# Remarks on Charney's Note on Geostropic Turbulence

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11 February 2000 and 18 January 2001

### ABSTRACT

Charney in 1971 generalized results for two-dimensional (2D) turbulence to quasigeostrophic (QG) turbulence and obtained two results that have important implications for the atmosphere. The first is an attempt to prove that, similar to 2D turbulence, energy in QG turbulence goes only upscale in the net. The second is a demonstration that 3D QG motion in terms of a 3D wavenumber in a stretched coordinate is isomorphic to 2D turbulence. Charney's proofs are shown here to be problematic.

## 1. Introduction

Charney's (1971) note, ''Geostrophic Turbulence,'' is generally credited as laying the foundation for the subject. Although three-dimensional (3D) in nature, large-scale motion in the atmosphere and oceans satisfying geostrophic scaling was shown to have more in common with the 2D turbulence of Kraichnan (1967) than with the 3D turbulence of Kolmogorov (1941a,b). As Charney (1971) demonstrated, the existence of a scalar invariant, the ''pseudo-potential vorticity,'' in addition to the energy invariant, provides a powerful constraint on energy transfers in quasigeostrophic (QG) turbulence, which is absent in 3D turbulence.

Charney's work was probably motivated by the observation available at the time (e.g., Wiin-Nielsen 1967), which showed an apparent $k^{-3}$ power-law behavior in the energy spectrum for horizontal wavenumbers $k$ in the synoptic scales (zonal wavenumbers 7–18), and its similarity to the $k^{-3}$ spectrum predicted by Kraichnan (1967) for 2D turbulence for wavenumbers higher than the excitation wavenumber. Charney's note contains two main results:

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- It attempts to prove that energy cascades upscale in the net in QG turbulence, similar to 2D turbulence.
- There is a demonstration of isomorphism between QG and 2D turbulence, and consequently the observed $k^{-3}$ spectrum over the synoptic scales was explained using Kraichnan's (1967) theory on isotropic and homogeneous 2D turbulence.

Both of these results contain major flaws, mathematical in nature in result 1 and quantitative in result 2. Although parts of the problem we will discuss may be known to some—for example, Merilees and Warn (1975) pointed out that the result of Fjørtoft (1953) on the direction of energy cascade in 2D turbulence, on which Charney relied, was in error—the implications for large-scale atmospheric turbulence probably have not been fully appreciated. Unproven ''folklore'' in 2D turbulence concerning the direction of energy cascades [see comments by Eyink (1996)] are often carried over to QG turbulence in the atmosphere without further proof. In this note, we point out some of the problem areas.

## 2. Mathematical aspects of Charney's proof

Large-scale atmospheric flows satisfying QG scaling conserve what Charney (1971) called ''pseudo-potential vorticity'' (which we will call ''potential vorticity''),

$$q = \nabla_H^2 \psi + \frac{f_0^2}{\overline{\rho}}\left(\frac{\overline{\rho}}{N^2}\psi_z\right)_z + \beta y,$$

where $\psi$ is the streamfunction, $\overline{\rho}(z)$ is the mean density, $f$ and $\beta$ are Coriolis parameters, and $N$ is the Bruut–Väisälä frequency. Its conservation equation takes the form

$$\frac{\partial}{\partial t}q + J[\psi, q] = 0. \tag{1}$$

It is seen that (1) is analogous to the 2D vorticity equation

$$\frac{\partial}{\partial t}\nabla_H^2\psi + J[\psi, \nabla_H^2\psi] = 0, \tag{2}$$

and so one may expect that QG flows have behaviors analogous to 2D flows.

Charney (1971) obtained an energy equation by multiplying Eq. (1) by $-\overline{\rho}\psi$ and integrating over $x$, $y$, and $z$ (The range of $z$ is semi-infinite, from 0 to $\infty$; the $x$ domain is periodic; the solution vanishes at two points in $y$):

$$\frac{d}{dt}E \equiv \frac{d}{dt}\iiint \frac{1}{2}\left[\boldsymbol{\nabla}_H\psi \cdot \boldsymbol{\nabla}_H\psi + \frac{f_0^2}{N^2}\psi_z^2\right]\overline{\rho}\ dx\ dy\ dz$$
$$= \iint \frac{f_0^2}{N^2}\overline{\rho}\psi\frac{\partial}{\partial t}\psi_z|_0^\infty\ dx\ dy = 0. \tag{3}$$

In arriving at energy conservation, Eq. (3), Charney used the vertical boundary conditions

$$\overline{\rho}\psi\psi_z \to 0 \quad \text{as } z \to \infty \quad \text{and} \quad \psi_z = 0 \quad \text{at } z = 0. \tag{4}$$

The upper-boundary condition in (4) equivalent to the assumption that the energy density vanishes at infinity. Without it there could be energy leakage to infinity and so there would not be energy conservation.[1]

Charney defined a 3D elliptic operator $L$ by

$$L(\psi) \equiv \nabla_H^2\psi + \frac{f_0^2}{\overline{\rho}}\left(\frac{\overline{\rho}}{N^2}\psi_z\right)_z,$$

and obtained the equation for the conservation of potential enstrophy by multiplying Eq. (1) by $\overline{\rho}L(\psi)$ and integrating over all $x$, $y$, and $z$ [although the $z$ integration is not necessary (see Salmon 1998)]:

$$\frac{d}{dt}F \equiv \frac{d}{dt}\iiint [L(\psi)]^2\overline{\rho}\ dx\ dy\ dz = 0. \tag{5}$$

Using the two derived conservation laws, (3) and (5), Charney then proceeded to derive the result that energy flows upscale. His (excerpted) argument follows:

"Now $L$ is a self-adjoint elliptic operator with a com-

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[1] The lower-boundary condition in (4) is unrealistic, because large-scale transient waves ("turbulence") are largely driven by temperature gradient near the surface. It is, however, necessary for Charney's energy conservation. One way to circumvent this problem is to use a trick first suggested by Bretherton (1966), to extend the lower boundary "underground," to $z = -z_0$, where the isothermal lower-boundary condition is applied.

plete orthonormal set of eigenfunctions $\psi_m$ and eigenvalues $\lambda_m(m = 1, 2, \ldots)$. . . .

"By virtue of the completeness property, we may set

$$\psi = \sum_1^\infty a_m\psi_m,$$

where

$$L(\psi_m) = -\lambda_m\psi_m. \tag{6}$$

"Substituting . . . we obtain

$$\left.\begin{array}{l}2E = \displaystyle\sum_1^\infty \lambda_m a_m^2 \equiv \sum_1^\infty b_m = \text{constant} \\[2mm] 2F = \displaystyle\sum_1^\infty \lambda_m^2 a_m^2 \equiv \sum_1^\infty \lambda_m b_m = \text{constant}\end{array}\right\}. \tag{7}$$

"It then follows that

$$\sum_M^\infty b_m < \frac{1}{\lambda_M}\sum_M^\infty \lambda_m b_m < \frac{2F}{\lambda_M}, \tag{8}$$

i.e., that $\Sigma_M^\infty b_m$ approaches zero with increasing $M$, and an energy cascade is impossible. All the other theorems pertaining to energy exchange among spectral components in two-dimensional flow may now be shown to apply to three dimensional quasigeostrophic flow as well . . . ."

There are several problems with Charney's proof. We will discuss the first in this section. It turns out that there is only one eigenfunction satisfying (6) and the boundary conditions (4). That eigenfunction is $\psi_z \equiv 0$; that is, it is a strictly 2D flow. Consequently, Charney did not prove anything more than what had already been shown for 2D flows by Fjørtoft (1953).

We can demonstrate this by explicitly solving Eq. (6) via separation of variables. Write

$$\psi_m(x, y, z) = \phi_m(z)e^{i(k_x x + k_y y)}$$

and substitute into Eq. (6) to yield (with $k^2 \equiv k_x^2 + k_y^2$)

$$f_0^2\frac{d}{dz}\left(\frac{\overline{\rho}(z)}{N^2}\frac{d}{dz}\phi_m\right) + \overline{\rho}(z)(\lambda_m - k^2)\phi_m = 0. \tag{9}$$

For $N^2$ constant (assumed later by Charney) and $\overline{\rho}(z) = \overline{\rho}(0)e^{-z/H}$, Eq. (9) can be solved explicitly as

$$\phi_m(z) = [A_m e^{-\alpha_m z} + B_m e^{\alpha_m z}]e^{z/2H},$$

where

$$\alpha_m^2 \equiv \frac{N^2}{f_0^2}(k^2 - \lambda_m) + \frac{1}{4H^2}.$$

Applying the boundary conditions (4) yields

$$\phi_m(z) = A_m, \quad \text{a constant, and thus } \psi_z \equiv 0.$$

This barotropic solution is the only eigenfunction of Charney's problem.

Equation (9) is indeed self-adjoint, as Charney

claimed. It is of the Sturm–Liouville form, but it is not a *regular* Sturm–Liouville system because the domain is semi-infinite, and one of the coefficients vanishes at one of the end points (viz. $z = \infty$) (see, e.g., Birkhoff and Rota 1969). The upper boundary, $z = \infty$, is a singular point of Eq. (9). The reason that Charney's system has only one eigenvalue is because his boundary condition (4) at the singular point overspecifies the problem. There should have also been a continuous spectrum

$$k^2 + \frac{f_0^2}{4H^2N^2} < \lambda_m < \infty$$

corresponding to the eigenfunctions

$$\phi_m(z) = C_m e^{z/2H} \left[ e^{ik_z z} + \frac{\left( ik_z + \dfrac{1}{2H} \right)}{\left( ik_z - \dfrac{1}{2H} \right)} e^{-ik_z z} \right].$$

where

$$k_z^2 \equiv \frac{N^2}{f_0^2}(\lambda_m - k^2) - \frac{1}{4H^2}.$$

These eigenfunctions satisfy the less restrictive upper condition

$$\overline{\rho}^{1/2}(z)\phi_m(z) \quad \text{bounded as } z \to \infty.$$

This set was eliminated by Charney's boundary condition (4).

The mathematical situation is analogous to the Legendre functions, governed by the following singular Sturm–Liouville system:

$$\frac{d}{dx}\left[(1 - x^2)\frac{d}{dx}\phi\right] + \lambda\phi = 0. \qquad -1 < x < 1,$$

subject to the proper boundary condition

$$\phi(x) \text{ bounded as } x \to \pm 1.$$

The eigenvalues are $\lambda = m(m + 1)$, $m = 0, 1, 2, \ldots$ and the complete set of orthonormal eigenfunctions are $P_m(x)$, where

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \quad \text{etc.}$$

If, instead, the more restrictive boundary condition

$$\frac{d}{dx}\phi(x) = 0 \quad \text{at } x = \pm 1$$

is used, all but one of the eigenfunctions are eliminated. The only eigenfunction that remains is the "barotropic" solution

$$P_0(x) = 1.$$

Given this problem, is there any way to rectify Charney's solution? A complete set of eigenfunctions could in theory be constructed by adding the continuous eigenvalues and eigenfunctions to the barotropic mode. However, these radiating solutions do not satisfy energy conservation. Without energy conservation [Eq. (3)] Charney's proof also fails. There does not appear to be a way to remedy the problem for the semi-infinite domain.

In oceanic applications, the vertical domain is bounded and a rigid-lid upper boundary condition can be imposed. For the atmosphere, such a lid can possibly be justified for motion trapped in the troposphere, for which case the height of the lid, *D,* is taken to be the height of the tropopause. (It is not physically justified in the atmospheric case to take the limit $D \to \infty$, because the upper stratosphere and mesosphere are very dissipative due to the presence of breaking planetary and gravity waves.) With this rigid upper boundary, the vertical eigenfunctions are (Flierl 1978; Hua and Haidvogel 1986)

$$\phi_m(z) = \cos\left(\frac{m\pi}{D}z\right), \qquad m = 0, 1, 2, 3, \ldots.$$

These eigenfunctions form a complete set. Equation (8) then is correct, but we note that it is still *not* a proof of upscale energy cascade. Equation (8) is nothing more than a statement about an a priori condition for convergence of the infinite series representation of $\psi$ and hence of $E$ and $F$. It must hold for any time, including the initial time. It does not imply that energy *will tend* to flow from high to low wavenumbers. The same criticism applies to Fjørtoft's proof for 2D turbulence, which Charney's proof mirrored.

## 3. Discussion

There is a common misconception that the relationship between enstrophy, *G,* and energy, *E,* that is, $G(\mathbf{k}) = k^2 E(\mathbf{k})$ for 2D and a similar relationship for QG turbulence, decides the direction of energy cascade and that the direction is upscale in the net. This line of argument originated with Fjørtoft on 2D turbulence with his "triad interaction" proof (separate from his "convergence" proof discussed in section 2). The triad interaction proof from 2D turbulence can plausibly be carried over to QG turbulence if one replaces enstrophy by potential enstrophy, as Charney seems to have done. However, Charney stated: "Fjørtoft found that a transfer of energy from one wavenumber to a higher one must be accompanied by still more energy toward a lower wavenumber," but it is precisely this finding by Fjørtoft that was in error. The corrected statement was given by Merilees and Warn (1975) and it reads: "energy and enstrophy in a 2D non-divergent flow cascade both to lower and higher wavenumbers," but "the majority of interactions are such that more energy flows *to and* from

smaller wavenumbers while more enstrophy flows to *and from* larger wavenumbers'' (emphases added).

To use Fjørtoft's own (mistyped) example, if we have three different scales $l_p > l_r > l_s$ in the ratios $l_r/l_s = 2$; $l_p/l_r = 2$, then we can find the ratio of energy change in the longest and shortest scales to be $\Delta E_p/\Delta E_s = 4$. However, both $\Delta E$'s could be positive or both negative; in the latter case, the longest scale loses more energy downscale to the intermediate scale than the shortest scale loses upscale to the intermediate scale, yielding a net downscale energy flow. Thus such triad-based reasoning for either 2D or QG turbulence can *not* determine the sign (direction) of net energy transfer. The fact that Charney's (and Fjørtoft's) proofs cannot indicate the cascade direction is because no causality was introduced in the arguments, which rely only on the conservation of energy and enstrophy. This ''reversibility'' problem is moot for real fluids or when viscosity is introduced.[2] The physical problem Charney (1971) was interested in, that is, the equilibrium energy spectrum of the atmosphere, is a *forced* dissipative system. There the direction of energy transfer can depend on the spectral location of forcing and dissipation.

Charney also noted a similarity between Eqs. (1) and (2). Equation (2), for 2D motion, conserves vorticity, $\nabla_H^2 \psi$, whose Fourier spectral component is

$$-(k_x^2 + k_y^2)\hat{\psi}(k_x, k_y) \equiv -k^2\hat{\psi}(\mathbf{k}). \qquad (10)$$

Equation (1) for QG motion, conserves, along horizontal trajectories, the potential vorticity, $\nabla_H^2\psi + (f_0^2/\overline{\rho})[(\overline{\rho}/N^2)\psi_z]_z$, whose spectral component in 3D is[3]

$$-\left(k_x^2 + k_y^2 + k_z^2\frac{f_0^2}{N^2}\right)\hat{\psi}\left(k_x, k_y, k_z\frac{f_0}{N}\right) \equiv -\hat{k}^2\hat{\psi}(\hat{\mathbf{k}}). \quad (11)$$

Charney used this isomorphism to say that Kraichnan's argument for the $k^{-3}$ spectrum in 2D turbulence could be carried over to explain the same in QG. However, crucial to this is the assumption of isotropy; in the QG case it means that $k_x^2$, $k_y^2$, and $k_z^2 f_0^2/N^2$ must be comparable in magnitude. Charney had shown previously (Charney 1947; Charney and Drazin 1961) that synoptic waves forced by baroclinic instability are mostly trapped in the troposphere. Letting the density scale height $H$ be the maximum vertical scale for these waves (Held 1978), and letting $k_x = 2\pi/L_x$ and $k_y = 2\pi/L_y$, we need, with $N/f_0 \sim 100$,

$$L_x, L_y < H\frac{N}{f_0} \equiv L_R \sim 700 \text{ km}$$

for isotropy, where $L_R$ is the Rossby radius of deformation. Therefore, *isotropy* is realizable in a 3D at-mosphere only for horizontal scales of motion much less than 700 km. However, the $k^{-3}$ part of the observed spectrum occurs for horizontal scales longer than 1000 km (see Charney's Fig. 1; Nastrom and Gage 1985; Wiin-Nielsen 1967).

Incidentally, the scales of motion with zonal wavenumbers 7–13 are the ''energy injection'' scales and therefore do not satisfy the conditions for an ''inertial subrange'' of Kraichnan. See Welch and Tung (1998) for an alternative explanation of their slope.

## REFERENCES

Batchelor, G. K., 1953: *The Theory of Homogeneous Turbulence.* Cambridge University Press, 197 pp.

Birkhoff, G., and G.-C. Rota, 1969: *Ordinary Differential Equations.* Blaisdell Publishing Company, 357 pp.

Bretherton, F. P., 1966: Critical layer instability in baroclinic flows. *Quart. J. Roy. Meteor. Soc., 92,* 325–334.

Charney, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor., 4,* 135–162.

——, 1971: Geostrophic turbulence. *J. Atmos. Sci., 28,* 1087–1095.

——, and P. G. Drazin, 1961: Propagation of planetary scale disturbances from the lower into the upper stratosphere. *J. Geophys. Res., 66,* 83–109.

Eyink, G. L., 1996: Exact results on stationary turbulence in 2D: Consequences of vorticity conservation. *Physica D, 91,* 97–142.

Fjørtoft, R., 1953: On the changes in the spectral distribution of kinetic energy for two-dimensional non-divergent flow. *Tellus, 5,* 225–230.

Flierl, G. R., 1978: Models of vertical structure and the calibration of two-layer models. *Dyn. Atmos. Oceans, 2,* 341–381.

Held, I. M., 1978: The vertical scale of an unstable baroclinic wave and its importance for eddy heat flux parameterizations. *J. Atmos. Sci., 35,* 572–576.

Hua, B. L., and D. B. Haidvogel, 1986: Numerical simulations of the vertical structure of quasi-geostrophic turbulence. *J. Atmos. Sci., 43,* 2923–2936.

Kolmogorov, A. N., 1941a: Dissipation of energy in locally isotropic turbulence (English translation 1991). *Proc. Roy. Soc. London A, 434,* 15–17.

——,1941b: The local structure in incompressible viscous fluid for very large Reynolds number (English translation 1991). *Proc. Roy. Soc. London A, 434,* 9–13.

Kraichnan, R. H., 1967: Inertial ranges in two-dimensional turbulence. *Phys. Fluids, 10,* 1417–1423.

Merilees, P. E., and H. Warn, 1975: On energy and enstrophy exchanges in two-dimensional non-divergent flow. *J. Fluid Mech., 69,* 625–630.

Nastrom, G. D., and K. S. Gage, 1985: A climatology of atmospheric wavenumber spectra of wind and temperature observed by commercial aircraft. *J. Atmos. Sci., 42,* 950–960.

Rhines, P. B., 1975: Waves and turbulence on a $\beta$-plane. *J. Fluid Mech., 69,* 417–443.

Salmon, R., 1998: *Lectures on Geophysical Fluid Dynamics.* Oxford University Press, 378 pp.

Welch, W. T., and K. K. Tung, 1998: On the equilibrium spectrum of transient waves in the atmosphere. *J. Atmos. Sci., 55,* 2833–2851.

Wiin-Nielsen, A., 1967: On the annual variation and spectral distribution of atmospheric energy. *Tellus, 19,* 540–559.

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[2] For the case of unforced, purely inviscid flows, causality can be introduced by adding an ad hoc probabilistic hypothesis [see Rhines (1975), Batchelor (1953), Salmon (1998) for 2D turbulence].

[3] Charney (1971) ignored the $\beta$ effect and the vertical variation of density $\overline{\rho}(z)$.