A Theory of Stationary Long Waves. Part III: Quasi-Normal Modes in a Singular Waveguide

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ABSTRACT

The present paper deals with the fundamental issue of whether one can treat waves as normal modes when critical surfaces, where the phase speed of the wave matches the zonal wind speed, are present. In particular the question of whether a Rossby critical level (such as the zero-wind line for stationary waves) is absorbing or reflecting is raised and subsequently addressed. It is found that the critical level is never totally absorbing; Rossby waves are partially reflected *even if* the critical layer is dominated by dissipative processes. The relevance of nonlinearity in planetary-scale Rossby wave critical layers is also discussed and it is found to be the dominant mechanism. With the relative magnitudes of nonlinearity versus viscosity relevant to the earth's atmosphere it is found that the *steady-state* critical level should be almost perfectly reflecting to incident Rossby waves. Consequently, normal-mode solutions can be found; the quantization condition for these waves is also derived.

1. Introduction

This paper is concerned with a discussion of Rossby wave critical levels or lines (CL) in general and the behavior of stationary planetary waves near the zero-wind line[1] in particular. During winter, planetary-scale stationary waves are mostly generated in the high-latitude westerly wind region, and propagate equatorward toward the zero-wind line, which is usually situated in the equatorial region. While shorter scale waves may be prevented from reaching the zero-wind line by the so-called "polar waveguide" formed between the pole and the strong westerly winds of the jet streams, which tend to confine these waves to the high-latitude region, it can be shown that such ducting is more imperfect for long waves with zonal wavenumbers $s = 1, 2, 3$ and 4. Through reflections and refractions by the wind shears, the wave rays are bent ultimately to a path incident normally on the zero-wind line. Here the linear inviscid wave equation becomes singular. A domain for wave propagation bounded by such a singular surface is called a singular waveguide.

What happens near the singular or critical surface is not yet well understood. A common approach taken by numerical modelers interested in studying Rossby waves in the Northern Hemisphere is to introduce enough numerical friction so that "no wave activity" appears to the south of the CL, thus decoupling the Northern Hemisphere from the Southern. In support of such an approach is a brief analytical study in Dickinson (1968), which suggests total absorption of the waves by the mean flow at the singular surface. A later study of the initial value problem, also by Dickinson (1970), reaches the same conclusion. However, both of these studies use a wind profile which extends linearly without bound. We shall show that the conclusion of total wave absorption is the consequence of the use of unbounded winds or local analyses. When bounded winds are used in a global model the absorption becomes partial. The presence of nonlinearity near the critical surface can also significantly alter the final state as indicated by the works of Murakami (1974) and Beland (1976). With sufficient nonlinearity the critical surface can evolve into a perfect "reflector",[2] a state predicted by Benney and Bergeron (1969). However, no estimate of the magnitude of nonlinearity versus viscosity in the real atmosphere has been given in the literature. As a result there has existed considerable uncertainty concerning reflective versus the absorbing nature of CL. An unresolved question has been: Is the critical layer in the real atmosphere a viscosity dominated one, or is it nonlinearity dominated? A probable answer is provided by the study in this paper.

[1] A zero-wind line separates, in the meridional plane, a region of westerly flow ($\bar{u} > 0$) from a region of easterly flow ($\bar{u} < 0$); thus, it is also the location of the CL for the stationary waves.

[2] T. Warn, in a private communication, suggested caution in using the word "reflection" when referring to a CL which does not actually reflect, but by not absorbing, merely permits the southern boundary to do the reflecting.

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A comprehensive study of wave absorptions and reflections by the CL will be undertaken here. We shall first reexamine Dickinson's (1968) wave absorption problem under the more realistic condition of bounded winds. A range of possible environmental conditions will be used in the present study in order to determine the response of the CL under various conditions. Both barotropic and nonbarotropic waves will be considered. Also both the case of one and two CL's will be investigated. Finally, nonlinearities of various magnitudes will be introduced near the critical layer and possible responses studied. The time evolution of the mean zonal flow under wave-mean flow interactions will, however, not be considered here [but see Geisler and Dickinson (1974), Beland (1976), Warn and Warn (1978) and Stewartson (1978)].[3]

In Section 2, we show how the three-dimensional primitive equations on a sphere can be reduced, under appropriate assumptions, to the two-dimensional barotropic vorticity equation commonly used in critical layer calculations. Prior to this, the question about the relevance of the idealized equation for a real atmospheric situation has often been raised. On the other hand, in the outer region (the region away from the critical layer), because of the fact that the domain of interest for the planetary waves includes the equatorial region, the geostrophic approximation does not seem appropriate. Also there is no *a priori* assurance that the sphericity of the earth may not play a significant part in the calculations. Consequently, the primitive equations on the sphere are again used for the outer region, though the waves in the outer region are assumed to be linear and inviscid. The resulting outer equations are nonseparable in the presence of both vertical and meridional shears. In the present study, we have obtained separability by assuming that the zonal wind is a function of only latitude in each layer of the atmosphere. This model cannot treat the problem of a horizontally aligned CL, for which presumably a different set of approximation can be used. In Section 10, the applicability of the present model to the more general problem of two-dimensional shears is discussed; it seems that our results on wave reflections from the zero-wind line are equally applicable to the more general case, provided that the zero-wind line is oriented more or less vertically. In Section 3, the solution near the CL is given. Here use has been made of the existing result of Haberman (1972) on nonlinear viscous critical layers for nonrotating flows; the relevance to the present problem is discussed in the Appendix, where appropriate modifications for rotating flows are also given. In Section 4 an estimate of the relative magnitude of nonlinearity versus viscosity in the critical layer is given. It is shown that, under conditions appropriate to the earth's atmosphere, nonlinearity is the dominant mechanism in the critical layer. That nonlinearity cannot be justifiably neglected in any calculation involving the zero-wind line in the atmosphere is the important conclusion of this section. The reflectivity of the zero-wind line to incident planetary waves is calculated in Sections 5 and 6. In Section 5 a very simple model on an equatorial β-plane is used to illustrate the procedure involved, and Section 6 modifies the results to include the more general case of a nonbarotropic wave on a sphere. Sections 7 and 8 contain the results of numerical evaluations of the reflectivity under a variety of atmospheric conditions and show that, contrary to common belief, almost perfect reflections can be achieved when nonlinearity of the relevant magnitude is introduced in the critical layer. The fact that the CL turns out to be nonabsorbing paves the way for a discussion of quasi-normal modes in the singular waveguide, and Section 9 is devoted to this purpose. In particular, an approximate quantization condition for the waves in the singular waveguide is derived and it is suggested that this approximate condition be used (in the absence of adequate resolution for the critical layer) instead of either the artificial wall or the absorbing side boundary conditions currently in use.

2. The governing equations

The following viscous nonlinear primitive equations of motion on a sphere (Phillips, 1966) (including the energy equation) will be used (a list of symbols is given in Appendix A):

$$\frac{d}{dt}u - \left(2\Omega + \frac{u}{a\cos\varphi}\right)\sin\varphi\, v = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi + F_\lambda \quad (1)$$

$$\frac{d}{dt}v + \left(2\Omega + \frac{u}{a\cos\varphi}\right)\sin\varphi\, u = -\frac{1}{a}\frac{\partial}{\partial\varphi}\Phi + F_\varphi \quad (2)$$

$$C_p\frac{d}{dt}\ln\theta = 0, \quad \theta \equiv T\left(\frac{p_0(0)}{p}\right)^{R/C_p}. \quad (3)$$

[3] The validity of the assumption made here that the steady state in the outer (away from the critical layer) region can be reached within a time shorter than the meteorological time scales of interest is still open to debate. However, a recent numerical calculation by Beland (1978) seems to give some support to the assumption: In less than a week, the waves in the outer regions settled down to a quasi-steady state and experienced a phase shift close to that predicted by steady-state theory, even though the critical layer may still be highly nonsteady.

In log p coordinates

$$z^* \equiv \ln\left(\frac{p_0(0)}{p}\right), \tag{4}$$

the substantial derivative can be written as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \frac{v}{a}\frac{\partial}{\partial\varphi} + w^*\frac{\partial}{\partial z^*}, \tag{5}$$

where $w^* \equiv dz^*/dt$ is the vertical "velocity" in the $\log p$ coordinate system. Φ is the geopotential height, and is related to the temperature T through

$$\frac{\partial}{\partial z^*}\Phi = RT \tag{6}$$

for a hydrostatic atmosphere. The law of conservation of mass takes the form

$$\nabla\cdot\mathbf{u} + \left(\frac{\partial}{\partial z^*} - 1\right)w^* = 0, \tag{7}$$

while the horizontal divergence $\nabla\cdot\mathbf{u}$ is given by

$$\nabla\cdot\mathbf{u} = \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}u + \frac{\partial}{\partial\varphi}(v\cos\varphi)\right].$$

Eqs. (1), (2), (3), (6) and (7) constitute a complete set of equations if the viscous forces can be parameterized in terms of the dependent variables. The form of parameterization that is used in this study is

$$F_\lambda = \nu\nabla^2 u + \nu\frac{1}{a^2}\frac{\partial^2}{\partial z^{*2}}u,$$

$$F_\varphi = \nu\nabla^2 v + \nu\frac{1}{a^2}\frac{\partial^2}{\partial z^{*2}}v,$$

where

$$a^2\nabla^2 = \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right).$$

Here ν denotes the *horizontal* eddy diffusivity and ν' that in the vertical direction. It is assumed, because of the difference in vertical and horizontal scales in the earth's atmosphere, that

$$\delta^2 \equiv \nu'/\nu \ll 1.$$

The viscous nonlinear equation applicable in the critical layer will now be derived from the primitive equations. We let ζ denote the vertical component of the relative vorticity, i.e.,

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{u} = \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}v - \frac{\partial}{\partial\varphi}(u\cos\varphi)\right].$$

Eqs. (1) and (2) can be combined to yield

$$\frac{d}{dt}[\zeta + f] + (\zeta + f)\nabla\cdot\mathbf{u} + \frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\varphi}(w^*\cos\varphi)\frac{\partial}{\partial z^*}u - \frac{\partial}{\partial\lambda}w^*\frac{\partial}{\partial z^*}v\right] = \nu\left(\nabla^2 + \delta^2\frac{\partial^2}{\partial z^{*2}}\right)\zeta, \tag{8}$$

where

$$f = 2\left(\Omega + \frac{u}{a\cos\varphi}\right)\sin\varphi \approx 2\Omega\sin\varphi$$

is the Coriolis parameter. To reduce Eq. (8) further (in particular we wish to show $\nabla\cdot\mathbf{u} \to 0$) one needs to make some assumptions concerning the orientation of the CL and the flow outside the critical layer.

We assume that the flow away from the critical layer is linear and inviscid and attempt to match this "outer" solution to the solution of Eq. (8) inside the critical layer. We make the additional assumption that the CL is vertically oriented and that the vertical shear of the mean zonal flow $\bar{u}_{z^*}$ is negligible near the CL. Under these assumptions, the linearized outer equations derived from Eqs. (1) and (2) are

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a\cos\varphi}\frac{\partial}{\partial\lambda}\right)u' - \left[2\Omega + \frac{2\bar{u}}{a\cos\varphi} - \cot\varphi\frac{\partial}{\partial\varphi}\left(\frac{\bar{u}}{a\cos\varphi}\right)\right]\sin\varphi v' = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi', \tag{9}$$

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a\cos\varphi}\frac{\partial}{\partial\lambda}\right)v' + \left[2\Omega + \frac{2\bar{u}}{a\cos\varphi}\right]\sin\varphi u' = -\frac{1}{a}\frac{\partial}{\partial\varphi}\Phi', \tag{10}$$

where $\bar{A}$ is the zonal average of A, and A' is given by

$$A = \bar{A} + \epsilon A',$$

where

$$0 < \epsilon \ll 1$$

is a small-amplitude parameter. Assuming *steady* wave solutions of the form $e^{i(\sigma t + s\lambda)}$, where σ is the frequency and s the zonal wavenumber of the wave, Eqs. (9) and (10) can be combined to yield

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{u}' = \frac{i\hat{\sigma}}{4\Omega^2 a^2}F_\mu[\Phi'], \tag{11}$$

where

$$\hat{\sigma} = \sigma + s\frac{\bar{u}}{a\cos\varphi}$$

is the Doppler-shifted frequency, and the operator F_μ is given by

$$F_\mu \equiv \frac{\partial}{\partial\mu}\left[\frac{(1-\mu^2)}{\Delta}\frac{\partial}{\partial\mu}\right]$$

$$-\left[\frac{1}{\hat{\sigma}/s}\frac{\partial}{\partial\mu}\left(\frac{f}{\Delta}\right)+\frac{s^2}{\Delta(1-\mu^2)}\right], \quad (12)$$

with

$$\mu \equiv \sin\varphi,$$

$$\Delta \equiv \frac{1}{4\Omega^2}[\hat{\sigma}^2 - fq],$$

$$f \equiv 2\left(\Omega + \frac{\bar{u}}{a\cos\varphi}\right)\sin\varphi,$$

$$q \equiv 2\left(\Omega + \frac{\bar{u}}{a\cos\varphi}\right)\sin\varphi - \cos\varphi\frac{\partial}{\partial\varphi}\left(\frac{\bar{u}}{a\cos\varphi}\right).$$

Since the CL is defined as the location where the Doppler-shifted frequency is zero, Eq. (11) would imply that the flow becomes nondivergent as the CL is approached, i.e.,

$$\nabla\cdot\mathbf{u}' \to 0 \quad \text{as} \quad \hat{\sigma} \to 0$$

provided that one can show $F_\mu[\Phi']$ is bounded near the CL. To show this, one needs the linearized form of the continuity and energy equations:

$$\frac{1}{a\cos\varphi}\left[\frac{\partial}{\partial\lambda}u' + \frac{\partial}{\partial\varphi}(v'\cos\varphi)\right] + \left(\frac{\partial}{\partial z^*} - 1\right)w^* = 0, \quad (13)$$

$$\left(\frac{\partial}{\partial t} + \frac{\bar{u}}{a\cos\varphi}\frac{\partial}{\partial\lambda}\right)\Phi'_{z^*} + Sw^* = 0. \quad (14)$$

In Eq. (14),

$$S \equiv RT\frac{\partial}{\partial z^*}\ln\theta$$

is called the stability parameter. The divergence $\nabla\cdot\mathbf{u}'$ can thus alternatively be expressed as

$$\nabla\cdot\mathbf{u}' = -\left(\frac{\partial}{\partial z^*} - 1\right)w^* = \left(\frac{\partial}{\partial z^*} - 1\right)\left(\frac{i\hat{\sigma}}{S}\Phi'_{z^*}\right). \quad (15)$$

Eqs. (11) and (15) then give

$$\frac{1}{4\Omega^2a^2}F_\mu[\Phi'] - \left(\frac{\partial}{\partial z^*} - 1\right)\left(\frac{1}{S}\Phi'_{z^*}\right) = 0. \quad (16)$$

Eq. (16) is separable if it is assumed that S is a function of z^* only. On writing

$$\Phi'(\mu,z^*) = Z(z^*)\hat{\Phi}(\mu),$$

Eq. (16) can be separated into the following horizontal and vertical equations:

$$F_\mu[\hat{\Phi}(\mu)] + \epsilon\hat{\Phi}(\mu) = 0, \quad (17)$$

$$\left(\frac{d}{dz^*} - 1\right)[Z_{z^*}/S] + \frac{\epsilon}{4\Omega^2a^2}Z = 0, \quad (18)$$

where the Lamb's parameter

$$\epsilon \equiv \frac{4\Omega^2a^2}{gh}$$

has been used as a separation constant; h defined by the above equality is called the equivalent depth of the wave mode.[4] Eq. (17) can be recognized as in the same form as the Laplace tidal equation. It shall be called the modified Laplace tidal equation, as it includes the effect of the meridional shear of the mean zonal flow.

Eq. (17) has a logarithmic singularity at the CL: $\mu = \mu_c$, where $\hat{\sigma}(\mu_c) = 0$. Therefore the solution has a singular derivative but $\hat{\Phi}$ itself is bounded at $\mu = \mu_c$. Hence

$$F_\mu[\hat{\Phi}] = -\epsilon\hat{\Phi}(\mu) \quad \text{is bounded at} \quad \mu = \mu_c$$

and so from Eq. (11),

$$\nabla\cdot\mathbf{u} \to 0 \quad \text{as} \quad \hat{\sigma} \to 0.$$

We have thus shown that the waves become nondivergent as the CL is approached, even if they are divergent away from it.

The waves also become two-dimensional, as Eq. (14) shows, i.e.,

$$w^* \to 0 \quad \text{as} \quad \hat{\sigma} \to 0.$$

Note that the above results do not apply to waves that are fully nonlinear away from the CL. This fully nonlinear case has recently been treated numerically by Ward (1974). These waves would presumably remain three-dimensional and nonbarotropic near the CL if they are so away from it. Also we have not ruled out the possibility that the waves again can become three-dimensional and divergent well within the critical layer, where the linear inviscid expression for the $\nabla\cdot\mathbf{u}$ used above long ceases to be valid. Such a three-dimensional nonlinear inner solution is not sought for here, as the simpler two-dimensional solution will satisfy the inner equation and the matching conditions. It should be kept in mind, however, that by not considering the three-dimensional case, one eliminates the mechanism by which the nonlinear flow inside the critical layer can become unstable to secondary inertial instabilities, which are three-dimensional in nature.

Using the results

$$\nabla\cdot\mathbf{u} \to 0 \quad \text{and} \quad w^* \to 0$$

[4] Eq. (17) implies $(i\hat{\sigma}/gh)\Phi' + \mathrm{div}\,\mathbf{u}' = 0$, which is just the continuity equation for an incompressible ocean of depth h. All the meridional structure equations can be alternatively derived using the two momentum equations and this equivalent continuity equation alone.

and the fact that the μ-variations are more important than the z^*-variations near the CL, the equation applicable in the critical layer can be derived from Eq. (8); the result is the nonlinear barotropic vorticity equation

$$\left[\frac{\partial}{\partial t} - \left(\frac{\partial}{a\partial\varphi}\Psi\right)\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda} + \left(\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Psi\right)\frac{\partial}{a\partial\varphi}\right](\nabla^2\Psi + f) - \frac{\partial}{a\partial\varphi}w^* \times \frac{\partial}{a\partial\varphi}\Psi = \nu\left(\nabla^2 + \delta^2\frac{\partial^2}{a^2\partial z^{*2}}\right)\nabla^2\Psi, \quad (19)$$

where Ψ is the streamfunction defined by

$$v\cos\varphi = \frac{\partial}{a\partial\lambda}\Psi, \quad u = -\frac{\partial}{a\partial\varphi}\Psi. \quad (20)$$

Furthermore, since Eq. (19) is to be used in a limited region only, local coordinates x and y can be used. With

$$dx = a\cos\varphi d\lambda \quad \text{and} \quad dy = ad\varphi,$$

Eq. (19) becomes

$$\left(\frac{\partial}{\partial t} - \Psi_y\frac{\partial}{\partial x} + \Psi_x\frac{\partial}{\partial y}\right) \times (\nabla^2\Psi + f) - w_y^*\Psi_{yz^*} = \nu\nabla_3{}^2\nabla^2\Psi, \quad (21)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla_3{}^2 = \nabla^2 + \delta^2\frac{\partial^2}{a^2\partial z^{*2}}.$$

These local approximations cannot be used on the outer equations since the outer solutions should have a greater region of validity; therefore the effects of spherical geometry cannot be neglected, especially for the long planetary waves under consideration. Except for the additional "turning and twisting" term $w_y^*\Psi_{yz^*}$ and the extra f term, Eq. (21) is almost the same as the two-dimensional barotropic vorticity equation considered by Benney and Bergeron (1969) and Haberman (1972). It reduces (in the absence of f) in its linear limit to the Orr-Sommerfeld equation in classical hydrodynamic stability theory (see Lin, 1955). It will be shown in Appendix B, where the asymptotic solution of the inner and outer equation is treated, that the "turning and twisting" term has no effect on the matched asymptotics to the orders considered. On the other hand, the Coriolis term does enter the matched asymptotics, but it turns out that the presence of the Coriolis term in the present equation does not alter the results significantly; it merely changes the curvature term $\bar{u}_{yy}$ to $(\bar{u}_{yy} - \beta)$, where β is df/dy evaluated at the CL. This will be shown in Appendix B.

For the outer region (i.e., away from CL) we shall use the separable linear inviscid equation [e.g., Eq. (17)] in the present study. The u' and v' are related to $\hat{\Phi}$ through

$$\hat{u} = \frac{1}{4\Omega^2a^2\Delta}\left[q(1-\mu^2)\frac{\partial}{\partial\mu}\hat{\Phi} - s\hat{\sigma}\hat{\Phi}\right], \quad (22)$$

$$\hat{v} = \frac{i}{4\Omega^2a^2\Delta}\left[\hat{\sigma}(1-\mu^2)\frac{\partial}{\partial\mu}\hat{\Phi} - sf\hat{\Phi}\right], \quad (23)$$

where

$$\hat{u} = u'\cos\varphi/a, \quad \hat{v} = v'\cos\varphi/a.$$

Equations involving $\hat{u}$ or $\hat{v}$ alone can also be derived. We find that in most of our analysis it is more convenient to use $\hat{v}$ as the dependent variable in the equation

$$\frac{d}{d\mu}\left[\frac{(1-\mu^2)}{\gamma}\frac{d}{d\mu}\hat{v}\right] + \left\{\frac{1}{\hat{\sigma}/s}\frac{d}{d\mu}\left(\frac{q}{\gamma}\right) + \frac{\epsilon\Delta}{\gamma} - \frac{s^2}{(1-\mu^2)\gamma}\right\}\hat{v} = 0, \quad (24)$$

where

$$\gamma \equiv 1 - (1-\mu^2)\frac{a^2(\hat{\sigma}/s)^2}{gh}.$$

The other variables are related to $\hat{v}$ through

$$\hat{\Phi} = \frac{a^2}{is\gamma}\left[\frac{\hat{\sigma}}{s}(1-\mu^2)\frac{d}{d\mu}\hat{v} + q\hat{v}\right]$$

and

$$\hat{u} = \frac{i(1-\mu^2)}{s\gamma}\left[\frac{d}{d\mu}\hat{v} + \frac{a^2(\hat{\sigma}/s)^2}{gh}\frac{q}{\hat{\sigma}/s}\hat{v}\right].$$

3. Outer solutions: Series expansion near CL

In this section and the following section, the linear inviscid equations in a meridionally sheared zonal flow possessing a vertical oriented CL are to be solved and matched to the nonlinear viscous solution in the critical layer. Eq. (24) is used as the governing equation, subject to the boundary conditions that the solutions are finite at the poles, which imply

$$\hat{v} = 0 \quad \text{at} \quad \mu = \pm 1. \quad (25)$$

We are interested in the case where the zero(es) of the Doppler-shifted frequency (or $\bar{u} - c_{ph}$, where c_{ph} is the phase speed of the wave) exists inside the domain $-1 < \mu < 1$. Eq. (24) then possesses logarithmic singularities at these points. For stationary waves (i.e., $\sigma = 0$) the singularity occurs at the zero-wind line, while for slowly moving waves, the singularity will be shifted somewhat. We let the singularity be located at $\mu = \mu_c$ and define

$$\chi = \mu - \mu_c.$$

We now let $f(\chi)$ and $g(\chi)$ be the two linearly independent fundamental solutions to Eq. (24). They have the following expansions near the singularity $\chi = 0$:

$$f(\chi) = \chi + \sum_{n=2}^{\infty} a_n\chi^n, \tag{26}$$

$$g(\chi) = 1 + \sum_{n=2}^{\infty} b_n\chi^n - \hat{\beta}_c \ln|\chi| f(\chi). \tag{27}$$

In (27), $\hat{\beta}_c$ is the dimensionless mean vorticity gradient at CL, and is given by

$$\hat{\beta}_c = \frac{1}{(1-\mu_c^2)} \left.\frac{\dfrac{d}{d\mu} q}{\dfrac{d}{d\mu}\bar{\omega}}\right|_{\mu_c}, \tag{28}$$

where

$$\bar{\omega} \equiv \frac{\bar{u}}{a\cos\varphi}$$

is the angular velocity of the mean zonal flow and

$$\frac{d}{d\mu} q = 2(\Omega + \bar{\omega}) - \frac{d}{d\mu}\left[(1-\mu^2)\frac{d}{d\mu}\bar{\omega}\right]$$

is the mean vorticity gradient; its counterpart on a β-plane is $\beta - \bar{u}_{yy}$. It is assumed here that

$$\bar{\omega}_c' \equiv \left.\frac{d}{d\mu}\bar{\omega}\right|_{\mu_c}$$

is nonzero. Without loss of generality $\bar{\omega}_c'$ is taken to be positive. The recursion relations for a_n and b_n can be obtained by substituting (26) and (27) into Eq. (24). Here we just note that a_n and b_n can be taken to be real, so that both $f(\chi)$ and $g(\chi)$ are real functions. A similar set of fundamental solutions has been used by Benney and Bergeron (1969) for the simple shear flow problem, and Tollmien (1935) gave the first several terms in the expansions, also for the simple shear flow problem.

The general solution to Eq. (24) can be expressed as a linear combination of the regular and irregular functions $f(\chi)$ and $g(\chi)$:

$$\hat{v} = \begin{cases} \hat{v}_+(\chi) = Af(\chi) + Bg(\chi), & \chi > 0 \quad (29) \\ \hat{v}_-(\chi) = A'f(\chi) + B'g(\chi), & \chi < 0. \quad (30) \end{cases}$$

In Eqs. (29) and (30), the solutions are allowed to be different on different sides of the critical layer. The relation between the two sets of constants (A,B) and (A',B') is to be found by matching (29) and (30) to the solution of the fourth-order, viscous, nonlinear equation (21).

For the case where nonlinearity is absent, the asymptotic results of Lin (1957), Wasow (1950) and many others suggest that the relations

$$\left.\begin{aligned} A' &= A - i(-\pi)\hat{\beta}_c B \\ B' &= B \end{aligned}\right\} \tag{31}$$

are appropriate. (Note that correction has been made for the effect of the Coriolis force here.) Eq. (31) implies that the solution remains continuous, i.e., $\hat{v}_+(0) = \hat{v}_-(0)$, but there is a discontinuity in the phase of the solutions when the singularity is crossed. The relations in (31) have traditionally been taken to mean that the solutions on different sides of the singularity can be connected by analytic continuation in an appropriately cut complex plane. That is, the solution to Eq. (24) can be rewritten in the complex plane as

$$\hat{v}(\chi) = Af(\chi) + B[1 + \sum_{n=2}^{\infty} b_n\chi^n - \hat{\beta}_c \ln\chi f(\chi)].$$

[Note that the absolute sign in $g(\chi)$ has been removed.] This is then taken to be valid on either side of the singularity. The $-\pi$ phase shift can be accounted for by the fact that $\ln(-\chi) = \ln\chi - i\pi$ in a cut plane where the $+\pi$ phase shift is excluded.

This result has been used so extensively in the literature that one tends to feel justified in demanding the analytic continuity of solutions whenever a singularity is encountered, forgetting that the phase shift has a real physical origin. That the solution cannot always be analytically continued is amply demonstrated by the case considered by Benney and Bergeron (1969). They showed that when nonlinearity is the dominant mechanism in the critical layer, the phase shift vanishes, giving $A' = A$ and $B' = B$.

The general problem of determining the relation between (A,B) and (A',B') when both viscous and nonlinear mechanisms are present is more difficult. In Appendix B, we have extended Haberman's treatment of this problem to include the Coriolis effect. The result indicates that the constants are related through

$$\left.\begin{aligned} A' &= A - i\phi\hat{\beta}_c B \\ B' &= B \end{aligned}\right\}. \tag{32}$$

Here ϕ can be interpreted as a phase shift across the critical layer. It is shown by Haberman to be a function of the viscosity/nonlinearity ratio λ_c only, i.e., $\phi = \phi(\lambda_c)$, where ϕ varies continuously and monotonically from the value zero at $\lambda_c \to 0$ (nonlinear limit) to the value $-\pi$ at $\lambda_c \to \infty$ (the viscous limit). Fig. 9, taken from Haberman (1972), depicts the dependence of ϕ on λ_c. An important question remains: What is the value of λ_c relevant to the earth's atmosphere? In the next section, λ_c will be redefined and its value estimated for the atmospheric case.

4. Combined effect of nonlinearity and viscosity at the critical layer

In Appendix B, the parameter λ_c, the ratio of viscosity to nonlinearity, is found to be

$$\lambda_c = \frac{1/\mathrm{Re}}{(V/U)^{3/2}}, \quad (33)$$

where V is the typical meridional velocity of the wave outside the critical layer and Re, the Reynold's number, is defined as

$$\mathrm{Re} = UL/\nu,$$

with $U \equiv |\bar{u}_c'|a/s$ being the velocity scale and $L \equiv a/s$ the length scale. The difficulty in giving an estimate for Re (and hence λ_c) lies in the fact ν, the coefficient of viscosity (or eddy diffusivity), is not a well-determined quantity from available observations. Values of ν that have been used in the literature range from 10^2 to 10^9 cm^2 s^{-1}, an exceedingly wide range. Nonetheless, it is known from numerical experiments that the upper limit for ν seems to be not much larger than 10^9 cm^2 s^{-1}, otherwise the simulated result would bear no resemblance to the observed atmosphere. We will show that for values of ν as high as 10^9 cm^2 s^{-1}, the nonlinearity is still at least an order of magnitude larger than the viscosity *in the critical layer*. The case of the longest waves ($s = 1$) is considered first. We take a typical value for the meridional shear $\bar{u}_c'$ to be 1 m s^{-1} per degree of latitude; then $U \approx 60$ m s^{-1} and $L = 6400$ km. Thus Re ≈ 4000 when the value 10^9 cm^2 s^{-1} is used for ν. To estimate the magnitude of the nonlinearity we take a typical value for the meridional velocity to be $V = 5$ m s^{-1}, so that $(V/U)^{3/2} \approx 2 \times 10^{-2}$. Comparing this with 1/Re, we see that $\lambda_c \approx 10^{-2}$, suggesting that nonlinearity should be the dominant mechanism in the critical layer for the long waves. The shorter waves are slightly more susceptible to the influence of viscosity. For $s > 1$, we rewrite (33) as

$$\lambda_c = s^{1/2}\left[\frac{\nu}{(|\bar{u}_c'|a)a} \Big/ \left(\frac{V}{|\bar{u}_c'|a}\right)^{3/2}\right]. \quad (34)$$

The term in the brackets in (34) is just the λ_c for $s = 1$, so that

$$\lambda_c(s) = s^{1/2}\lambda_c(s = 1). \quad (35)$$

Thus

$$\lambda_c \approx s^{1/2} \times 10^{-2}.$$

It is seen that for those long waves that are able to reach the zero-wind line (*viz.*, $s = 1, 2, 3$ and 4), nonlinearity is at least an order of magnitude greater than viscosity.

For small values of λ_c, Haberman (1972) derived an asymptotic formula for the phase shift ϕ:

$$\phi \approx -4.2 \times \lambda_c \quad \text{for} \quad \lambda_c \ll 1.$$

It is thus seen that $-\phi$ is quite small for long waves under typical atmospheric conditions, implying that Benney and Bergeron (1969)'s nonlinear limit is closer to the conditions of the real atmosphere than the viscous limit of $\phi = -\pi$ taken over from classical hydrodynamics.

5. The reflectivity of the critical layer

In this and the following section, the reflection coefficient of the CL will be calculated for a full range of values of the phase shift ϕ. One would have expected to recover the absorption result of Dickinson (1968) when ϕ is set to $-\pi$, but it does not turn out to be so. Even in this linear limit, the reflectivity can take on a range of values depending on the mean wind profile. To examine this apparent discrepancy, the simple case of a barotropic fluid on a β-plane is treated first. This is the same case considered by Dickinson. The more general cases (which include spherical geometry, nonbarotropicity and more general wind profiles) will be considered in Section 6 as modifications to this case.

The governing equation is the barotropic vorticity equation on an equatorial β-plane

$$\frac{d^2}{dy^2}v' + \left[\frac{\beta - \bar{u}_{yy}}{\bar{u} - c_{ph}} - s^2\right]v' = 0, \quad (36)$$

where y is the northward tangential coordinate made dimensionless by a and $\beta \equiv 2\Omega a$. The following simple profile for $\bar{u}(y)$ is used:

$$\bar{u}(y) - c_{ph} = \begin{cases} \bar{u}_c'\cdot(y - y_c), & y_2 \le y \le y_1 \\ \bar{u}(y_1) - c_{ph} \equiv U_1, & y \ge y_1 \\ \bar{u}(y_2) - c_{ph} \equiv U_2, & y \le y_2. \end{cases} \quad (37)$$

It is assumed that the CL is located between y_1 and y_2, i.e., $y_2 < y_c < y_1$. In the constant westerly wind region $y \ge y_1$, the solution is simply

$$v' = v_1(y) = A_1 e^{i\lambda_1(y-y_1)} + B_1 e^{-i\lambda_1(y-y_1)}, \quad (38)$$

where

$$\lambda_1 \equiv \left\{\frac{\beta}{\bar{u}(y_1) - c_{ph}} - s^2\right\}^{1/2},$$

and $A_1 e^{i\lambda_1(y-y_1)}$ represents a wave propagating to the north, and $B_1 e^{-i\lambda_1(y-y_1)}$ a southward propagating wave. Treating the southward propagating wave as an incident wave, the reflectivity of the CL is then measured by

$$R \equiv \left|\frac{\text{amplitude of the reflected wave}}{\text{amplitude of the incident wave}}\right| = \left|\frac{A_1}{B_1}\right|. \quad (39)$$

To calculate R, one needs to know the behavior of the solution near the CL and the boundary condition applied at the southern end of the domain.

For waves on an infinite β-plane, the southern boundary condition is that the solution is bounded, implying the following solution for $y \le y_2$:

$$v = v_2(y) = A_2 e^{-\kappa_2(y_2-y)}, \tag{40}$$
$$\kappa_2 \equiv \left\{\frac{\beta}{c_{ph} - \bar{u}(y_2)} + s^2\right\}^{1/2}.$$

In the shear zone, $y_2 \le y \le y_1$, the solution can be expressed in terms of the f and g functions defined in (26) and (27), i.e.,

$$v = \begin{cases} v_+(y) = Af(y-y_c) + Bg(y-y_c), y > y_c, & (41)\\ v_-(y) = A'f(y-y_c) + B'g(y-y_c), y < y_c, & (42)\end{cases}$$

where

$$A' = A - i\phi\hat{\beta}_c B,$$
$$B' = B,$$
$$\hat{\beta}_c = \left.\frac{\beta - \bar{u}_{yy}}{\bar{u}_y}\right|_{y_c},$$
$$f(\chi) = \chi + \sum_{n=2}^{\infty} a_n \chi^n,$$
$$g(\chi) = 1 + \sum_{n=2}^{\infty} b_n \chi^n - \hat{\beta}_c \ln|\chi| f(\chi).$$

For a linear profile,[5] the recursion relations for a_n and b_n have only three terms:

$$n(n+1)a_{n+1} + \hat{\beta}_c a_n - s^2 a_{n-1} = 0 \quad \text{for} \quad n \ge 2 \tag{43}$$

with

$$a_1 = 1 \quad \text{and} \quad a_2 = -\tfrac{1}{2}\hat{\beta}_c$$

and

$$n(n+1)b_{n+1} + \hat{\beta}_c \cdot b_n - s^2 b_{n-1} = \hat{\beta}_c(2n+1)a_{n+1}, \tag{44}$$

where

$$b_0 = 1, \quad b_1 = 0 \quad \text{and} \quad b_2 = \tfrac{1}{2}(s^2 - \tfrac{3}{2}\hat{\beta}_c^2).$$

The matching of the solutions in various regions is accomplished by requiring that

$$v \quad \text{and} \quad \frac{d}{dy}v - \frac{\bar{u}_y}{\bar{u} - c_{ph}} v' \tag{45}$$

be continuous across y_1 and y_2, yielding

$$\left|\frac{A_1}{B_1}\right|^2 = R^2 = 1 + \frac{4\lambda_1\phi\hat{\beta}_c}{r^2}, \tag{46}$$

where

$$r^2 \equiv \left[\frac{1}{\sigma'}p_1 - q_1 - \phi\lambda_1\hat{\beta}_c f_1\right]^2 + \left[\lambda_1\left(\frac{1}{\sigma'}f_1 - g_1\right) + \phi\hat{\beta}_c p_1\right]^2$$

with subscript 1 referring to evaluation of the function at $y = y_1$, e.g., $f_1 \equiv f(y_1 - y_c)$. We have also defined

$$\left.\begin{aligned} p(\chi) &\equiv \frac{d}{d\chi}f(\chi) - \frac{\bar{u}_y}{\bar{u} - c_{ph}}f(\chi) \\ q(\chi) &\equiv \frac{d}{d\chi}g(\chi) - \frac{\bar{u}_y}{\bar{u} - c_{ph}}g(\chi) \end{aligned}\right\} \tag{47}$$

and

$$\sigma' \equiv -B'/A'$$
$$= (p_2 - \kappa_2 f_2)/(q_2 - \kappa_2 g_2). \tag{48}$$

Note that the expression for R [Eq. (46)] is actually more general than it seems from the way it is derived. The form of (46) is unchanged for a profile more general than the linear profile assumed. For any monatomic $\bar{u}(y)$ in the shear zone $y_2 \le y \le y$, all the steps leading to Eq. (46) are not altered, except the recursion relations for a_n and b_n [Eqs. (43) and (44)] which become of orders in general higher than 3. In a similar manner, the effects of sphericity and nonbarotropicity can also be introduced. This will be the subject of next section, where it will be shown that the form of Eq. (46) is not altered even in the presence of these effects. In particular, the following conclusions seem to be of general applicability:

1) For $\hat{\beta}_c > 0$ (a barotropically stable mean state), R^2 is always less than 1 as long as there is some viscosity in the system so that $\phi < 0$. The wave is (partially) absorbed by the mean flow near the CL.

2) As nonlinearity becomes dominant in the critical layer, $\phi \to 0$, and so Eq. (46) implies $R^2 \to 1$. In other words, a nonlinearity dominated CL is a perfectly reflecting surface.

3) Even for a viscosity dominated critical layer, for which $\phi = -\pi$ a value used by Dickinson (1968), R is *not* zero in general. This can be seen more clearly in the following expression for R^2:

$$R^2 = \{[\sigma'^{-1}, p_1 - q_1 + \phi\lambda_1\hat{\beta}_c f_1]^2 + [\lambda_1(\sigma'^{-1}, f_1 - g_1) - \phi\hat{\beta}_c p_1]^2\}/r^2,$$

which is zero only if the sum of terms in each pair of brackets is separately zero. This in general does not happen. Incidentally, the fact that $R > 0$ is *not* due to the discontinuity in shear at $y = y_1$ in the present simple model. If the flow is continuously dif-

[5] Alternatively, Whitaker functions can be used as fundamental solutions instead of f and g.

ferentiable, one simply replaces p by df/dy and q by dg/dy (see next section for a more general derivation of R^2). The conclusion of $R^2 > 0$ is unaffected.

4) If the mean potential vorticity gradient is negative at the CL, i.e., $\hat{\beta}_c < 0$, the reflectivity R is *greater* than 1, as long as there is some viscosity in the critical layer. The wave is said to be *overreflected* from the CL. It turns out that wave overreflection is intimately related to instability.[6] A separate paper (Lindzen and Tung, 1978) discusses the relation of overreflection to instability. In this paper we will restrict our discussions to the stable case, i.e. $\hat{\beta}_c > 0$.

6. Modifications for the spherical and nonbarotropic cases

For nonbarotropic waves on the sphere, the governing equation [Eq. (24)] can be written in the Mercator coordinate y defined by $\tanh y = \mu$ as

$$\frac{d}{dy}\left[\frac{1}{\gamma}\frac{d}{dy}\hat{v}\right] + \left\{\frac{1}{\bar{\omega} - c}\frac{d}{dy}\left(\frac{q}{\gamma}\right) + \frac{\epsilon\Delta\,\mathrm{sech}^2 y}{\gamma} - \frac{s^2}{\gamma}\right\}\hat{v} = 0 \quad (49)$$

subject to the boundary conditions

$$\hat{v} \to e^{-s|y|} \quad \text{as} \quad y \to \pm\infty. \quad (50)$$

The equation reduces to a rather simple form for the case of a barotropic fluid for which $\epsilon = 0$, *viz.*,

$$\frac{d^2}{dy^2}\hat{v} + \left[\frac{\frac{d}{dy}q}{\bar{\omega} - c} - s^2\right]\hat{v} = 0. \quad (51)$$

The form of Eq. (51) is almost identical to the equation on an equatorial β-plane [cf. Eq. (36)], except now q_y, the mean vorticity gradient, is more complicated in form:

$$\frac{d}{dy}q = 2(\Omega + \bar{\omega})\,\mathrm{sech}^2 y - \bar{\omega}_{yy}. \quad (52)$$

Near the CL (i.e., the location where $\bar{\omega} - c = 0$), the solution to Eq. (49) can again be written in a series form [see Eqs. (26), (27), (28), (29), (30) and (32)]. The recursion relations for a_n and b_n are in general more complicated. For our purpose, the series solutions (29) and (30) suffer two drawbacks. First, it is difficult to define a reflectivity R, as the incident and reflected waves cannot be easily separated. Second, the radius of convergence for the series is usually rather limited. Matching to solutions with different domains of validity is necessary. For convenience we pick the patching points to be at the westerly and easterly wind maxima y_1 and y_2, respectively. For $y < y_2$, the solution to Eq. (49) that satisfies the south polar boundary condition is

$$\hat{v} = \hat{v}_2(y) = \gamma^{1/2}(y)A_2 \exp\left[-\int_y^{y_2} K(y)dy\right], \quad (53)$$

where $K(y)$ satisfies

$$K^2 + \frac{d}{dy}K = -\Lambda^2(y), \quad \mathrm{Re}K > 0, \quad (54)$$

where

$$\Lambda^2(y) \equiv \frac{\gamma}{\bar{\omega} - c}\frac{d}{dy}\left(\frac{q}{\gamma}\right) - s^2 + \epsilon\Delta\,\mathrm{sech}^2 y + \left[\frac{1}{\gamma^{1/2}}\frac{d^2}{dy^2}(\gamma^{1/2}) - \frac{(\gamma')^2}{2\gamma^2}\right]. \quad (55)$$

The zonal flow may or may not be discontinuously differentiable. Allowing for a discontinuity in $d/dy\,\bar{\omega}$ at $y = y_2$, the matching conditions

$$\hat{v} \quad \text{and} \quad \frac{d}{dy}\hat{v} - \frac{\bar{\omega}_y}{\bar{\omega} - c}\hat{v} \quad \text{continuous} \quad (56)$$

give

$$[\gamma^{1/2}(y_2)A_2] = A'f(y_2 - y_c) + B'g(y_2 - y_c), \quad (57)$$

$$K(y_2)[\gamma^{1/2}(y_2)A_2] = A'p(y_2 - y_c) + B'q(y_2 - y_c). \quad (58)$$

Here

$$p(\chi) \equiv \frac{d}{d\chi}f(\chi) - \frac{[\bar{\omega}_\chi]}{\bar{\omega}(\chi) - c}f(\chi), \quad (59)$$

$$q(\chi) \equiv \frac{d}{d\chi}g(\chi) - \frac{[\bar{\omega}_\chi]}{\bar{\omega}(\chi) - c}g(\chi), \quad (60)$$

and $[\bar{\omega}_\chi]$ is the jump in $\bar{\omega}_\chi$ as one crosses the discontinuity from the side nearer the CL to the other side of the discontinuity.

On the westerly side of the CL, we want to write the solution in terms of an incident and reflected wave. Since no northern boundary condition will be applied (as we are interested only in calculating R, and not in solving an eigenvalue problem), a local solution will suffice. Assuming the region near y_1^+ is smooth, the solution can be written as

$$\hat{v} = \hat{v}_1(y) = \gamma^{1/2}(y)\left\{A_1 \exp\left[i\int_{y_1}^{y} \Lambda(y_1)dy\right] + B_1 \exp\left[-i\int_{y_1}^{y} \Lambda(y_1)dy\right]\right\} \quad (61)$$

[6] Incidentally, it can be shown that barotropic instability disappears when $\phi \to 0$. It is an implicit assumption in the classical theories of hydrodynamic instability that viscosity is present at least in the critical layer.

for y sufficiently close to y_1^+.

Matching at $y = y_1$ yields

$$[\gamma^{1/2}(y_1)B_1][(A_1/B_1) + 1] = Af(y_1 - y_c) + Bg(y_1 - y_c) \quad (62)$$

and

$$i\Lambda(y_1)[\gamma^{1/2}(y_1)B_1][(A_1/B_1) - 1] = Ap(y_1 - y_c) + Bq(y_1 - y_c). \quad (63)$$

Note that in Eq. (63), as well as in (58), the fact that $d\bar{u}/dy$ vanishes at y_1, the westerly jet maximum, and y_2, the easterly jet maximum, has been helpful in simplifying the expressions because γ' vanishes. $\Lambda(y_1)$ is also simpler, i.e.,

$$\Lambda^2(y_1) = \frac{1}{\bar{\omega}_1 - c}\frac{d}{dy}q(y_1) - s^2 + \epsilon\Delta\,\text{sech}^2 y_1 + \frac{\gamma''}{2\gamma}.$$

Defining the reflectivity to be

$$R \equiv \left|\frac{A_1}{B_1}\right|,$$

one obtains from the matching conditions the desired formula

$$R^2 - 1 = \frac{4\phi\hat{\beta}_c\Lambda_1}{\left[\frac{1}{\sigma'}p_1 - q_1 - \phi\Lambda_1\hat{\beta}_c f_1\right]^2 + \left[\Lambda_1\left(\frac{1}{\sigma'}f_1 - g_1\right) + \phi\hat{\beta}_c p_1\right]^2}, \quad (64)$$

where

$$\sigma' \equiv -B'/A' = (p_2 - K_2 f_2)/(q_2 - K_2 g_2). \quad (65)$$

Here $K_2 \equiv K(y_2^-)$. If the region $y < y_2^-$, is sufficiently smooth, so that

$$\left|\frac{d}{dy}K/K^2\right| \ll 1,$$

then K can be replaced by its WKB solution of Eq. (55), i.e.,

$$K_2 \approx [-\Lambda^2(y_2)]^{1/2}. \quad (66)$$

Eq. (65) has the same form as Eq. (46) for the simple case on a β-plane with λ and κ replaced by Λ and K.

7. Numerical evaluation of R, the linear case

In this section, the reflectivity R will be evaluated using the formulas given by Eqs. (46) and (64). Aside from the factor $(1 - \mu^2) = \text{sech}^2 y$ multiplying various terms, the governing equation for the spherical case is, if written in Mercator coordinates, not much different from the corresponding equation for the case of an equatorial β-plane geometry. Here in evaluating the series for $f(\chi)$ and $g(\chi)$ using the recursion relations, we shall approximate $\text{sech}^2 y$ by 1, so that the results obtained for the β-plane case can be carried over to the spherical case. This approximation seems to be valid because it has been shown in the previous section that the reflectivity is a local function dependent on quantities only in the range $y_2 \leq y \leq y_1$. In the calculations we will present, y_1 and y_2 are taken to be not too far from the equator, so in a test calculation where $\text{sech}^2 y$ is not approximated, the results are not noticeably different from the approximated case.

In examining the numerical results, it is helpful to remember that the following identifications should be made for the spherical case:

$$\left.\begin{aligned} y &\equiv \tanh^{-1}\mu \\ U_1 &\equiv a\bar{\omega}(y_1), \quad U_2 \equiv a\bar{\omega}(y_2) \\ \beta\text{-effective} &\equiv a^{-1}(2\Omega - \bar{\omega}_{yy})_{y_c} \end{aligned}\right\}$$

and a "linear profile" means $\bar{\omega}(y) = \bar{\omega}_c'\cdot(y - y_c)$.

The reflectivity R as given by Eq. (46) is first evaluated for a profile that is linear between y_1 and y_2. It is the simplest possible continuous profile that contains a CL; yet it is found to possess many of the features of the other more complicated profiles, insofar as the reflectivity is concerned. This is due to the fact that R, as given by Eq. (46), does not depend on any quantity beyond the range $y_2 \leq y \leq y_1$, and in this range, because of the presence of the CL, the most significant contribution is from the linear part of the profile. Dickinson (1968) also treated the linear wind case as the limiting form near the singularity. However, his main conclusion that the waves are totally absorbed is a result of the implicit assumption that the wind magnitudes increase linearly without bound. This rather unrealistic assumption is removed from the present study by placing bounds $\bar{u}_1$ and $\bar{u}_2$ on the zonal wind, with $\bar{u}_1$ being the westerly wind maximum and $\bar{u}_2$ the easterly wind maximum.

Fig. 1 shows the result for zonal wavenumbers $s = 1$–5. The phase shift ϕ is set to be $-\pi$, the viscous limit, in this figure. The easterly jet maximum is taken to be -10 m s^{-1} relative to the wave, i.e., $U_2 = -10$ m s^{-1}, with the westerly jet maximum taking on a range of possible tropospheric and stratospheric wind values. The width of the shear zone (i.e., the region of linear variation), is taken to be 25° of latitude. Narrower shear zones

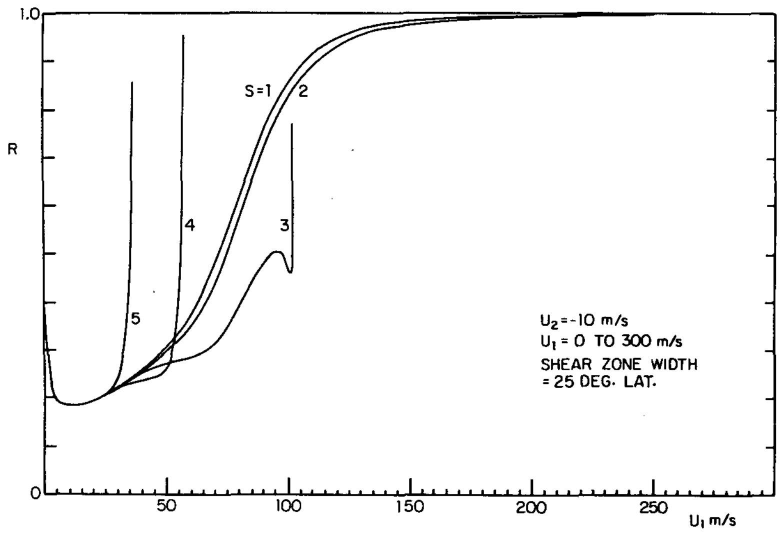

FIG. 1. Reflectivity R vs U_1 (the westerly wind maximum) for U_2 (the easterly wind maximum) fixed and the width of the shear zone fixed, and zonal wavenumber $s = 1, 2, 3, 4$ and 5. $\phi = -\pi$.

would give higher reflectivities, with $R \rightarrow 1$ as the width goes to zero (giving a Helmoltz profile). With the width of the shear zone fixed, an increase in the westerly jet speed U_1 increases the local shear at the CL, and R is in general increased as a result. It is seen in Fig. 1 that for low values of U_1 commonly observed in the troposphere, the reflectivity is generally very low: $R \sim 20\%$. As U_1 takes on higher values appropriate for the lower stratosphere ($U_1 \sim 50$ m s^{-1}), the shorter waves ($s \geq 4$) cease to propagate meridionally. Hence these waves will not see their CL at higher wind speeds. There is a rather rapid increase in reflectivity for longer waves when U_1 takes on still higher values. For waves with zonal wavenumbers $s = 1$ and $s = 2$, the CL becomes an almost perfect reflector when U_1 exceeds 100 m s^{-1}. These wind magnitudes are rather high[7] but they are nevertheless often observed in the upper stratosphere preceding the onset of sudden warming events. In any case, the conclusion reached by Dickinson concerning the nonexistence of normal modes continues to hold for the present case, since under realistic wind conditions, the reflectivity of the CL cannot be made to be *uniformly* high throughout both the stratosphere and troposphere. Introducing nonlinearity in the critical layer will change this conclusion. Before we present the nonlinear results, however, we would like to vary other parameters and see if the reflectivity can be improved in the linear viscous case.

Fig. 2 is the same for $s = 1$ as Fig. 1, but now β-effective $\equiv \beta - \bar{u}_{yy}(y_c)$ is allowed to be different from β, so that we can assess the effects of wind curvature on the reflectivity. Strictly speaking, β-effective should be the same as β for a linear profile, but if the linear profile is taken to be a first approximation to the real zonal wind near the CL, then β-effective can have values different from β depending on the local curvature of the wind. For U_1 with stratospheric values ($U_1 \gtrsim 40$ m s^{-1}), the reflectivity generally increases rapidly with decreasing values of β-effective. Thus it seems that quasi-normal modes can still form for $U_1 < 100$ m s^{-1} provided that β-effective is small. For U_1 between $\sim$30–40 m s^{-1}, the behavior is rather unexpected: R decreases as β-effective is decreased, until β-effective becomes

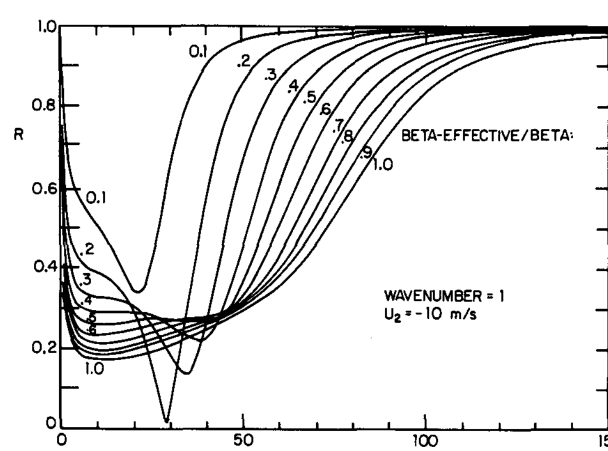

FIG. 2. Reflectivity R vs β-effective $[\equiv(\beta - \bar{u}_{yy})_{y_c}]$ for $s = 1$. The case β-effective/β = 1.0 is the same as in Fig. 1. $\phi = -\pi$.

[7] When a value of, say, 100 m s^{-1} is used for U_1, one should bear in mind that the actual velocity $\bar{u}_1 = U_1 \cos\varphi_1$ is lower on the sphere; e.g., $\bar{u}_1 = 87$ m s^{-1}, if the jet maximum is located at 30°N.

very small; then R would approach 1 as dictated by Eq. (46). For smaller U_1's, the variation of R with β-effective returns to normal, but the reflectivities are in general still quite low, unless (β-effective)/β becomes much less than 10%. Geisler and Dickinson (1974) studied the quasi-linear evolution of the CL and found that a positive β-effective will decrease in response to the absorption of wave energy, which tends to increase the curvature of the zonal wind near the CL. Similarly, a negative β-effective will become less negative locally at the CL due to the extraction of energy from the mean flow by the wave. They speculated that an end result of such feedback mechanisms might be a statistically small value of β-effective near the zero-wind line. Objections to the aforementioned result have recently been raised (Beland, 1976; Warn and Warn, 1977) on the ground that nonlinear wave-wave interactions ignored by Geisler and Dickinson may substantially change the evolution of the CL.

So far only the case of one CL has been considered. This is the relevant case for waves in the stratosphere and mesosphere. In the troposphere, however, there usually exist two CL's, with easterly wind in the equatorial region and westerlies in both the northern and southern latitudes. The calculation for the reflectivity in the presence of two CL's has been given in Tung (1977), where it is found, using a parabolic profile, that the presence of an additional westerly zone in the Southern Hemisphere does not change the reflectivity in any significant manner. This is a consequence of the fact that the evanescent region (i.e., the easterly zone) in the atmosphere is wide enough so that changes to the south of this region have only minimal effect on the reflectivity to the north.

The nonbarotropic case is next considered. The details of the calculation will not be given here, but they can also be found in Tung (1977). The results show that positive finite equivalent depths in general *lower* the reflectivity, while negative finite equivalent depths *increase* it, i.e.,

$$R(\epsilon > 0) < R(\epsilon = 0) < R(\epsilon < 0).$$

This trend is depicted in Fig. 3 for $s = 1$. For sufficiently small negative equivalent depths, R can be increased appreciably. Unfortunately, the equivalent depth is not a free parameter since its value is determined from the vertical structure equation. It turns out that ϵ is usually positive for most of the planetary scale slow moving waves that we are considering. ϵ can occasionally become slightly negative when the westerly wind is very strong. In general there do not exist solutions with large negative ϵ's (or small negative h's) for westerly zonal winds. We can therefore conclude that for the waves that are of interest to us, the reflectivity is not improved when the nonbarotropicity of the waves is taken into account.

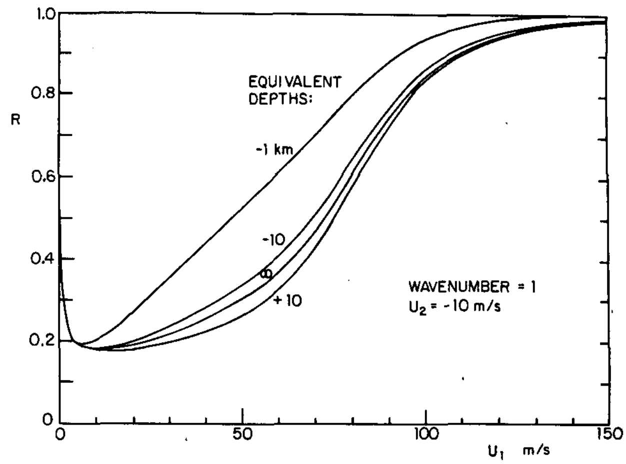

FIG. 3. Reflectivity R vs U_1 for various positive and negative equivalent depths. The infinite equivalent depth case is the same as in Fig. 1. $\phi = -\pi$, $s = 1$.

The problem becomes much more complicated when ϵ is allowed to be *complex*, as in the case of *forced* waves. In a forced problem, ϵ is to be determined as an *eigenvalue* in the horizontal structure equation [Eq. (17) or (24)] when both of the polar boundary conditions are applied. The details of the calculation will not be given here and interested readers are again referred to Tung (1977). In Fig. 4, some of the results are depicted. It seems that if ϵ_i, the imaginary part of ϵ, were at one's disposal, the reflectivity could be made arbitrarily large. Indeed R is in principle no longer restricted to values less than unity as in the real ϵ case. The fact is, however, that neither ϵ_i nor ϵ_r is a free parameter. It is found that in order to satisfy the polar boundary conditions, ϵ_i turns out to be positive and of the order of unity for planetary waves of interest. Referring to Fig. 4 it is seen that

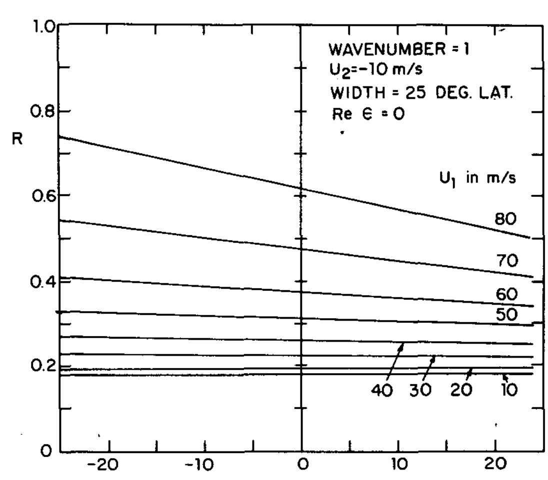

FIG. 4. Variation of R as a function of Imϵ, the imaginary part of the Lamb's parameter, for values of U_1 from 10 to 80 m s^{-1}.

for ϵ_i of order of unity, the change in reflectivity is almost unnoticeable, and in any case, R cannot be improved over that of the real ϵ case.

The effect of wave absorption at the CL is not as apparent for the forced waves as for the free waves. The existence of normal modes is no longer an issue for the forced waves. What is affected, however, is the vertical propagation of these waves. One would expect that, as the wave propagates upward from the lower boundary where it is forced, its energy density will decrease with height if wave absorption occurs at one of its lateral boundaries. Simmons (1974) calculates the "vertical penetration" of the stationary planetary waves and finds that the vertical distance penetrated by a wave in the presence of an absorbing zero-wind line is only slightly less than that achieved by a wave in the absence of such a singularity. The numerical results of Matsuno (1970) also tend to indicate that a forced wave's vertical propagation is not drastically different from what one may expect if the zero-wind line is replaced by a reflecting wall. The question of why it is so has rarely been raised. The answer seems to be the following: First, the wave absorption at the zero-wind line is never total and for partially reflected waves, ϵ_i is finite and turns out to be of order unity. Second, since the vertical propagation of the wave is governed by an equation such as (18),

$$\left(\frac{d}{dz^*} - 1\right)\left[\frac{d}{dz^*} Z\right] + \frac{S\epsilon}{4\Omega^2 a^2} Z = 0,$$

the wave solution has the form

$$Z \propto e^{z^*/2} e^{i\lambda z^*},$$

where

$$\lambda = \left[\frac{S\epsilon_r}{4\Omega^2 a^2}\left(1 + i\,\frac{\epsilon_i}{\epsilon_r}\right) - \frac{1}{4}\right]^{1/2},$$

and the effect of a nonzero ϵ_i is to introduce a decaying part

$$e^{-\lambda_i z^*}.$$

For an order 1 ϵ_i, ϵ_i/ϵ_r is usually much less than 1 for the waves of interest; thus so one has

$$\lambda_i \approx \left(\frac{S}{4\Omega^2 a^2}\right)^{1/2} \frac{\epsilon_i}{2\epsilon_r^{1/2}}$$

$$\approx 10^{-1} \times \frac{\epsilon_i}{\epsilon_r^{1/2}}.$$

Thus the effect of an order 1 ϵ_i on a forced wave is felt only after the wave has propagated many scale-heights.

It seems reasonable to conclude here that due to the fact that the adsorption at the CL is not total as commonly believed, but at least 20% of the incident waves are reflected, neither the lifetimes[8] of the free waves nor the propagation properties of the forced waves are significantly affected. However, with a major portion of the wave energy absorbed, as in the tropospheric case, no wave energy buildup is possible, and hence *resonant waves cannot exist*. The existence of resonant waves requires almost perfect reflections from the CL, and we have seen that such a condition is not uniformly met in our model atmosphere, especially in the lower atmosphere which contains most of the wave energy.

All the conclusions reached so far are based on the assumption that the dominant physical mechanism in the critical layer is viscosity or eddy diffusion; all nonlinear effects have been neglected. The effect of the presence of nonlinearity at the critical layer will be treated next.

8. The nonlinear case

The calculations presented in Section 7 are repeated for values of ϕ between $-\pi$ and 0. The numerical calculations are rather easy, since the series $f(\chi)$ and $g(\chi)$ are independent of ϕ and their values can thus be stored from the linear calculations.

In Figs. 5, 6, 7 and 8, we have plotted the variation of R vs U_1 for various values of the phase shift ϕ. The case of $\phi = -\pi$ has been given in Fig. 1. It is seen that, with a few insignificant exceptions, the reflectivities generally increase with decreasing values of $(-\phi/\pi)$. For $(-\phi/\pi) < 0.1$, the CL becomes a good reflector. For $(-\phi/\pi)$ of the order of 10^{-2}, a valued reached for $\lambda_c \approx 10^{-2}$, the reflectivity

[8] It can be shown (see Tung, 1977) that the lifetime T_e of a wave in the presence of absorption at the CL can be estimated to be

$$T_e \approx \left(\frac{100}{s}\right) \text{days} \times \frac{1}{\ln R}.$$

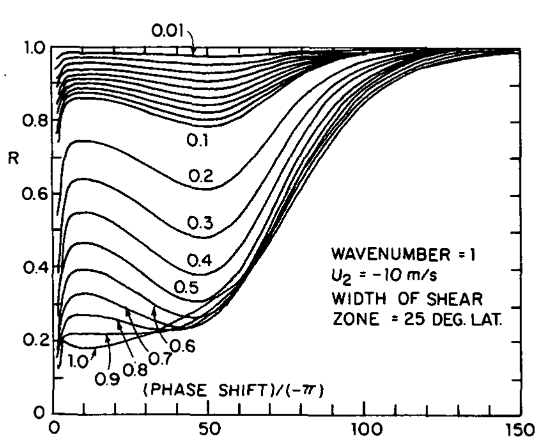

FIG. 5. Reflectivity R vs U_1 for different phase shifts ϕ: $0.01 \leq (-\phi/\pi) \leq 1.0$. $s = 1$.

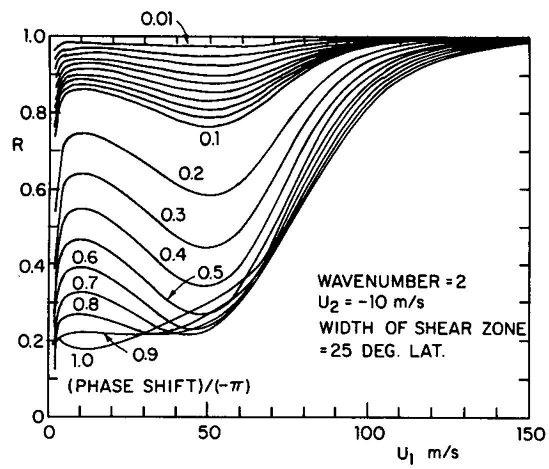

FIG. 6. As in Fig. 5 except for $s = 2$.

is almost indistinguishable from 1 *uniformly* for all values of U_1. Since the value of λ_c in the real atmosphere is usually less than 10^{-2}, the critical surface for planetary waves in the atmosphere can be treated as a reflector instead of as an absorber of wave energy, assuming the waves outside the critical layer have reached a steady state.

So far the only form of damping considered is eddy diffusion. It is well known that there exist other forms of damping, e.g., Ekman pumping and Newtonian cooling, that have shorter damping time scales than diffusion. A question that can be raised is: Does the presence of these other dampings in the real atmosphere alter the previous result that nonlinearity dominates over viscosity in the critical layer? Having repeated the matched asymptotics for these dampings, we find that unlike diffusion, whose effect becomes increasingly larger as the critical surface is approached, the magnitudes of other linear dampings remain the same order in the critical layer as away from it. Thus even though dampings like the Ekman pumping have shorter damping time scales than diffusion outside the critical layer, sufficiently far inside the critical layer diffusion should be more important. It is found that our previous results of matched asymptotics are not altered unless dampings due to Ekman pumping, etc., are so large that they significantly change the outer solutions so that we no longer have inviscid waves away from the CL.

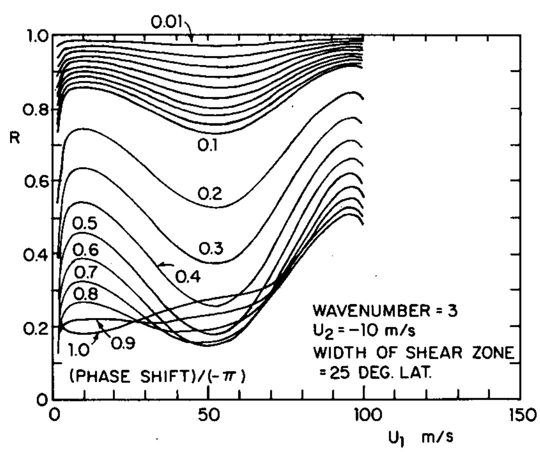

FIG. 7. As in Fig. 5 except for $s = 3$.

9. Nonlinear quantization and the virtual southern boundary

Now that it has been determined that the critical layer for planetary waves in the atmosphere is probably dominated by nonlinear processes and so is reflecting, it is meaningful to discuss the eigenfunctions, which can now be treated as "quasinormal modes."

The Rossby waves in the singular waveguide of the atmosphere are bounded on one side by the North Pole, and on the other side by the singular surface. The boundary condition near the Pole involves turning points. We denote y_t to be the northernmost turning point, so that

$$\begin{aligned} \Lambda^2(y) &= 0 \quad \text{at} \quad && y = y_t, \\ \Lambda^2(y) &< 0 \quad \text{for} \quad && y > y_t, \\ \Lambda^2(y) &> 0 \quad \text{for some} \quad && y < y_t. \end{aligned}$$

The solution on the north side of the turning point should tend asymptotically to the decaying exponential $\exp(-s|y|)$ as $y \to \infty$, in order to satisfy the polar boundary condition [Eq. (50)]. The controversial problem of the connection of oscillating to exponential solutions on the two sides of a turning point has been discussed in Dingle (1973). Utilizing his results, it can be shown that the exact solution satisfying the north polar boundary condition can be written for $y < y_t$, as

$$\hat{v}(y) = A\gamma^{1/2}(y)\left[\exp\left(i\int_y^{y_t} k_- dy + i\pi/4\right) - \exp\left(-i\int_y^{y_t} k_+ dy - i\pi/4\right)\right], \quad (67)$$

where k_+ and k_- satisfy the Riccatti equations

$$k_+^2 - i\frac{d}{dy}k_+ = \Lambda^2(y), \quad \mathrm{Re}k_+ \geq 0, \quad (68)$$

$$k_-^2 + i\frac{d}{dy}k_- = \Lambda^2(y), \quad \mathrm{Re}k_- \geq 0, \quad (69)$$

as can be shown by substituting (67) into Eq. (49).

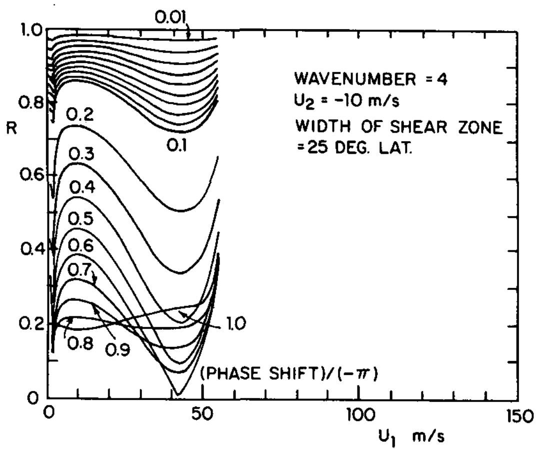

FIG. 8. As in Fig. 5 except for $s = 4$.

For $\Lambda^2(y)$ real,[9] Eqs. (68) and (69) imply that k_+ and k_- are complex conjugates of each other, so they can be rewritten as

$$k_+ = k_r + ik_i \quad \text{and} \quad k_- = k_r - ik_i,$$

where k_r and k_i are real functions. Observing the relationship

$$k_i = \frac{d}{dy} k_r/2k_r,$$

the solution [Eq. (67)] can be put into a simpler form

$$\hat{v}(y) = C\left[\frac{\gamma(y)}{k_r(y)}\right]^{1/2} \sin\left[\int_y^{y_t} k_r dy + \pi/4\right]. \tag{70}$$

To obtain the quantization condition, the southern boundary condition has to be applied to (70). Strictly speaking, the only southern boundary at one's disposal is the South Pole, where one has

$$\hat{v}(y) \to e^{-s|y|} \quad \text{as} \quad y \to -\infty.$$

However, various "virtual southern boundary" conditions have been used by those who are only interested in waves in the westerly region and do not wish to have to treat the whole domain between the two poles. One of the common practices is to put a wall at $y = y_0$, with y_0 usually taken as the equator, i.e., $y_0 = 0$. Applying the artificial southern boundary condition

$$\hat{v}(y) = 0 \quad \text{at} \quad y = y_0$$

to (70) then yields the quantization condition[10]

$$\int_{y_0}^{y_t} k_r dy + \pi/4 = n\pi, \quad n = 1, 2, 3. \ldots \tag{71}$$

Some of the well-known dispersion formulas are contained in Eq. (71). For example, one can obtain Rossby's dispersion formula by considering the case of a barotropic wave in the presence of an uniform zonal wind on an equatorial β-plane. For this case, we can solve for k_r and obtain

$$k_r = \left\{\frac{2\Omega a}{\bar{u} - c_{ph}} - s^2\right\}^{1/2},$$

so that (71) gives

$$c_{ph} - \bar{u} = -\frac{2\Omega a}{s^2 + l^2},$$

where

$$l = l_n \equiv (n\pi - \pi/4)/(y_t - y_0),$$

with

$$(y_t - y_0)a$$

interpreted, in the spirit of the β-plane approximation, as the distance between the two boundaries. Since the artificial wall-boundary condition has no physical basis[11] for a general zonal wind profile, it is desirable to obtain an alternate virtual boundary condition. The CL seems to be a convenient candidate for the location of such a boundary, if it can be assumed that the critical layer has evolved into a state in which planetary waves are (almost) perfectly reflected. If the CL were a solid wall, the quantization condition would be simply Eq. (71) with y_0 replaced by y_c. But the problem is not so simple: the CL is *not* a rigid wall, even though it is assumed to be *reflecting*. Since $\hat{v}$ is not necessarily zero at $y = y_c$, we modify Eq. (71) to include the correction

$$\int_{y_c}^{y_t} k_r dy + \pi/4 = n\pi + \theta, \quad n = 1, 2, 3 \ldots, \tag{72}$$

where $-3/4\pi < \theta < \pi$. As far as the waves to the north of the CL are concerned, the parameter θ contains all the information about the region to the south of the singularity. In principle, θ is different for different wavenumbers and for different atmospheric conditions south of the CL, and thus an extensive tabulation of its values is required. Fortunately, because of the nature of the singularity at the CL, the value of θ does not seem to depend too strongly on the particular wind profile chosen and hence an approximate universal value seems to be obtainable. We proceed by examining

[9] It can be shown that c_i, the imaginary part of the phase speed, is infinitesimally small for an almost perfectly reflecting CL.

[10] Note that unlike the case of gravity waves in stratified fluids, the integral $\int^y k_r dy$ for the present case is well behaved at the CL. There is no rapid oscillation near the singularity.

[11] The equatorial boundary condition can be justified by symmetry arguments for waves in a constant zonal wind.

the asymptotic solution near the CL. In that neighborhood the governing equation (49) reduces to

$$\frac{d^2}{d\chi^2}\hat{v} + \frac{\hat{\beta}_c}{\chi}\hat{v} = 0 \quad \text{for} \quad |\chi| \ll 1, \tag{73}$$

where

$$\chi \equiv y - y_c.$$

For $\chi > 0$, two linearly independent solutions to Eq. (73) are

$$\hat{v}_1(\chi) = 2(\hat{\beta}_c\chi)^{1/2}J_1[2(\hat{\beta}_c\chi)^{1/2}],$$
$$\hat{v}_2(\chi) = 2(\hat{\beta}_c\chi)^{1/2}Y_1[2(\hat{\beta}_c\chi)^{1/2}],$$

where J_1 and Y_1 are the Bessel functions (of order 1) of the first and second kind, respectively. The first solution has a node at the CL, i.e.,

$$\hat{v}_1(\chi) \to 2\hat{\beta}_c\chi \quad \text{as} \quad \chi \to 0.$$

If the solution near the critical surface consists of $\hat{v}_1$ only, then the singularity would act as a wall, yielding [cf. Eqs. (71) and (72)]

$$\theta = 0. \tag{74}$$

If instead, the dominant local solution is $\hat{v}_2$, one would then have

$$\theta \approx -\pi/2, \tag{75}$$

since Y_1 is almost $\pi/2$ out of phase with J_1. The task now is to determine which is the more dominant solution for $0 < \chi \ll 1$. The general solution near $\chi = 0$ can be written as a linear combination of the two solutions

$$\hat{v}(\chi) = C_1\hat{v}_1(\chi) + C_2\hat{v}_2(\chi).$$

We would like to show that $|C_2/C_1| \gg 1$, so that Y_1 becomes the dominant solution and (75) together with (72) is the relevant quantization rule. To do this, the behaviors of $\hat{v}_1(\chi)$ and $\hat{v}_2(\chi)$ are examined for *negative* χ. If viscosity were the dominant mechanism in the critical layer, the solutions for $\chi < 0$ could be obtained by analytic continuation as

$$\hat{v}_1(\chi) = \begin{cases} 2(\hat{\beta}_c\chi)^{1/2}J_1[2(\hat{\beta}_c\chi)^{1/2}], & \chi > 0 \\ -2(\hat{\beta}_c|\chi|)^{1/2}I_1[2(\hat{\beta}_c|\chi|)^{1/2}], & \chi = |\chi|e^{-i\pi} \end{cases}$$

$$\hat{v}_2(\chi) = \begin{cases} 2(\hat{\beta}_c\chi)^{1/2}Y_1[2(\hat{\beta}_c\chi)^{1/2}], & \chi > 0 \\ -\pi^{-1}\{(-i\pi)2(\hat{\beta}_c|\chi|)^{1/2}I_1[2(\hat{\beta}_c|\chi|)^{1/2}] + 2\cdot 2(\hat{\beta}_c|\chi|)^{1/2}K_1[2(2\hat{\beta}_c|\chi|)^{1/2}], & \chi = |\chi|e^{-i\pi} \end{cases},$$

where I_1 and K_1 are the modified Bessel functions. For the more relevant case of a nonlinearity dominated critical layer, the phase shift should be very close to zero (instead of the value $-\pi$ used above). The continuation to $\chi < 0$ becomes

$$\hat{v}_1(\chi) = \begin{cases} 2(\hat{\beta}_c\chi)^{1/2}J_1[2(\hat{\beta}_c\chi)^{1/2}], & \chi > 0 \\ -2(\hat{\beta}_c|\chi|)^{1/2}I_1[2(\hat{\beta}_c|\chi|)^{1/2}], & \chi < 0 \end{cases}$$

$$\hat{v}_2(\chi) = \begin{cases} 2(\hat{\beta}_c\chi)^{1/2}Y_1[2(\hat{\beta}_c\chi)^{1/2}], & \chi > 0 \\ -2\pi^{-1}\cdot 2(\hat{\beta}_c|\chi|)^{1/2}K_1[2(\hat{\beta}_c|\chi|)^{1/2}], & \chi < 0. \end{cases}$$

The solutions remain real when the critical surface is crossed. For $\chi < 0$, $\hat{v}_1(\chi)$ is exponentially growing and $\hat{v}_2(\chi)$ is exponentially decaying away from the critical surface. [Numerically, $K_1(z)$ resembles an exponentially decreasing function for z around 2 and $I_1(z)$ becomes an exponentially increasing function for z around 3.7 (see 9.8.8 and 9.8.4 of Abramowitz and Stegun, 1970). Since the relevant value for $\hat{\beta}_c$ is around 20, $z \equiv 2(\beta|\chi|)^{1/2}$ would become 2 when $|\chi|$ is around 0.05, and 3.7 when $|\chi| \approx 0.17$. Both values are sufficiently close to the critical surface so that $\Lambda^2(y)$ can still be approximated by $\hat{\beta}_c/\chi$.] To match to the solution away from the critical surface one should have $|C_1/C_2|$ of the order of $K_1[2(\hat{\beta}_c|\chi_2|)]^{1/2}/I_1[2(\hat{\beta}_c|\chi_1|)^{1/2}]$ or *less*, which is a very small quantity. Therefore, one should usually have

$$|C_2/C_1| \gg 1$$

and

$$v(\chi) \approx C_2v_2(\chi) \quad \text{for} \quad \chi > 0.$$

[Note that $C_1v_1(\chi)$ cannot be neglected from the general solution for $\chi < 0$.] Thus it seems that the relevant approximate value for θ is $-\pi/2$ and the appropriate quantization condition is

$$\int_{y_0}^{y_t} k_r dy + \pi/4 = n\pi - \pi/2, \quad n = 1, 2, \ldots. \tag{76}$$

Eq. (76) shows that the critical surface cannot be replaced by a rigid wall even if it is a perfect reflector for the wave energy. The phase of the reflected wave differs by $\pi/2$ from that reflected from a wall. Note also that there is no zero for the eigenfunction corresponding to the lowest mode $n = 1$, i.e.,

$$\int_{y_c}^{y_t} (k_r)_{n=1}dy + \pi/4 = \pi/2.$$

This mode has a quarter of its wavelength between the turning point and the CL.

10. Two-dimensional shears

In the presence of both meridional and vertical shears, i.e., $\bar{\omega} = \bar{\omega}(y,z^*)$, the linearized primitive equations that we have considered in Section 2 are no longer separable. We have obtained separability in Section 2 by assuming that the vertical shears vanish. The only way our model can be made to handle the vertical variation of the mean wind is to divide the wind into several uniform layers, i.e., to approximate the profile by

$$\bar{\omega}(y,z^*) = \bar{\omega}_m(y)$$

for

$$z_m^* \leq z^* < z_{m+1}^*, \quad m = 0, 1, 2, 3, \ldots . \quad (77)$$

In each layer, the mean wind varies with latitude only, and so the calculations presented in the previous sections apply. Eq. (77) is the only separable wind profile for the equations which are valid over the whole sphere.

For a $\bar{\omega}$ more general than that given by (77), the primitive equations are no longer separable. However, our results concerning the reflectivity R and the quantization parameter θ are still applicable, since these are local quantities which can be calculated by considering only the region near the CL. It has been shown (Matsuno, 1970) that as the waves approach the critical surface, the wave vectors become increasingly perpendicular to it. This is due to the fact that near the singularity the normal direction is the direction of dominant variations of the solutions, and variations in other directions become unimportant. The governing equation becomes in effect one-dimensional. For a vertically oriented CL (as we have assumed), the equation in the normal direction is the y equation used in this study.

The above arguments become clearer when an explicit example is considered. Dickinson (1968) has shown that for long waves the governing equation can be well approximated by the "geostrophic" equation given below [cf. Eq. (2.5) of Dickinson] valid on the sphere except in the equatorial region:

$$\left[\frac{\partial^2}{\partial y^2} - s^2 + \tanh^2 y \operatorname{sech}^2 y e^{z^*} \times \frac{\partial}{\partial z^*}\frac{e^{-z^*}}{\$}\frac{\partial}{\partial z^*}\right]\Psi + \frac{\bar{q}_y}{\bar{\omega} - c}\Psi = 0, \quad (78)$$

where

$$\bar{q}_y = 2\Omega \operatorname{sech}^2 y - \bar{\omega}_{yy} - \tanh^2 y \operatorname{sech}^2 e^{z^*} \times \frac{\partial}{\partial z^*}\frac{e^{-z^*}}{\$}\bar{\omega}_{z^*}$$

is the gradient of the mean potential vorticity and $\$ \equiv S/(2\Omega a)^2$. In (78), y is the Mercator coordinate and z^* the log$p$ coordinate defined previously. The equation is nonseparable except for a very special wind profile given by

$$\frac{\bar{q}_y}{\bar{\omega}(y,z^*)} = \hat{\beta}\left[\frac{1}{U(y)} + \frac{\operatorname{sech}^2 y \tanh^2 y}{V(z^*)}\right]. \quad (79)$$

For this profile and for stationary waves, Eq. (78) can be separated, by assuming

$$\Psi = Y(y)Z(z^*),$$

into

$$\left[\frac{d^2}{dy^2} - s^2 + \frac{\hat{\beta}}{U(y)} - \operatorname{sech}^2 y \tanh^2 y \bar{\gamma}^2\right]Y = 0, \quad (80)$$

$$\left[e^{z^*}\frac{d}{dz^*}\frac{e^{-z^*}}{\$}\frac{d}{dz^*} + \frac{\hat{\beta}}{V(z^*)} + \bar{\gamma}^2\right]Z = 0 \quad (81)$$

where $\bar{\gamma}^2$ is the separation constant.

The applicability of this separable model to the atmosphere is limited by the form of $\bar{\omega}$ required by (79). What we will now show is, for a general velocity profile not necessarily in the form of (79), that the nonseparable equation (78) can be reduced, in a region near the CL, to a form similar to the meridional equation (80). For convenience, we assume that the CL is oriented vertically (otherwise a set of suitably oriented orthogonal coordinates has to be defined). Then near the CL, one can Taylor expand

$$\bar{\omega}(y,z^*) - c = \bar{\omega}_{y_c}(z^*)(y - y_c) + \tfrac{1}{2}\bar{\omega}_{yy_c}(z^*)(y - y_c)^2 + O((y - y_c)^3)$$

$$\bar{q}_y(y,z^*) = [2\Omega \operatorname{sech}^2 y - \bar{\omega}_{yy}]_{y_c} - \tanh^2 y_c \operatorname{sech}^2 y_c \times e^{z^*}\frac{d}{dz^*}\left[\frac{e^{-z^*}}{\$}\frac{d}{dz^*}\bar{\omega}_{y_c}(z^*)\right] \times (y - y_c) + O((y - y_c)^2).$$

Then

$$\frac{q_y(y,z^*)}{\bar{\omega}(y,z^*) - c} \approx \frac{[2\Omega \operatorname{sech}^2 y_c - \bar{\omega}_{yy_c}(z^*)]}{\bar{\omega}_{y_c}(z^*)(y - y_c)}. \quad (82)$$

It is seen that the vertical shear terms in the potential vorticity gradient is of the order of $(y - y_c)$ and hence drop out in (82), and the z^* dependence enters only through the meridional shear terms $\bar{\omega}_{y_c}(z^*)$ and $\bar{\omega}_{yy_c}(z^*)$. Also, since (82) varies most rapidly in the y direction as $y \to y_c$, and this variation is to be balanced by the $\partial^2/\partial y^2$ term in (78), the vertical derivatives of the solutions can be shown to be negligibly small compared with the y derivatives. Therefore, the nonseparable equation (78) can be reduced to

$$\left[\frac{\partial^2}{\partial y^2} - s^2 + \frac{\hat{\beta}_c(z^*)}{(y - y_c)}\right]\Psi = 0, \quad (83)$$

where

$$\hat{\beta}_c(z^*) \equiv \frac{2\Omega \operatorname{sech}^2 y_c - \bar{\omega}_{yy_c}(z^*)}{\bar{\omega}_{y_c}(z^*)}.$$

Eq. (83) is similar to the meridional structure equation (80) and is also the same as the meridional barotropic equation we have considered in previous sections, except here $\hat{\beta}_c$ is allowed to have different values at each height z^*. This z^* dependence is only parametric, as no z^* derivative appears in Eq. (83).

For the special case where (79) does apply, one then has the separated meridional and vertical equations as given by Eqs. (80) and (81). Eq. (80) can be treated in the same way as done in this paper, and Eq. (81) can be solved along the lines given in Part II. In particular, Eq. (81) possesses internal wave solutions (i.e., solutions which are propagating in the lower atmosphere and become evanescent above a certain height). The existence of such solutions has been shown in Part II to be an important aspect in the theory of resonant Rossby waves.

Before concluding this section, let us point out that this section has shown only that the *outer linear* equations can reduce to the form used in earlier sections even when the assumption of one-dimensional shear is relaxed. The nonlinear terms in the *inner nonlinear* equation are more complicated than, and in general do not reduce to, the simple barotropic vorticity equation used in the previous sections unless the CL is oriented predominately in the vertical direction. In particular, our results concerning the reflectivity of the CL do not apply to a horizontally oriented CL in the upper stratosphere, first because the nonlinear terms in the inner equation near the critical layer can be shown to be different, and second, the strong photochemical damping mechanisms present in these high altitudes may significantly affect our assessment of the relative importance of the nonlinear verses viscous effects.

11. Conclusion

Unlike the fast-moving waves considered by Haurwitz (1940), who first obtained the quantization of the planetary waves between the poles on the sphere, stationary or slowly moving waves are significantly affected by the variations of the mean wind from westerlies in the winter hemisphere to easterlies in the summer hemisphere. To study these waves in the westerly region, a common practice has been to introduce two vertical walls as the northern and southern boundaries, so that the waves can be confined in a meridionally limited channel (for purposes of using local approximations to the mean wind and geometry), and quantization conditions are derived for the waves that fit into this artificial waveguide. Obviously, these walls do not exist in the real atmosphere, but their justifications seem to have been found in the theory of "polar waveguides" (Dickinson, 1968; Matsuno, 1970), whereby the waves are confined by the pole to the north and the jet stream to the south. The pole serves as a good boundary as it can be shown that the waves are strongly evanescent near it. The jet stream, on the other hand, does not seem to be as effective in reflecting the long waves. For stationary or slowly moving waves, those of planetary scales can leak some of their energy to the south through tunneling and, in addition, since these long waves are not confined vertically to the troposphere, they can propagate first vertically to the lower stratosphere and then southward through the relatively weak westerly region between the tropospheric and stratospheric jet maxima. The breakdown of the theory of polar waveguides for these long waves presents a dilemma concerning the existence of normal modes in the real atmosphere. Without the confining "southern wall" the waves can now reach the region near the zero-wind line where they find their critical surfaces. There, along the lines of classical hydrodynamics, it was thought that viscosity becomes important even if it is negligibly small elsewhere, and as a result the waves are dissipated in a thin layer (called the critical layer) around the critical surface. That the waves are totally absorbed near the critical surface has almost become a common belief. On the other hand, observational studies of the phase speeds of the Rossby waves in the winter hemisphere (Eliasen and Machenhauer, 1965, 1969; Diky and Golitsyn, 1968) seem to show that normal modes satisfying the Rossby-Haurwitz dispersion relation appear to exist in the atmosphere. Some of these observed waves have phase speeds slow enough to have encountered a critical surface. The recent interest in the phenomena of blocking and sudden warming also brings into attention the existence of large-scale stationary waves that are observed to amplify at certain times during some winters. These amplifying waves cannot be described within the existing framework of "singular waveguides" with an absorbing southern boundary.

We have shown in this paper that the critical layer is never totally absorbing; there is always some partial reflection even with a viscosity-dominated critical layer. This fact alone may be sufficient for the existence of "quasi-normal modes", which can retain their identity not for an infinite period of time as implied by the mathematical definition of an ideal normal mode, but for a duration perhaps only longer than two north-south traverses of the wave in the waveguide. However, we have not capitalized here on this possibility to build a theory of quasi-normal modes, but have instead gone on further and introduced nonlinearity of the magnitude relevant to the real atmosphere. It is shown that a nonlinearity-dominated critical layer should be almost perfectly reflecting, at least in its steady state. The small amount of wave energy that is absorbed cause the wave amplitude to decay in time, but the e-folding time scale is so much longer than the time scales due to other damping processes, that it is of no practical importance to consider the amplitude change due to absorption. These results indicate that "quasi-normal mode waves" can exist in the atmosphere. The dispersion relation satisfied by these waves has also been determined. Surprisingly, it is found to be only slightly different from that obtained by imposing an artificial southern wall. With respect to the amplifying waves that are believed to play some part in the phenomena of sudden warming and blocking, the study in this paper lays the basic framework upon which a theory of resonant Rossby waves can be developed.

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APPENDIX A

List of Symbols

a	radius of the earth
λ	longitude
φ	latitude
φ_1	latitude of the westerly wind maximum
φ_c	latitude of the critical surface
φ_2	latitude of the easterly wind maximum
$\bar{\omega}$	angular frequency of the zonal flow $[=\bar{u}/(a\cos\varphi)]$
Ω	angular frequency of the earth's rotation
f	Coriolis parameter $[=(2\Omega+\bar{\omega})\sin\varphi]$
q	$2(\Omega+\bar{\omega})\sin\varphi - \cos\varphi \dfrac{\partial}{\partial\varphi}\bar{\omega}$
S	$RT\dfrac{\partial}{\partial z^*}\ln\theta$
$\mathcal{S}$	$S/(2\Omega a)^2$
s	zonal wavenumber
σ	frequency of the wave
$\hat{\sigma}$	Doppler-shifted frequency $[=\sigma + s\bar{\omega}]$
c_{ph}	phase speed of the wave $[=-a\cos\varphi \times \sigma/s]$
c	$-\sigma/s$
μ	$\sin\varphi$
Δ	$\dfrac{1}{4\Omega^2}[\hat{\sigma}^2 - fg]$
γ	$1-(1-\mu^2)a^2\dfrac{(\hat{\sigma}/s)^2}{gh}$, where h is the equivalent depth
ϵ	Lamb's parameter $[=4\Omega^2a^2/gh]$
x, y	local coordinates used in Section 2; $dx \equiv a\cos\varphi d\lambda$, $dy \equiv a d\varphi$
β	$\left(\dfrac{d}{dy}f\right)_{y_c}$, used in Section 2
χ	$\mu-\mu_c$ in Section 3 where $\mu_c=\sin\varphi_c$; $y-y_c$ in Section 6
$\hat{\beta}_c$	$\dfrac{1}{(1-\mu_c^2)}\left(\dfrac{d}{d\mu}\bar{\omega}\right)_c^{-1}\left(\dfrac{d}{d\mu}q\right)_c$
$f(\chi), g(\chi)$	regular and irregular Tollmien functions defined in Eqs. (26) and (27)
ϕ	phase shift across the critical surface
y	Mercator coordinates $\tanh y = \mu$, except in Sections 2 and 5 where it is the local northward Cartesian coordinate
q_y	mean vorticity gradient $[=dq/dy]$; the β-plane equivalent is $\beta - \bar{u}_{yy}$
$k_\pm$	satisfying Eqs. (68) and (69) and having the physical interpretation of being the meridional wavenumbers
k_r, k_i	real and imaginary parts of $k_\pm$
y_t	turning point in Mercator coordinate: $\Lambda = 0$ at $y = y_t$
R	reflection coefficient, defined as the ratio of the reflected to incident wave amplitudes
y_1	location of the westerly jet maximum
y_2	location of the easterly jet maximum
T_e	e-folding damping time scale due only to the absorption of the wave energy at the critical surface
$p(\chi)$	$\dfrac{d}{dx}f(\chi) - \dfrac{(d/dx)\bar{\omega}}{\bar{\omega}-c}f(\chi)$
$q(\chi)$	$\dfrac{d}{dx}g(\chi) - \dfrac{(d/dx)\bar{\omega}}{\bar{\omega}-c}g(\chi)$
σ'	$-B'/A'$
U_1	$a[\bar{\omega}(y_1)-c] = (\bar{u}_1 - c_{ph})/\cos\varphi_1$, where $\bar{u}_1$ is the westerly wind maximum
U_2	$a[\bar{\omega}(y_2)-c] = (\bar{u}_2 - c_{ph})/\cos\varphi_2$, where $\bar{u}_2$ is the easterly wind maximum
λ_c	ratio of viscosity to nonlinearity [Eq. (138)].

There are also some temporary dummy symbols defined where they are used.

APPENDIX B

Matched Asymptotics at the Critical Layer

The governing equation valid in the critical layer is given by Eq. (21) to be

$$\left[\frac{\partial}{\partial t} - \psi_y \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial y}\right](\nabla^2\psi + f) = \nu\nabla^2\nabla_3{}^2\psi + w_y^*\psi_{yz^*}, \quad \text{(B1)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\nabla_3{}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\delta^2}{a^2}\frac{\partial^2}{\partial z^{*2}},$$

$$f = 2\Omega \sin\varphi.$$

The z^*-derivative terms and the turning and twisting term can be dropped from the outset without affecting the results to be obtained. They are nevertheless retained here.

The purpose of this Appendix is twofold: First we want to show that the presence of the Coriolis term in (B1) and in the outer solutions does not alter Haberman's (1972) result that the combined effect of viscosity and nonlinearity at the critical layer is to introduce a phase shift in the outer solutions and that the phase shift is a function only of one parameter, λ_c. Second, the important parameter λ_c is carefully rederived, using quantities relevant to our problem, thus enabling an estimation of its value for the real atmosphere.

To proceed, we express ψ as a sum of its zonal mean[12] and deviation from the mean as

$$\Psi = -\int^y \bar{u}dy + \epsilon(|\bar{u}_c'|a^2)\psi, \quad \text{(B2)}$$

where $\epsilon\psi$ is the perturbation streamfunction made dimensionless by $|\bar{u}_c'|a^2$, with $\bar{u}_c'$ being the meridional shear of the zonal wind at the CL; ϵ has been assumed to be small. Eq. (B1) becomes

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \epsilon(|\bar{u}_c'|a^2) \times \left(\psi_x\frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial x}\right)\nabla^2\psi + (\beta - \bar{u}_{yy})\psi_x$$

$$= \nu\nabla^2\nabla_3{}^2\psi - \frac{\nu(\bar{u}_{yyy})}{(\epsilon|\bar{u}_c'|a^2)} + \epsilon(|\bar{u}_c'|/a)w_y\psi_{yz^*}, \quad \text{(B3)}$$

where

$$\beta = \frac{d}{dy}f \quad \text{and} \quad w = w^*/(\epsilon|\bar{u}_c'|/a).$$

Following Benney and Bergeron (1969) we translate the coordinates into a frame moving with the phase speed c_{ph} of the wave by letting

$$x_1 = x - c_{ph}t.$$

Then (B3) is

$$(\bar{u} - c_{ph})\frac{\partial}{\partial x_1}\nabla'^2\psi + \epsilon(|\bar{u}_c'|a^2) \times \left(\psi_{x_1}\frac{\partial}{\partial y} - \psi_y\frac{\partial}{\partial x_1}\right)\nabla'^2\psi + (\beta - \bar{u}_{yy})\psi_{x_1}$$

$$= \nu\nabla'^2\nabla_3'^2\psi - \frac{\nu(\bar{u}_{yyy})}{(\epsilon|\bar{u}_c'|a^2)} + (\epsilon|\bar{u}_c'|/a)w_y\psi_{yz^*}, \quad \text{(B4)}$$

with

$$\nabla'^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y^2}$$

and

$$\nabla_3'^2 = \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y^2} + \frac{\delta^2}{a^2}\frac{\partial^2}{\partial z^{*2}}.$$

In the critical layer, we introduce the "inner coordinates" as

$$Y = \frac{(y - y_c)}{a}\epsilon^{-1/2}\,\mathrm{sign}(\bar{u}_c'),$$

$$X = \frac{x_1}{a},$$

$$Z = z^*.$$

Near the critical layer, we expand the prescribed mean flow as

$$\bar{u}_0 - c_{ph} = \bar{u}_c'(y - y_c) + \tfrac{1}{2}\bar{u}_c''(y - y_c)^2 + \ldots$$

and the mean flow modification as

$$\epsilon^{1/2}\bar{u}_1 = \epsilon^{1/2}\bar{u}_c'a[H_1 + 2H_2(y - y_c)/a + \ldots].$$

H_1 can be set to zero without any loss of generality, as its only effect is to shift the location of the CL by an $O(\epsilon^{1/2})$ distance. The total mean flow is thus expanded as

[12] We shall let the zonal mean flow $\bar{u}$ include any possible "mean flow modification" due to the wave. That is, if $\bar{u}_0(y)$ is the prescribed zonal flow and $\epsilon^{1/2}u_1(y)$ is the flow induced by the wave, then here $\bar{u} = \bar{u}_0 + \epsilon^{1/2}\bar{u}_1$. In a steady-state problem it is meaningless to separate the two.

$$\begin{aligned}\bar{u} - c_{ph} &= \bar{u}_0 - c_{ph} + \epsilon^{1/2}\bar{u}_1(y)\\ &= \bar{u}_c'(y - y_c) + \tfrac{1}{2}\bar{u}_c''(y - y_c)^2 + \ldots + \epsilon^{1/2}\bar{u}_c'a[2H_2(y - y_c)/a + \ldots]\\ &= |\bar{u}_c'|aY\epsilon^{1/2} + |\bar{u}_c'|a\left[\frac{1}{2}\frac{\bar{u}_c''a}{|\bar{u}_c'|}Y^2 + 2H_2Y\right]\epsilon + O(\epsilon^{2/3}).\end{aligned}$$

Eq. (B4) becomes

$$\left[Y + \epsilon^{1/2}\left(\frac{1}{2}\frac{\bar{u}_c''a}{|\bar{u}_c'|}Y^2 + 2H_2Y\right) + O(\epsilon)\right]\frac{\partial}{\partial X}\left[\frac{\partial^2}{\partial Y^2} + \epsilon\frac{\partial^2}{\partial X^2}\right]\psi$$
$$+ \left(\psi_X\frac{\partial}{\partial Y} - \psi_Y\frac{\partial}{\partial X}\right)\left[\frac{\partial^2}{\partial Y^2} + \epsilon\frac{\partial^2}{\partial X^2}\right]\psi + \epsilon^{1/2}\hat{\beta}_c\psi_X - \epsilon^{1/2}w_Y\psi_{YZ}$$
$$= \lambda\left[\frac{\partial^2}{\partial Y^2} + \epsilon\frac{\partial^2}{\partial X^2}\right]\left[\frac{\partial^2}{\partial Y^2} + \epsilon\frac{\partial^2}{\partial X^2} + \epsilon\delta^2\frac{\partial^2}{\partial z^2}\right]\psi - \lambda\epsilon a^2|\bar{u}_c'|^{-1}(\bar{u}_{yyy})_{y_c}, \tag{B5}$$

where

$$\hat{\beta}_c \equiv \frac{a}{|\bar{u}_c'|}(\beta - \bar{u}_{yy})_{y_c},$$
$$\lambda \equiv \left(\frac{\nu}{|\bar{u}_c'|a^2}\right)\epsilon^{-3/2}.$$

Eq. (B5) is the "inner equation" valid in the critical layer; its solution is to be matched to the "outer solution" valid away from the critical layer. In the outer region, the solution is expanded as

$$\psi = \psi_1 + \epsilon^{1/2}\psi_{3/2} + \epsilon\psi_2 + O(\epsilon^2).$$

Substituting this into Eq. (B4) and equating like orders in ϵ, the following set of asymptotic equations are obtained:

$$(\bar{u}_0 - c_{ph})\frac{\partial}{\partial x_1}\nabla'^2\psi_1 + (\beta - \bar{u}_{0yy})\frac{\partial}{\partial x_1}\psi_1 = 0, \tag{B6a}$$

$$(\bar{u}_0 - c_{ph})\frac{\partial}{\partial x_1}\nabla'^2\psi_{3/2} + (\beta - \bar{u}_{0yy})\frac{\partial}{\partial x_1}\psi_{3/2} = \bar{u}_{1yy}\frac{\partial}{\partial x_1}\psi_1 - \bar{u}_1\frac{\partial}{\partial x_1}\nabla'^2\psi_1, \tag{B6b}$$

$$(\bar{u}_0 - c_{ph})\frac{\partial}{\partial x_1}\nabla'^2\psi_2 + (\beta - \bar{u}_{0yy})\frac{\partial}{\partial x_1}\psi_2 = \bar{u}_{1yy}\frac{\partial}{\partial x_1}\psi_{3/2} - \bar{u}_1\frac{\partial}{\partial x_1}\nabla'^2\psi_{3/2} - (|\bar{u}_c'|a^2) \times \left(\psi_{1x_1}\frac{\partial}{\partial y} - \psi_{1y}\frac{\partial}{\partial x_1}\right)\nabla'^2\psi_1. \tag{B6c}$$

Eq. (B6a) is the linear inviscid two-dimensional barotropic equation whose solutions we wish to match across the critical layer. The solution to Eq. (B6a) can be expressed in terms of the regular and irregular functions $f(\chi)$ and $g(\chi)$, where $\chi \equiv (y - y_c) \times \mathrm{sign}(\bar{u}_c')/a$, as

$$\psi_1 = [Af(\chi) + Bg(\chi)]\cos(sX) + [Cf(\chi) + Dg(\chi)]\sin(sX), \tag{B7}$$

when $f(\chi)$ and $g(\chi)$ are given previously in the text; their first few terms are

$$f(\chi) = \chi - \tfrac{1}{2}\beta_c\chi^2 + \tfrac{1}{6}[\tfrac{1}{2}\alpha\hat{\beta}_c + s^2 + \delta_2]\chi^3 + O(\chi^4),$$

$$g(\chi) = 1 + \tfrac{1}{2}[(s^2 + \delta_2) + \hat{\beta}_c(\tfrac{1}{2}\alpha - 2\hat{\beta}_c)]\chi^2 + O(\chi^3) - \hat{\beta}_c f(\chi)\ln|\chi|, \tag{B8}$$

with

$$\alpha \equiv a^2\beta/(|\bar{u}_c'|a), \quad \delta_2 \equiv (a^3\bar{u}_c''')/(|\bar{u}_c'|a).$$

Eqs. (B7) and (B8) give the first few terms of ψ_1 as

$$\psi_1 = [B\cos(sX) + D\sin(sX)][1 - \hat{\beta}_c\chi\ln|\chi| + \ldots] + [A\cos(sX) + C\sin(sX)][\chi + \ldots]. \tag{B9}$$

In terms of the inner variable $Y = \chi\epsilon^{-1/2}$, (B9) is

$$\psi_1 = [B\cos(sX) + D\sin(sX)] - \epsilon^{1/2}Y\ln\epsilon^{1/2}\hat{\beta}_c \times [B\cos(sX) + D\sin(sX)] + \epsilon^{1/2}\{-[B\cos(sX) + D\sin(sX)]\hat{\beta}_cY\ln|Y| + [A\cos(sX) + C\sin(sX)]Y\} + O(\epsilon\ln\epsilon). \tag{B10}$$

The constants A, B, C, D are allowed to take different values for $Y > 0$ and $Y < 0$. We are primarily interested in the matching relations for these four constants and it is seen that they are determinable when terms up to order $\epsilon^{1/2}$ only are retained in the inner expansions. However, to this order the higher order terms, $\psi_{3/2}$ and ψ_2, also contribute. The leading term in the homogeneous solution of (B6b) contribute an $O(\epsilon^{1/2})$ term [cf. (B10)],

$$\epsilon^{1/2}\psi_{3/2} \sim \epsilon^{1/2}[B_{3/2}\cos(sX) + D_{3/2}\sin(sX)] + O(\epsilon\ln\epsilon).$$

The forced solution of (B6b) is of $O(\epsilon\ln\epsilon)$ in the inner variable Y. The homogeneous solution of (B6c) is of $O(\epsilon)$ and therefore does not contribute, while the particular solution for ψ_2 and higher order terms contribute only if they are singular in χ, i.e.,

$$\epsilon\psi_2 + \epsilon^{3/2}\psi_{2\,1/2} + \sim\epsilon^{1/2}[O(Y^{-1})] + O(\epsilon\ln\epsilon).$$

Thus,

$$\psi \sim [B\cos(sX) + D\sin(sX)] - \epsilon^{1/2}Y\ln\epsilon\tfrac{1}{2}\hat{\beta}_c[B\cos(sX) + D\sin(sX)] + \epsilon^{1/2}\{-[B\cos(sX) + D\sin(sX)]\hat{\beta}_cY\ln|Y| + [A\cos(sX) + C\sin(sX)]Y + B_{3/2}\cos(sX) + D_{3/2}\sin(sX) + O(Y^{-1})\} + O(\epsilon\ln\epsilon). \tag{B11}$$

Eq. (B11) is to be matched asymptotically to the inner solution for $Y \to \pm\infty$. To $O(\epsilon^{1/2})$, (B5) is, assuming $\lambda \leqslant O(1)$,

$$\left[Y + \epsilon^{1/2}\left(\frac{1}{2}\frac{\bar{u}_c''a}{|\bar{u}_c'|}Y^2 + 2H_2Y\right)\right]\frac{\partial^3}{\partial X\partial Y^2}\psi + \left(\psi_X\frac{\partial}{\partial Y} - \psi_Y\frac{\partial}{\partial X}\right)\frac{\partial^2}{\partial Y^2}\psi + \epsilon^{1/2}\hat{\beta}_c\psi_X - \epsilon^{1/2}w_Y\psi_{YZ} = \lambda\frac{\partial^4}{\partial Y^4}\psi. \quad \text{(B12)}$$

The solution to (B12) can be expanded in a form suggested by (B11), i.e.,

$$\psi = \psi^{(0)} + \epsilon^{1/2}\ln\epsilon\psi^{(1)} + \epsilon^{1/2}\psi^{(2)} + O(\epsilon\ln\epsilon). \quad \text{(B13)}$$

Substituting (B13) into (B12) and equating like orders in ϵ, one obtains an hierarchy of equations

$$Y\frac{\partial^3}{\partial X\partial Y^2}\psi^{(0)} + \left[\psi_X^{(0)}\frac{\partial}{\partial Y} - \psi_Y^{(0)}\frac{\partial}{\partial X}\right]\frac{\partial^2}{\partial Y^2}\psi^{(0)} - \lambda\frac{\partial^4}{\partial Y^4}\psi^{(0)} = 0, \quad \text{(B14)}$$

$$Y\frac{\partial^3}{\partial X\partial Y^2}\psi^{(1)} + \left[\psi_X^{(0)}\frac{\partial}{\partial Y} - \psi_Y^{(0)}\frac{\partial}{\partial X}\right]\frac{\partial^2}{\partial Y^2}\psi^{(1)} - \lambda\frac{\partial^4}{\partial Y^4}\psi^{(1)} = -\left(\psi_X^{(1)}\frac{\partial}{\partial Y} - \psi_Y^{(1)}\frac{\partial}{\partial X}\right)\frac{\partial^2}{\partial Y^2}\psi^{(0)}, \quad \text{(B15)}$$

$$Y\frac{\partial^3}{\partial X\partial Y^2}\psi^{(2)} + \left[\psi_X^{(0)}\frac{\partial}{\partial Y} - \psi_Y^{(0)}\frac{\partial}{\partial X}\right]\frac{\partial^2}{\partial Y^2}\psi^{(2)} - \lambda\frac{\partial^4}{\partial Y^4}\psi^{(2)}$$
$$= -\left[\psi_X^{(2)}\frac{\partial}{\partial Y} - \psi_Y^{(2)}\frac{\partial}{\partial X}\right]\frac{\partial^2}{\partial Y^2}\psi^{(0)} - \left[\hat{\beta}_c + \left(\frac{1}{2}\frac{\bar{u}_c''a}{|\bar{u}_c'|}Y^2 + 2H_2Y\right)\frac{\partial^2}{\partial Y^2}\right]\frac{\partial}{\partial X}\psi^{(0)} + w_Y\psi_{YZ}^{(0)}. \quad \text{(B16)}$$

The solutions to (B14) and (B15) that satisfy the outer matching conditions can be shown to be

$$\psi^{(0)} = B\cos(sX) + D\sin(sX), \quad \text{(B17)}$$

$$\psi^{(1)} = -\frac{\hat{\beta}c}{2}[B\cos(sX) + D\sin(sX)]Y, \quad \text{(B18)}$$

implying that the constants B and D are the same for $Y > 0$ and $Y < 0$, i.e.,

$$B_+ = B_-, \quad \text{and} \quad D_+ = D_-. \quad \text{(B19)}$$

The aim now is to determine the jump condition on A and C from (B16) which is

$$Y\frac{\partial^3}{\partial X\partial Y^2}\psi^{(2)} + \frac{\partial}{\partial X}[B\cos(sX) + D\sin(sX)] \times \left[\frac{\partial^3}{\partial Y^3}\psi^{(2)} + \hat{\beta}_c\right] = \lambda\frac{\partial^4}{\partial Y^4}\psi^{(2)}, \quad \text{(B20)}$$

subject to matching to the $\epsilon^{1/2}$ term in (B11). The "turning and twisting" term disappears since $\psi_Y^{(0)} = 0$.

It is convenient to have the parameters in the above equation reduced to as few as possible. To do so, we define[13]

$$\psi^{(2)} = \hat{\psi} - \frac{a}{|\bar{u}_c'|}\frac{\beta}{a}Y^3/6 + \left[\frac{\bar{u}_c''}{|\bar{u}_c'|}Y^3/6 + H_2Y^2\right],$$

$$B\cos(sX) + D\sin(sX) = -\hat{B}\cos\hat{X},$$

$$A\cos(sX) + C\sin(sX) = \hat{A}\cos\hat{X} + \hat{C}\sin\hat{X},$$

$$\hat{Y} = Y/\hat{B}^{1/2},$$

$$\lambda_c = \lambda/(s\hat{B}^{3/2}).$$

Eq. (B20) then reduces to

$$\hat{Y}\frac{\partial^3}{\partial\hat{X}\hat{Y}^2}\hat{\psi} + \sin\hat{X}\frac{\partial^3}{\partial\hat{Y}^3}\hat{\psi} = \lambda_c\frac{\partial^4}{\partial\hat{Y}^4}\hat{\psi}, \quad \text{(B21a)}$$

subject to the asymptotic matching condition as $Y \to \pm\infty$:

$$\hat{\psi} \to \hat{\beta}_cY^3/6 - H_2Y^2 - \hat{\beta}_cY\ln|Y|[B\cos(sX) + D\sin(sX)] + Y[A\cos(sX) + C\sin(sX)] + B_{3/2}\cos(sX) + D_{3/2}\sin(sX)] + O(Y^{-1}). \quad \text{(B21b)}$$

Except for $\beta - \bar{u}_c''$ replacing $-\bar{u}_c''$ of Haberman, $\hat{\psi}$ satisfies the same equation and matching conditions as Haberman's $\psi^{(2)}$. Thus Haberman's solution applies, with only a change in mean flow vorticity gradient to include earth's rotation. Eq. (B21a) depends on a single parameter λ_c, which we have defined to be

$$\lambda_c = \frac{\nu}{|\bar{u}_c'|a^2s\hat{B}^{3/2}\epsilon^{3/2}} = \frac{\nu(|\bar{u}_c'|as)^{1/2}/a}{(\epsilon|\bar{u}_c'|as\hat{B})^{3/2}}. \quad \text{(B22)}$$

Since the meridional velocity of the wave (in dimensional form) is

$$v = \epsilon(|\bar{u}_c'|a)\frac{\partial}{\partial X}\psi$$
$$\approx \epsilon(|\bar{u}_c'|a)(s\hat{B}\sin\hat{X}),$$

[13] Note that aside from the β term, $\hat{\psi}$ is the same as $\psi^{(2)}$ in Haberman, which is the total streamfunction at the same order.

the denominator in (B22) is of the order of $\nu^{3/2}$. Letting $V \equiv \epsilon|\bar{u}_c'|as\hat{B}$ be the typical meridional velocity of the wave, (B22) can be rewritten as

$$\lambda_c = \frac{\left[\dfrac{\nu}{|\bar{u}_c'|(a/s)}\right]}{\left[\dfrac{V}{|\bar{u}_c'|a/s}\right]^{3/2}} . \tag{B23}$$

Thus if we interpret $U \equiv |\bar{u}_c'|a/s$ as a typical velocity scale, $L \equiv a/s$ as a typical length scale, then the numerator of (B23) is just the reciprocal of the Reynold's number, Re, i.e.,

$$\mathrm{Re}^{-1} = \frac{\nu}{UL} ,$$

while the denominator

$$\left(\frac{V}{U}\right)^{3/2}$$

is a measure of the nonlinearity of the wave. Therefore

$$\lambda_c = \mathrm{Re}^{-1} \Big/ \left(\frac{V}{U}\right)^{3/2} \tag{B24}$$

is a measure of the relative magnitude of viscosity to that of nonlinearity.

The solution of Eq. (B21) is given in Haberman (1972) and so will not be repeated here. The result that is of relevance to us is that the matched asymptotics give the following relations concerning the constants A and C for $Y > 0$ and $Y < 0$:

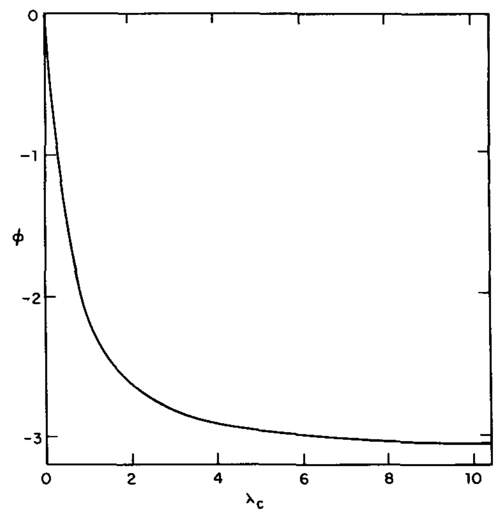

FIG. 9. The phase shift ϕ as a function of the viscosity/nonlinearity ration λ_c taken from Haberman (1972).

$$\left.\begin{aligned} A_+ &= A_- \\ C_+ &= C_- - \hat{\beta}_c\phi B_+ \end{aligned}\right\} , \tag{B25}$$

where $\phi = \phi(\lambda_c)$ is plotted in Fig. 9, taken from Haberman (1972). For small λ_c, ϕ is given by

$$\phi \approx -4.2\lambda_c. \tag{B26}$$

The function ϕ can be interpreted as a phase shift in the following manner: For $Y > 0$, we can set $C_+ = 0$ and $D_+ = 0$ without loss of generality and write the outer solution (B7) as

$$\psi_+ = \mathrm{Re}\{[A_+f(\chi) + B_+g(\chi)]e^{isX}\}. \tag{B27}$$

Using (B25), the outer solution for $Y < 0$ is

$$\begin{aligned} \psi_- &= [A_-f(\chi) + B_-g(\chi)] \cos(sX) + [C_-f(\chi)] \sin(sX) \\ &= [A_+f(\chi) + B_+g(\chi)] \cos(sX) + \hat{\beta}_c\phi B_+f(\chi) \sin(sX) \\ &= \mathrm{Re}\{[(A_+ - i\phi\hat{\beta}_cB_+)f(\chi) + B_+g(\chi)]e^{isX}\}. \end{aligned} \tag{B28}$$

Eq. (B28) suggests that if the outer solutions are written in the complex notation as

$$\left.\begin{aligned} \psi_+ &= [Af(\chi) + Bg(\chi)]e^{isX}, \quad Y > 0 \\ \psi_- &= [A'f(\chi) + B'g(\chi)]e^{isX}, \quad Y < 0 \end{aligned}\right\} , \tag{B29}$$

then

$$A' = A - i\phi\hat{\beta}_cB \quad \text{and} \quad B' = B \tag{B30}$$

are the required continuation formulae across the CL. Eq. (B29) can also be written as

$$\left.\begin{aligned} \psi_+ &= e^{isX}\{Af(\chi) + B[1 + \sum_{n=2}^{\infty} b_n\chi^n - \hat{\beta}_c \ln\chi f(\chi)]\} \\ \psi_- &= e^{isX}\{Af(\chi) + B[1 + \sum_{n=2}^{\infty} b_n\chi^n - \hat{\beta}_c(\ln|\chi| + i\phi)f(\chi)]\} \end{aligned}\right\} , \tag{B31}$$

where

$$f(\chi) = \chi + \sum_{n=2}^{\infty} a_n\chi^n.$$

Eq. (B31) suggests that the continuation condition (B30) is equivalent to introducing a "phase shift" ϕ in $\ln\chi$ for $\chi < 0$ as

$$\ln\chi = \ln|\chi| + i\phi,$$

in an analogous manner to the analytic continuation of

$$\ln\chi = \ln(|\chi|e^{-i\pi}) = \ln|\chi| - i\pi$$

that is used for the case $\lambda_c \to \infty$ (viscous limit).

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