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SIMPLE CLIMATE MODELING

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When he was at the University of Washington, Fred Wan established a tradition of teaching mathematical modeling as part of the applied mathematics education. This paper is an expanded version of a topic on mathematical modeling I have taught to undergraduates following Fred's footsteps.

Abstract. We consider a simple climate model of global warming to help understand and constrain predictions from the more comprehensive General Circulation Models (GCMs). By using observations to constrain the climate gain factor, which presents the greatest uncertainty in GCMs, we discuss the atmosphere's response to a doubling of carbon dioxide concentration in the atmosphere in both equilibrium and time-dependent states.

1. **Introduction.** Carbon dioxide is but one of many greenhouse gases naturally occurring in our atmosphere; the others are methane, nitrous oxide, and more importantly, water vapor. These greenhouse gases are responsible for our current global temperature of 15°C. Without them, our global temperature would have been a chilly −17°C. There is no controversy concerning the fact that the carbon dioxide concentration in the atmosphere is increasing steadily. Measurements at the pristine mountain top of Mauna Loa shows a steady increase from 310 parts per million of air in 1958 to our current concentration of 375 ppm. There is even evidence from ice cores that the atmospheric carbon dioxide concentration hovered around 280 ppm for over a thousand years prior to 1800, and then increased rapidly since.

Like a greenhouse, which admits short-wave radiation from the sun through its glass, but traps within the greenhouse the infra-red reemission from inside the greenhouse, the greenhouse gases in the atmosphere warm the lower atmosphere of the earth by keeping in more of the infra-red reemission from the ground. The controversy centers around the following quantitative question: If the carbon dioxide concentration in the atmosphere is doubled, say, from its pre-industrial value of 280 ppm, how much warmer will the global temperature be? It has been estimated that this is equivalent to an additional net radiative heating of the lower atmosphere of 2.6 watts per square meter (Hansen et al., 2005). A back-of-the envelope response can be obtained at equilibrium by balancing this heating with the infrared cooling from a warmer planet. Since the latter is temperature dependent and is approximately equal to $B\delta T$ (which is a linear approximation of the Stefan-Boltzmann law) with $B = 1.9$ watt m^{-2}/°C, we find $\delta T \sim 2.6/1.9 \sim 1.4$°C. This estimate, however,

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does not include the effect of feedbacks involving water in its various forms. Since an increase in surface temperature may lead to more evaporation and more greenhouse gas—namely water vapor—in the atmosphere, the initial warming may be amplified. Similarly, clouds may form and change the radiative balance. At the edges of the ice and snow cover, an initial warming may expose darker surfaces underneath the white ice or snow, leading to more absorption of the sun's radiation. These are called water-vapor feedback, cloud feedback, and albedo feedback, respectively. These processes occur at small scales, below the resolution of the current GCMs, and need to be parameterized as subgrid processes. Such parameterizations are highly uncertain. Their effects differ from model to model, and, even within the same model, from one version to another and from one resolution to another. As a consequence, the model predictions of global warming due to a doubling of carbon dioxide has a rather large range: $\delta T \sim 1.5$ to 4.5°C. Despite intense efforts of hundreds of climate modelers and two Intergovernment Panels on Climate Change (IPCC (1990) and IPCC(2001)), this large range of uncertainty has remained almost unchanged for more than two decades. While a warming of the globe of 4.5°C may be alarming and would be a cause of concern that would prompt a call for action, an eventual warming of 1.5°C may be more benign to human society. Given the high cost of the proposed remedy—involving drastic curbs on the burning of fossil fuels—to nations' industrial production and development, the large scientific uncertainty from model predictions fuels political debates on whether nations should undertake immediate action to curb carbon dioxide emission despite the cost. Recently, efforts have been made to vary model parameters to determine where the likely range should be for models (see Murphy et al., 2004), and to understand better the physical feedback processes involved (see Bony et al., 2006).

In this paper, we will use a simple climate model to understand the source of the uncertainty and discuss a possible way to reduce it. Because the perturbation to the radiative forcing of the atmosphere is small, about 1%, even for doubling carbon dioxide scenarios, a perturbation approach can be used to isolate the feedback factors.

2. **A simple energy balance model.**

The energy balance at the surface of the earth can be stated as: the heat capacity of the atmosphere, expressed as a function of the near-surface temperature $T(y,t)$, is increased by the solar heating that the earth receives at the surface, after taking into account reflection, decreased by the infrared reemission back to space, and increased by the amount of heat transported from other latitudes to the latitude under consideration. Mathematically, it can be written as, in annual mean:

$$R\frac{\partial}{\partial t}T = Qs(y)(1-\alpha(y)) - (A+BT) + \nabla\cdot(\text{HeatFlux}), \tag{1}$$

where Q is the mean solar radiation per unit surface area and is equal to a quarter of the solar constant, $s(y)$ is its distribution with respect to latitude, globally normalized to unity, $y = \sin$ (latitude), and $\alpha(y)$ is the albedo—the fraction of the sun's radiation reflected back to space by clouds and the surface. $(A+BT)$ is the linearized form of the infrared emission from the earth to space fitted from observational data on outgoing longwave radiation (Graves et al, 1993), with $A = 202$ watts m^{-2} and $B = 1.90$ watts m$^{-2\circ}C^{-1}$ for our current climate. The parameter R in Eq.(1) represents the thermal capacity of the atmosphere-ocean climate system. Its value is uncertain and that has prevented the use of the time-dependent version of this equation. Dynamical transport of heat is written in the more general form of a divergence of heat fluxes, so that latitudinal redistribution of heat vanishes when globally integrated. We will consider here the global average to Eq. (1):

$$R\frac{\partial}{\partial t}\overline{T} = Q(1-\overline{\alpha}) - (A + B\overline{T}), \tag{2}$$

where an overbar denotes global average, and $\overline{\alpha} = \frac{1}{2}\int_{-1}^{1}\alpha(y)s(y)dy$ is the weighted global average albedo. Missing in Eq. (2) is the vertical heat flux into the oceans. This will be included later. Henceforth the overbar is dropped for convenience.

Considering a small radiative perturbation δQ in $Q = Q_0 + \delta Q$, we find that the equation governing the small temperature perturbation can be obtained from the first variation of the above equation. We write

$$T = T_0 + \delta T,$$

where T_0 is the unperturbed temperature and δT is the perturbation temperature response to the perturbation in heating δQ. Linearizing Eq. (2) (using a Taylor series expansion of T about T_0) then leads to the following perturbation equation:

$$R\frac{\partial}{\partial t}\delta T = (1-\alpha)\delta Q - B\delta T - (\frac{\partial}{\partial T}A)_0\delta T - (T\frac{\partial}{\partial T}B)_0\delta T - (Q\frac{\partial}{\partial T}\alpha)_0\delta T.$$

This can be rewritten as

$$B\tau\frac{\partial}{\partial t}\delta T = (1-\alpha)\delta Q - B\delta T/f,$$

where

$$\begin{aligned} &\tau \equiv R/B \\ &f = 1/(1-g), \\ &g = g_1 + g_2, \\ &g_1 = (-T\frac{\partial}{\partial T}B - \frac{\partial}{\partial T}A)_0 \\ &g_2 = (-Q\frac{\partial}{\partial T}\alpha)_0 \end{aligned} \tag{3}$$

The factor f is the controversial climate gain; it amplifies any response to radiative perturbation by a factor f (see below). g_1 contains the water-vapor feedback and g_2 contains the ice and snow albedo feedback. Cloud feedback has effects in both g_1 and g_2. The water-vapor feedback factor is potentially the largest and therefore it is the most controversial. The cloud feedback is the most uncertain; even its sign is under debate. Note that various feedback processes can be superimposed in g (but not in f).

Water-vapor feedback. When the surface warms, it is natural to expect that there will be more evaporation and hence that more water vapor will be present in the atmosphere. Since water vapor is a natural greenhouse gas, one expects that the initial warming may be amplified. That is, the factor f should be greater than 1. In one of the earliest models of global warming in 1967, Manabe and Weatherald of Princeton's Geophysical Fluid Dynamics Laboratory made the simplifying assumption that the relative humidity of the atmosphere remains unchanged when the atmosphere warms. This is in effect saying that the atmosphere can hold more water vapor if it is warmer. The presence of this additional greenhouse gas (i.e. water vapor) would amplify the initial warming and double it. That is, the climate gain factor is $f_1 \sim 2$ due to water-vapor feedback alone. This implies a feedback factor of $g_1 \sim 0.5$. This result appears to have stood the test of time. Most modern models yield water-vapor amounts consistent with this prediction. Nevertheless, the fact that most models tend to have similar water-vapor feedback factors does not necessarily mean that they are all correct.

Cloud feedback. Cloud tops reflect visible sunlight back to space. Therefore, more clouds imply higher albedo and cooling. However, clouds also behave like greenhouse gases in trapping infra-red radiation from below. Clouds are actually the second most important greenhouse gas, after water vapor but ahead of carbon dioxide. The degree of cancellation of the albedo effect and the greenhouse effect of the clouds differs in climate models. As a consequence, even the sign of the cloud feedback is uncertain, although typical values in some climate models are around ~ 0.1 for the g factor. It is probably close to zero.

Ice-snow albedo feedback. As the surface warms, snow or ice melts, exposing the darker surface underneath, thus lowering the albedo and increasing the absorption of sun's radiation. This is a positive feedback process and is probably more important at high latitudes than at low latitudes. It may explain the higher sensitivity of the polar latitudes to global warming. On a globally averaged basis it is probably about 0.1 to 0.2 for the g factor.

Total climate gain. Adding all the feedback processes yields $g \sim 0.7$ in most climate models. This then yields a climate gain factor of $f = 1/(1-g) \sim 3$. As noted previously, this number is uncertain.

3. **Using observation to infer climate gain.** There are other forcings to our climate, and studies of these can help us understand how the atmosphere-ocean system responds. The 11-year solar cycle is better measured in terms of its forcing at the top of the atmosphere. Recently the atmosphere's response to such a decadal forcing has also been obtained with increased statistical confidence (see Coughlin and Tung, 2004).

The sun's radiation is observed to vary slightly over an 11-year cycle. This is related to the appearance of darker sunspots and the brighter faculae on the surface of the sun. Sunspots have been observed since ancient times, but an accurate measurement of its radiative variation was not available until recently, when starting in 1979 satellites could directly measure the solar constant S above the earth's atmosphere. It is found that the solar constant varies by about 0.07% over a solar cycle. The atmosphere's response near the surface to this solar cycle variation has also been measured to be about 0.2°C on a global average. This information can be used to infer a climate gain factor. This leads to $f \sim 3$. The calculation is outlined briefly below.

The sun's radiant output fluctuates on an 11-year periodic cycle, which is modeled by:

$$Q = Q_0 + \delta Q, \quad \text{where } \delta Q(t) = a\cos(\omega t), \quad \text{and } Q_0 = 343 \text{ watts m}^{-2}$$

with $\omega = 2\pi/(11 \text{ years})$. The time-dependent equation (3) has a solution for the periodic temperature response of the atmosphere near the surface, $\delta T(t)$, of the form:

$$\delta T(t) = \frac{\delta Q(t-\Delta)\cdot(1-\alpha)f/B}{\sqrt{1+\epsilon^2}}, \tag{4}$$

where $\epsilon = f\omega\tau$, and $\omega\Delta = \tan^{-1}(\epsilon)$. Δ is the time lag of the response, and the factor in the denominator gives the reduction in amplitude from the equilibrium value because of the periodic nature of the response.

The variability of the sun's radiation through the 11-year solar cycle has been measured since 1979 by earth orbiting satellites. We know that the solar constant varies by 0.07% from solar minimum to solar maximum. Thus we know that

$$2a/Q_0 = 0.07\%.$$

So $2a \sim 0.24$ watts m^{-2}. This value should be reduced by a factor of 0.85 because 15% of the total solar radiation is in the UV wavelengths and these are absorbed by ozone in the stratosphere. Since the albedo is about 30%, the amount of solar radiation reaching the surface is probably ~ 0.14 watts m^{-2}. The atmosphere's temperature response is found to lag only slightly (less than a year) and its magnitude is measured near the surface to be about 0.2°C on a global average from minimum to maximum. These values are substituted into Eq. (4) to deduce the climate gain factor f, and we find that it should be about $f \sim 3$.

The observational evidence of atmosphere's response to the decadal radiative forcing arising from solar cycles points to the conclusion that our climate system's response includes a net positive feedback, which amplifies the warming by a factor of 3.

4. **Equilibrium global warming.** Let's now return to the global warming problem. Setting the time derivative to zero, the steady state solution to (3) is:

$$(\delta T)_{\text{eq}} = \frac{(1-\alpha)\delta Q}{B} f. \tag{5}$$

The solution shows prominently the climate gain factor f in amplifying the equilibrium response to a given radiative forcing. For an "adjusted radiative forcing" of $(1-\alpha)\delta Q = 2.6$ watts m^{-2} due to doubling CO_2, the expected global warming is 1.4°C without the amplifying factor but 4.1°C with the amplifying factor of $f \sim 3$.

The range of current model predictions of 1.5-4.5°C indicates that the various models have different feedback mechanisms implying that the climate gain factor in these models ranges from $f \sim 1$ to 3.

5. **Time-dependent global warming.**

Growth phase. As a model for the increase in greenhouse gases we assume that their radiative forcing has increased linearly since 1800, which we call $t = 0$:

$$(1-\alpha)\delta Q(t) = bt, \quad \text{for } t > 0 \quad \text{and} \quad \delta Q(t) = 0, \quad \text{for } t < 0. \tag{6}$$

This is the model considered by Hartmann (1994). It leads to a linearly increasing global warming. Given the recent accelerated warming trend, a model giving rise to an exponentially increasing temperature may be more appropriate. That model is discussed in the Appendix. Staying with (6), the solution to the time-dependent equation (3) is now obtained. In view of the form of the forcing term, we assume the solution to consist of a homogeneous solution plus a particular solution. The particular solution is of the form: $\delta T_{\text{particular}} = at - c$, and the homogeneous solution is of the form: $\delta T_{\text{homogeneous}} = c\exp\{-t/(g\tau)\}$. (The two c's are of opposite sign so as to satisfy the initial condition that the total temperature perturbation is zero at

$t = 0$.) The constants a and c are found by substituting this assumed solution into Eq. (3). This yields, for the sum of the homogeneous and the particular solutions:

$$\delta T(t) = (bf/B)(t - f\tau) + (bf^2\tau/B)\exp\{-t/f\tau\}, \quad \text{for } t > 0. \tag{7}$$

Eq. (7) can be written in the following more interesting form:

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t-\Delta)}{B} f,$$

where $\Delta \equiv f\tau(1 - \exp\{-\frac{t}{f\tau}\})$ is the time delay. The delay is initially zero at $t = 0$ and increases steadily to a maximum of $f\tau$ for $t \gg f\tau$. This form of the solution looks just like the equilibrium solution (5) except that it is evaluated at time t using the value of the radiative forcing at $t - \Delta$. We call this the quasi-equilibrium solution with delay. For $t > f\tau$, $\Delta \to f\tau$. Then the solution can be written in the following simple form:

$$\delta T(t) \cong \frac{(1-\alpha)\delta Q(t-f\tau)}{B} f, \quad \text{for } t > f\tau. \tag{8}$$

Curbs in effect. Suppose at some $t = t_s$ in the future, all nations decide to implement a curb on emission of greenhouse gases. For simplicity, we assume that the emission curbs are such that the concentration of the greenhouse gases in the atmosphere remains constant:

$$\delta Q(t) = \text{constant for } t > t_s. \tag{9}$$

The solution for the constant forcing case can be found using a particular solution which is a constant. This constant is found by substituting this trial particular solution into Eq. (3):

$$\delta T_{\text{particular}} = (1-\alpha)\delta Q(t_s) f/B.$$

The homogeneous solution is the same as before: $\delta T_{\text{homogeneous}} = c\exp\{-t/(f\tau)\}$, but now the constant c needs to be evaluated so that the solution at ts matches that from (8). This yields:

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t_s) f}{B}\left[1 - (\frac{f\tau}{t_s})\exp\{-\frac{t-t_s}{f\tau}\}\right]. \tag{10}$$

We see that eventually the warming will approach the equilibrium value predicted by Eq. (5), and that warming will be amplified by the climate gain factor f. However, it takes a time longer than $f\tau$ to reach that equilibrium. We now have a conclusion that is consistent with what other scientists have found using more complex computer model simulations (see Hansen et al., 1985), and is rather general:

> The more sensitive the climate response (the larger climate gain factor), the larger the global warming at equilibrium will be. However it also takes longer to reach that equilibrium.

Next we will try to determine how long is “long”.

6. **Thermal inertia of the atmosphere-ocean system.** Before we can gain any insight from the time-dependent solution, we need to estimate the thermal capacity $R = B\tau$ of the atmosphere-ocean system. This is very uncertain because we do not know how deeply the warming would penetrate into the ocean. If the response of the climate system involves deep ocean circulations, the climate response time may be of the order of centuries. This is currently a subject of intense study using state-of-the-art coupled atmosphere-ocean general circulation computer models.

Because of the inertia, the radiative budget of our climate system at present is not balanced. That is, the earth currently receives more solar energy (in the first term on the right-hand side of Eq. (3)) than it radiates back to space (in the second term in that same equation). This radiative imbalance is estimated using a combination of observations and a GCM to be 0.85 ± 0.15 watts per square meter by Hansen et al (2005) for 2003. The uncertainty is probably larger than what was implied by the error bar. Nevertheless, since we don't have any better measurements we will use this value and see what it implies. The imbalance is due to the thermal inertia of our climate system. This we have modeled by the left-hand side of Eq. (3). Since the right hand side of Eqs.(3) represents the difference between the radiative input and output of the earth, the left hand side can be estimated from this imbalance, yielding, for 2003 values:

$$R\frac{\partial}{\partial t}\delta T \approx 0.85 \text{ watts m}^{-2}.$$

The earth has warmed globally by 0.6C$\pm$0.2°C from 1880 to 2003 (IPCC 2001). The time-like quantity τ can now be assigned a value:

$$\tau \equiv R/B \approx 0.85/[(1.90)(0.6/123)] \approx 90 \text{ years}.$$

Taking the extremes of the error bars in the parameter inputs, this estimate of τ can range from 57 years to 162 years.

As can be seen in the time-dependent solution (8) and (10), the lag time for the climate system response is not τ, but rather $f\tau$, which is $\sim$ 170-490 years for a climate gain factor of $f \sim 3$.

> It probably takess more than 200 years after the greenhouse gases have been curbed for our climate system to reach the predicted equilibrium! If the carbon dioxide is doubled and maintained at that level for more than 200 years, we will reach a global warming of 4°C. In the meantime, that predicted equilibrium warming is less relevant.

This estimate of τ depends somewhat on our assumption of linear increase of temperature and greenhouse heating. An exponentially increasing model is considered in the Appendix.

7. **Heat flux into the ocean.** Global averaging of Eq. (2) eliminates the horizontal fluxes of heat, but not its vertical flux. Therefore, there is a term missing in Eq. (3). The full equation is:

$$R\frac{\partial}{\partial t}\overline{T} = Q(1-\overline{\alpha}) - (A + B\overline{T}) + \frac{\partial}{\partial z}\overline{F}_z, \tag{11}$$

where the overbar denotes global averaging. Eq.(11) states that the heat content of the atmosphere is increased by radiative forcing (first term on the right) and by heat flux to the oceans below (the last term), and decreased by infrared emission to space above (second term). Henceforth the overbar is dropped for convenience. Considering a small radiative perturbation δQ in $Q = Q_0 + \delta Q$, the equation governing the small temperature perturbation can be obtained from the first variation of the above equation, with B and α expanded in Taylor series in T. This leads to the following perturbation equation:

$$B\tau\frac{\partial}{\partial t}\delta T = (1-\alpha)\delta Q - B\delta T/f + \frac{\partial}{\partial z}\delta F_z, \tag{12}$$

where

$$f = 1/(1-g),$$
$$g = (\frac{T}{B}\frac{\partial}{\partial T}B - \frac{Q}{B}\frac{\partial}{\partial T}\alpha)_0$$

The last term in Eq.(12) is difficult to model simply. In simple climate models another box is added at the bottom of the atmosphere to represent the ocean. Often the ocean box needs to have two layers, the upper mixed layer, where heat is diffused, and a lower deep-water layer, where a cold temperature is maintained. When the atmosphere and ocean reach an equilibrium, there will be no net heat flux into the oceans, and the last term in Eq. (12) will vanish. Therefore, in a way, the heat flux term is a transient term, and by ignoring it in our previous Eq. (3), we are in effect lumping it with the transient term on the left-hand side of Eq. (12). For this reason, the heat capacity R in Eq. (3), and hence the inertia time-scale τ, would have to be different for different phenomenon under consideration. That is, we cannot use one phenomena to find R and use the same value of R for another phenomenon when the ocean heat flux is not explicitly modeled. For example, using phase-lags observed for solar cycle response, we have found that τ for that case to be less than a year. Yet, we have found that τ for the case of global warming should be around 90 years.

8. **Conclusion.** There are a number of limitations with the simple climate models we have highlighted in this paper: the use of a single temperature, the near surface temperature, to characterize the state of the atmosphere; and the treatment of heat exchanges between the atmosphere and the ocean. Nevertheless, as an educational tool, it helps isolate where the controversy in the global warming debate lies and suggests ways to constrain the uncertain parameters using observations. Because of the available simple analytical solution, the simple model also gives a much clearer grasp to a larger portion of the scientific community of the important parameter dependence of the solutions. In particular, it makes clear how the solution changes depending on how rapid the greenhouse gases are increasing, that the time scale involved in approaching equilibrium should be multiplied by the climate gain factor, and that the effective time scale is about 200 years.

Appendix. Exponentially increasing temperature.

Time-dependent global warming. We consider the scenario of a period of radiative perturbation growing with rate b:

$$\delta Q(t) = a\exp(bt) \quad \text{for} \quad -\infty < t < t_s,$$

before a policy action to curb the growth at a future time $t = t_s$:

$$\delta Q(t) = \delta Q(t_s) \quad \text{for } t > t_s.$$

By solving Eq.(3), we find that the atmosphere's response to this forcing, subject to the initial condition $\delta T(-\infty) = 0$, is:

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t)}{B} f \frac{1}{(1+\gamma)} \quad \text{for} \quad t < t_s \tag{13}$$

and

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t_s)}{B} f[1 - \exp\{-\frac{(t-t_s)}{f\tau}\}\frac{\gamma}{1+\gamma}] \quad \text{for } t > t_s, \tag{14}$$

where $\gamma = b(R/B)f$.

Asymptotic limits of the global warming solution. The nature of the solution obtained here depends on the non-dimensional quantity $\gamma = b(R/B)f = bf\tau$. This is a measure of how fast the forcing is increasing relative to the natural response time of the atmosphere-ocean system.

To help understand the exact solution, we next consider the solution in different asymptotic limits with respect to γ.

a. *The slow growth limit,* $\gamma \ll 1$: The solution is given approximately by:

$$\delta T = \frac{(1-\alpha)\delta Q(t)}{B} f \quad \text{for } t < t_s, \quad \text{and for } t > t_s,$$

which is in the same form as the equilibrium solution, except with instantaneous forcing. We call this the quasi-equilibrium solution.

b. *The rapid growth limit,* $\gamma \gg 1$. The solution becomes:

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t)}{B} \cdot \frac{1}{b\tau} \quad \text{for } t < t_s$$

$$\delta T(t) = \frac{(1-\alpha)\delta Q(t_s)}{B} f[1 - \exp\{-\frac{(t-t_s)}{f\tau}\}] \quad \text{for } t > t_s.$$

The surprising result is that the climate's response to rapidly increasing radiative forcing is independent of the climate gain factor f. For $t > t_s$, the time-dependent solution is independent of γ and the time scale for approach to equilibrium is given by $f\tau$.

The current emission values seem to place the real value of $\gamma \sim 2.3$ to be somewhere between these two asymptotic limits, closer to (b) than (a). This explains why it is so difficult to use the observed rise in temperature due to increases in greenhouse gases to constrain the climate gain factor f.

The constant CO_2 phase, $t > t_s$. After the increases in the greenhouse gases have been capped, the slow approach to the steady state takes the form of (14) for any value of γ. It is seen that the solution will eventually approach the equilibrium value:

$$\delta T(t) \to (\delta T)_{\text{eq}} = \frac{(1-\alpha)\delta Q(t_s)}{B} f,$$

but the approach is very slow. The ratio of actual temperature at any time $t > t_s$ and the predicted equilibrium solution is given by

$$\frac{\delta T(t)}{(\delta T)_{\text{eq}}} = [1 - \exp\{-(t-t_s)/(f\tau)\} \frac{\gamma}{1+\gamma}]$$

When this factor approaches one, the equilibrium prediction is approached. For large value of γ this ratio becomes independent of γ dependent only on the e-folding time of $f\tau$. For $f\tau = 280$ years, this asymptotic ratio is 30% in 100 years and 83% after 500 years.

For the case of $\gamma = 2.3$, we have the following estimates: The time-dependent warming at double CO_2 is 1.2°C from (13), a factor of 3.3 less than the predicted equilibrium solution of 4.1°C. If the level of CO_2 is then maintained at that constant level, the warming will become (from (14)) 2.1°C in 100 years, increasing to 3.6°C in 500 years.

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