ is greater or smaller than λ_s.

APPENDIX B

The Eigenfunction with a Node at the Critical Level

Suppose one has a neutral eigenfunction $v(y) = v_p(y)$ that has a node at the critical level y_c, i.e.,

$$v_p(y_c) = 0. \quad \text{(B1)}$$

From the Frobenius solution near y_c we know that $v_p(y)$ is analytic and expressible as $v_p(y) = Av_1(y)$, where $v_1(y) = (y - y_c) + O[(y - y_c)^2]$. Eq. (2.1) can be integrated with respect to y from $y = y_c$ to y to yield

$$(U-c)\frac{d}{dy}v_p(y) - \frac{d}{dy}Uv_p(y) = \int_{y_c}^{y}[\alpha^2(U-c)-\beta]v_p dy, \quad \text{(B2)}$$

where (B1) has been used to eliminate the part of the left-hand side evaluated at y_c. Let y be in the direction of decreasing $U(y)$ away from y_c, and let y_0 be the next zero of $v_p(y)$ in this direction. Then for y between y_c and y_0, $v_p(y)$ is of one sign, which we can set to $-$ sign $(y - y_c)$, without loss of generality. Thus the right-hand side of Eq. (B2) is positive, and we have

$$(U-c)\frac{d}{dy}v_p(y) - \frac{d}{dy}Uv_p(y) > 0. \quad \text{(B3)}$$

If we assume that sign $(y - y_c)$ is positive, then the second term in (B3), $(dU/dy)v_p(y)$, is positive, and hence $(U - c)(d/dy)v_p(y) > 0$ which implies, since $U - c < 0$, that

$$\frac{d}{dy}v_p(y) < 0, \quad y_c < y < y_0. \quad \text{(B4)}$$

This, together with the assumption

$$v_p(y) < 0, \quad y_c < y < y_0,$$

implies that one of the boundary conditions in (2.2), *viz.*,

$$v(y) = 0 \quad \text{at} \quad y = y_2 > y_c$$

cannot be satisfied. If sign $(y - y_c)$ is negative, i.e., $(dU/dy) > 0$ near y_c, we then have

$$\left.\begin{array}{ll} v_p(y) > 0, & y_c > y > y_0 \\ \dfrac{d}{dy}v_p(y) < 0, & y_c < y < y_0 \end{array}\right\}, \quad \text{(B5)}$$

and hence the boundary condition at $y_1 < y_c$ cannot be satisfied. Thus, *for monotonic profiles no neutral solution can have a node at the critical level located in the interior of the domain.*[16] An alternate proof can be found in Howard (1964) for the single inflection-point case.

Incidentally, the same conclusion can also be applied to the baroclinic problem. The argument is actually easier to make as the lower boundary condition in that problem is

$$(U-c)\frac{d}{dz}\phi - \frac{d}{dz}U\cdot\phi = 0. \quad \text{(B6)}$$

[16] The arguments presented here do *not* eliminate the possibility of a node at a critical level located at one of the boundaries.

Comparing (B6) with (B3) then shows directly that the boundary condition cannot be satisfied. The following important result is obtained for the baroclinic problem:

For the baroclinic problem of a monotonic zonal wind with no interior inflection point, the neutral solution, if it exists, must have its critical level located at the (lower) boundary. For profiles that do not permit the phase speed of the disturbance to match the surface wind speed, no neutral solutions exist.

For nonmonotonic profiles, Eqs. (B4) and (B5) can be made to hold only for the part of the domain between the critical level and the location of the nearest minimum of $U(y)$.[17] Through a slight refinement of the arguments presented above, which involves using $F = v(U - c)$ instead of v, the result can be extended past the minimum of U into the region in which $U(y)$ is increasing until $U(y)$ again attains the value c. If one of the boundary is located in this (extended) region, the boundary condition again cannot be satisfied and the original assumption of $v_p(y_c) = 0$ has to be abandoned. If, however, the boundary is located outside this region, the possibility of $v_p(y_c) = 0$ is not excluded.

[17] If $\beta < 0$, change "minimum of $U(y)$" to "maximum of $U(y)$." If $\beta \equiv 0$, either extremum will do.

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