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ON THE DIFFERENCES BETWEEN 2D AND QG TURBULENCE

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Abstract. Due to their mathematical tractability, two-dimensional (2D) fluid equations are often used by mathematicians as a model for quasi-geostrophic (QG) turbulence in the atmosphere, using Charney's 1971 paper as justification. Superficially, 2D and QG turbulence both satisfy the twin conservation of energy and enstrophy and thus are unlike 3D flows, which do not conserve enstrophy. Yet 2D turbulence differs from QG turbulence in fundamental ways, which are not generally known. Here we discuss ingredients missing in 2D turbulence formulations of large-scale atmospheric turbulence. We argue that there is no proof that energy cannot cascade downscale in QG turbulence. Indeed, observational evidence supports a downscale flux of both energy and enstrophy in the mesoscales. It is suggested that the observed atmospheric energy spectrum displays such a downscale energy cascade of QG turbulence, and is therefore inconsistent with 2D turbulence theories. A simple solved example is used to illustrate some of the ideas discussed.

1. **Introduction.**

Two-dimensional incompressible fluid flows satisfy the following equation in the absence of forcing and dissipation:

$$\frac{\partial}{\partial t}\omega + J(\psi, \omega) = 0, \tag{1}$$

where

$$\omega = \nabla^2\psi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi$$

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is the vertical component of vorticity, (x, y) are the horizontal coordinates, and ψ is the streamfunction for the horizontal velocities $(u, v) = \left(-\frac{\partial}{\partial y}\psi, \frac{\partial}{\partial x}\psi\right)$. $J(A, B) \equiv \frac{\partial}{\partial x}A\frac{\partial}{\partial x}B - \frac{\partial}{\partial x}B\frac{\partial}{\partial y}A$. The mechanism of vortex stretching, which is paramount in transferring energy from large to small scales in 3D turbulence, is absent in 2D turbulence.

Large-scale atmospheric motion is confined in a shallow layer of fluid, whose horizontal dimension is of the order of the radius of the earth ($a = 6,400$km), while its vertical dimension is measured, depending on application, by either the scale height ($H = 7$km, the scale over which the density of air decreases by a factor of e) or by the thickness of the troposphere ($D = 10$km). Consequently most of the kinetic energy of large-scale motion is contained in the horizontal velocities, the vertical velocity, w, being smaller by at least a factor of the aspect ratio: $\delta = H/a \sim 10^{-3}$. This fact is often used by mathematicians to proclaim, näively as it turns out, that large-scale geophysical flows should be "quasi two-dimensional", and hence could be modeled by 2D equations such as Eq. (1). The mathematical issue is more subtle than that, and this was pointed out by Charney (1947) in his now classic paper on quasi-geostrophic scaling.

For QG dynamics, the vorticity equation (1) is modified to

$$\frac{D}{Dt}(\omega + f) = -f\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v\right), \tag{2}$$

where f is the vertical component of the planetary vorticity, $\omega \equiv \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u$ is the vertical component of the fluid vorticity relative to the rotating frame of reference, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

In a truly 2D flow, the right-hand side of Eq. (2) is zero if the fluid is incompressible ($\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$). In that case, Eq. (2) expresses the conservation of absolute vorticity, and it reduces to Eq. (1) if the y-variation of f is ignored. A QG flow, however, has three dimensions, and thus the incompressibility condition is

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = -\frac{\partial}{\partial z}w. \tag{3}$$

[Ambient density variation is ignored here, but will be inserted later.] The right-hand side is not negligible in the large-scale atmosphere even though $w = O(\delta \cdot u)$[1], because $\frac{\partial}{\partial z} = O\left(\frac{1}{\delta}\frac{\partial}{\partial x}\right)$. Eq. (2) becomes

$$\frac{D}{Dt}(\omega + f) = f\frac{\partial}{\partial z}w. \tag{4}$$

[1]For large-scale flows on a fast rotating planet, like the earth, w should actually be $O(\delta \cdot Ro \cdot u)$, where $Ro = (U/L)/f$ is the Rossby number ($Ro \sim 0.1$), U is a typical horizontal velocity and L a typical horizontal length scale. Nevertheless, the horizontal divergence, i.e. the left-side of Eq. (3), turns out to be zero at leading order in Ro, and so (3) is balanced at $O(Ro)$.

The right-hand side of (4) is the vortex stretching term mentioned earlier, which is important in 3D turbulence. A horizontally convergent flow $\left(\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v < 0\right)$ stretches the vortex in the vertical direction $\left(\frac{\partial w}{\partial z} > 0\right)$, tending to increase its relative vorticity $\left(\frac{D}{Dt}\omega > 0\right)$. In this respect, QG turbulence is akin to 3D turbulence.

On the other hand, like 2D turbulence, QG turbulence conserves a vorticity-like quantity, q, called the potential vorticity (Charney originally called it pseudo-potential vorticity):

$$\frac{D}{Dt}q = \frac{\partial}{\partial t}q + J(\psi, q) = 0, \tag{5}$$

where

$$q = \nabla^2\psi + f + \frac{f^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial}{\partial z}\psi\right) \equiv L(\psi) + f \tag{6}$$

$\rho_0(z)$ the ambient density of air,

$N^2(z)$ the Brunt-Väisälá frequency.

In deriving (5), the anelastic approximation is used because air is not really incompressible. This involves replacing the right-hand side of the incompressibility condition (3), by $-\frac{1}{\rho_0}\frac{\partial}{\partial z}\rho_0 w$. Also, the temperature (T) equation is used.

$$\frac{D}{Dt}T = -\frac{T_0 N^2}{g}w, \tag{7}$$

which says that in a stably stratified atmosphere, sinking air ($w < 0$) is adiabatically warmed $\left(\frac{D}{Dt}T > 0\right)$. The vorticity equation then becomes

$$\frac{D}{Dt}\left(\omega + f + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0 g}{N^2 T_0}T\right)\right) = 0.$$

The conserved quantity, $(\omega + f + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\frac{\rho_0}{N^2}\frac{g}{T_0}T))$, is seen to be q when the hydrostatic relation

$$T = \frac{T_0 f}{g}\frac{\partial}{\partial z}\psi \tag{8}$$

is used to express temperature T in terms of the streamfunction ψ.

Herein lies the dilemma: Is QG turbulence more like 3D turbulence in its ability to transfer energy from large to small scales — since the vortex stretching mechanism remains intact in QG equation (4) — or is QG turbulence more like 2D turbulence in its *inability* to transfer energy from large to small scales since the twin conservation of energy and some form of vorticity has been argued to prevent such a downscale transfer?

The prevailing view has been that QG turbulence is more like 2D turbulence. As a result, intensive studies of the analytical theories of 2D turbulence have been justified by their applicability to the atmosphere. Atmospheric data are routinely analyzed and presented in 2D-like quantities (e.g. kinetic energy, relative enstrophy and their fluxes), treating each atmospheric layer as if it were a 2D fluid and lumping baroclinic terms as forcing (Boer and Shepherd, 1983; Straus and Ditlevsen, 1999; Lindborg, 1999). Much of this perspective has its origin in a Note by Charney (1971), which laid the foundation for QG turbulence.

However, as will be discussed here, QG turbulence has a rich range of behaviors, from 2D-like to 3D-like. The fact that there is the twin conservation of total energy (kinetic plus potential energy) and potential enstrophy (half the square of potential energy) does not play as important a role as previously thought in determining the direction of energy cascade.

In section 2 the conservation of energy and enstrophy is investigated mathematically for the case of 2D turbulence, and previous arguments for the direction of energy cascade are investigated. The same appears for QG turbulence in section 3. Section 4 discusses the issue of anomalous dissipation for 2D, 3D, and QG turbulence and its implications for the cascade direction. An example of a forward energy cascade from surface quasi-geostrophy (a special case of QG) is presented in section 5. Finally, the historical progression of observations and arguments for atmospheric turbulence is included in section 6 and conclusions in section 7.

2. **Energy and enstrophy conservation in 2D.**

In 2D turbulence, there is conservation of both (kinetic) energy and enstrophy (half the square of vorticity) (Batchelor, 1953). It was thought that nonlinear (triad) interactions among the different scales satisfying this twin constraint could only direct energy upscale. The proof originated in an important study by Fjørtoft (1953), but it can now be shown to be inadequate in determining the direction of energy flow, as we will outline below.

However, it does turn out that – for reasons not discussed by Fjørtoft – 2D flows do not transfer energy to the small scales *when viscosity vanishes.* These facts are not commonly known and Fjørtoft's proof has been repeated and the misunderstanding carried over to QG turbulence. QG flows, however, do not in general behave like 2D with respect to the direction of energy cascade.

Multiplying Eq. (1) by ψ and integrating over x and y yields the energy constraint:

$$\frac{d}{dt}\mathcal{E} \equiv \frac{d}{dt}\iint \frac{1}{2}\nabla\psi\cdot\nabla\psi dxdy = 0, \tag{9}$$

if the domain either (i) is doubly periodic, (ii) has no normal flows at the boundaries, or (iii) satisfies a combination of the two. In the following, we shall assume a doubly periodic boundary condition for simplicity and for later adoption of a Fourier spectrum.

2.1. **Relation between energy and enstrophy densities.**

By the Parseval's theorem:

$$\iint \frac{1}{2}\nabla\psi\cdot\nabla\psi dxdx = \iint \frac{1}{2}(k_x^2+k_y^2)|\hat{\psi}(\mathbf{k})|^2 dk_x dk_y,$$

where

$$\psi(x,y,t) = \frac{1}{4\pi^2}\iint \hat{\psi}(k_x,k_y,t)e^{i(k_x x+k_y y)}dk_x dk_y.$$

The Fourier integrals are interpreted as a sum when the wavenumbers $\mathbf{k} = (k_x, k_y)$ are discrete.

Therefore the spectral energy density

$$E(\mathbf{k}) = \frac{1}{2}(k_x^2 + k_y^2)|\hat{\psi}(\mathbf{k})|^2 \equiv \frac{1}{2}k^2|\hat{\psi}(\mathbf{k})|^2 \tag{10}$$

satisfies

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\iint E(\mathbf{k})dk_x dk_y = 0, \tag{11}$$

and so energy density can only be redistributed among wavenumbers without net gain or loss.

Similarly, if we multiply Eq. (1) by $\nabla^2\psi$ and integrate over the x, y domain, we arrive at enstrophy conservation:

$$\frac{d}{dt}\mathcal{G} \equiv \frac{d}{dt}\iint \frac{1}{2}(\nabla^2\psi)^2 dxdy = 0 \tag{12}$$

In spectral form, we have

$$\frac{d}{dt}\mathcal{G} = \frac{d}{dt}\iint G(\mathbf{k})dk_x dk_y = 0, \tag{13}$$

where

$$G(\mathbf{k}) = \frac{1}{2}k^4|\hat{\psi}(\mathbf{k})|^2 \tag{14}$$

is the spectral enstrophy density. $G(\mathbf{k})$ and $E(\mathbf{k})$ are related by:

$$G(\mathbf{k}) = k^2 E(\mathbf{k}). \tag{15}$$

2.2. **Fjørtoft's proofs.** The relationship (15) and the fact that in any triad exchange, $G(\mathbf{k})$ and $E(\mathbf{k})$ are individually conserved, were thought to determine the direction of energy cascade in 2D turbulence. Fjørtoft's proof in (1953), later repeated by Charney (1971) for QG turbulence and in textbooks, is really just a convergence requirement for a spectral representation of enstrophy. That is, since the enstrophy is conserved, the integral

$$\mathcal{G} = \iint k^2 E(\mathbf{k})dk_x dk_y$$

must converge. This in turn requires that the energy spectrum, $E(\mathbf{k})$, must decay for large k faster than $O(k^{-3})$. However, it does not say, as Fjørtoft claimed, that energy therefore cannot be transferred to the small scales.

Furthermore, it is clear that in forced-dissipative systems, energy should flow from where it is forced to where it is dissipated; neither of these two pieces of information is contained in (15). Unforced inviscid flow, on the other hand, is completely reversible. So, in the unforced inviscid case, unless an additional *ad hoc* (probabilistic) statement is advanced which determines the direction of the time arrow (e.g. Rhines, 1979, Batchelor, 1953), any proof which purports to show that energy can go in one direction can be reversed to "prove" that energy should go in the opposite direction by reversing time.

The essence of these arguments can be seen from a simple example used by Fjørtoft himself. (Fjørtoft's presentation contains a typo; Salmon (1998)'s version is used here.)

Suppose initially the wavenumber k_1 has energy E_1, which is subsequently transferred via triad interaction to two other wavenumbers: $k_0 = k_1/2$ and $k_2 = 2k_1$. By conservation of energy,

$$E_1 = E_0 + E_2, \tag{16}$$

and by conservation of enstrophy,

$$k_1^2 E_1 = k_0^2 E_0 + k_2^2 E_2. \tag{17}$$

Solving, we get

$$E_0 = \frac{4}{5} E_1 \text{ and } E_2 = \frac{1}{5} E_1.$$

It therefore appears that 80% of the energy ends up in the lower wavenumber. Hence it was concluded that if any fraction of the initial energy is to be transferred downscale, "a greater fraction simultaneously has to flow to components with larger scales", resulting in a net upscale energy transfer.

Suppose now we switch what is "initial" and what is "subsequent". This involves nothing more than switching the left- and right-hand sides of Eq. (16) and of Eq. (17). But now we have the scenario of two wavenumbers, k_0 and k_2, with initial energies E_0 and E_2, interacting with a third intermediate wavenumber k_1, which ends up with $E_1 = \frac{5}{4} E_0 = 5E_2$. Suppose E_0 has more energy than E_2 initially, say $E_0 = 4E_2$. So of the $5E_2$ units of energy ending up in k_1, one E_2 unit was transferred upscale from k_2, while $4E_2$ units were transferred downscale from k_0. This would have implied a net *downscale* energy cascade of $3E_2$ units!

The above example shows that conservation principles alone are inadequate in showing the direction of energy transfer in either direction. In addition, it presupposes a certain triad which transfers the entire initial energy. In general, the initial energy E_1 at k_1 can be transferred by many possible triads, each taking away a fraction, ΔE_1, of the initial energy, while at the same time other triads may move energy into k_1. For this reason as well, ΔE_1 can be of either sign.

What the constraints of twin conservation and (15) actually imply is that (Merilees and Warn, 1972): "energy and enstrophy in 2D non-divergent flow cascade both to lower and higher wavenumbers", but "the majority of interactions are such that more energy flows to *and from* smaller wavenumbers while more enstrophy flows to and from larger wavenumbers." [Emphasis added.] The net direction depends on other factors. For forced-dissipative cases it should depend on the dissipation operator, to be discussed later.

3. **Energy and enstrophy conservation in QG.** The conservation laws for QG flows are due to Charney (1971). There have been concerns that the derivation given by him is too restrictive, as the isothermal boundary

condition adopted precludes the important atmospheric processes of baroclinic instability and frontogenesis. To avoid this, the derivation given here is more general.

Charney obtained an energy equation by multiplying Eq. (5) by $-\rho_0\psi$ and integrating over the x, y, z domain. The range in z is semi-infinite, from 0 to ∞; the x-domain is periodic; the y-domain is either periodic or satisfies no normal velocity at the y-boundaries. This yields:

$$\begin{aligned}\frac{d}{dt}\mathcal{E} &\equiv \frac{d}{dt}\iiint \frac{1}{2}[\nabla\psi\cdot\nabla\psi + \frac{f^2}{N^2}(\frac{\partial}{\partial z}\psi)^2]\rho_0 dxdydz \\ &= \iint \frac{f^2}{N^2}\rho_0\psi\frac{\partial}{\partial t}\frac{\partial\psi}{\partial z}\bigg|_0^\infty dxdy = 0 \end{aligned} \tag{18}$$

We note that, in setting the boundary terms above at $z = 0, \infty$ to zero, Charney (1971) used the following vertical boundary conditions:

$$\rho_0\psi\frac{\partial}{\partial z}\psi \to 0 \text{ as } z \to \infty \text{ and } \frac{\partial}{\partial z}\psi = 0 \text{ at } z = 0. \tag{19}$$

The Neumann boundary condition at $z = 0$ is equivalent to assuming that $z = 0$ is an isothermal surface (see Eq. (8)), which is unrealistic for atmospheric applications. A more general lower boundary condition is that of no normal flow, which is $w = 0$ at $z = 0$ in the absence of topography. With topography, it should be

$$w = J(\psi, h), \quad \text{at} \quad z = 0 \tag{20}$$

where

$$z = h(x, y)$$

is the topographic elevation of the lower surface. (Consistent with QG scaling, the boundary condition (20) is evaluated at $z = 0$ instead of $z = h$.) The presence of a topography should not affect energy conservation, because the boundary condition of $w = 0$ is still homogeneous and does not permit flow of energy through the boundary. Using the temperature equation (7), we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z}\psi = -J(\psi, \frac{\partial}{\partial z}\psi + \frac{N^2}{f_0}h), \text{ at } z = 0.$$

This, when integrated over the x,y domain, vanishes. Thus energy conservation (18) is recovered even for the more general lower boundary condition (20), the same as for the specific isothermal condition (19b). The energy integral constraint has previously been derived in the general case by Liu *et al.* (1996).

In QG dynamics, the total energy (from (18)) consists of two parts, the kinetic energy:

$$\frac{1}{2}\rho_0\nabla\psi\cdot\nabla\psi = \frac{1}{2}\rho_0(u^2+v^2),$$

and the available potential energy:

$$\frac{1}{2}\frac{f^2}{N^2}\rho_0(\frac{\partial}{\partial z}\psi)^2 = \frac{1}{2}\rho_0\frac{g^2}{N^2}(\frac{T}{T_0})^2.$$

It is the sum of these two which is conserved in QG flows.

A potential enstrophy conservation law is obtained by multiplying Eq. (5) by q and integrating over the x, y, z domain (the z-integration, it turns out, is not necessary):

$$\frac{d}{dt}\mathcal{G} \equiv \frac{d}{dt}\iiint \rho_0 (L(\psi))^2 dxdydz = 0. \tag{21}$$

This involves only the relative potential vorticity $L(\psi)$. (The elliptic operator L was defined in (6).)

3.1. **Relation between total energy and potential enstrophy densities.** We suppose for the moment that the z-domain is either infinite (instead of semi-infinite), or periodic, and that the z-variations of ρ_0 and N^2 are ignored. Then with the Fourier representation

$$\psi(x,y,z,t) = \frac{1}{8\pi^3}\iiint \hat{\psi}(\mathbf{k})e^{i(k_x x + k_y y + k_z z)}dxdydz$$

(where the integral is to be interpreted as a sum for discrete wavenumbers), the spectral conservation laws are:

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\iiint E(\mathbf{k})dk_x dk_y dk_z = 0 \tag{22}$$

with

$$E(\mathbf{k}) = \frac{1}{2}(k_x^2 + k_y^2 + \frac{f_0^2}{N^2}k_z^2)|\hat{\psi}(\mathbf{k})|^2 \tag{23}$$

being the spectral energy density, and

$$\frac{d}{dt}\mathcal{G} \equiv \frac{d}{dt}\iiint G(\mathbf{k})dk_x dk_y dk_z = 0, \tag{24}$$

with

$$G(\mathbf{k}) = \frac{1}{2}(k_x^2 + k_y^2 + \frac{f_0^2}{N^2}k_z^2)^2|\hat{\psi}(k)|^2 \tag{25}$$

being the spectral potential enstrophy density. (23) and (25) suggest the relationship between $E(\mathbf{k})$ and $G(\mathbf{k})$:

$$G(\mathbf{k}) = \tilde{k}^2 E(\mathbf{k}), \tag{26}$$

where

$$\tilde{\mathbf{k}} = (k_x, k_y, \frac{f_0}{N}k_z)$$

is a "stretched" wavenumber (N/f_0 is typically 100 in the atmosphere).

3.2. **Isomorphism between 2D and QG.** Eq. (26) is strikingly similar to its 2D version (15). It is in fact mathematically identical for QG flows which are *isotropic* in the three stretched coordinate directions $(x, y, \frac{N}{f_0}z)$, and 2D flows isotropic in (x, y). [The isotropic assumption allows for dependence only on the magnitude of the wavenumber, k (or $\tilde{k}$) instead of the components of the vector $\mathbf{k}$ (or $\tilde{\mathbf{k}}$). The isotropy in the three stretched coordinates is termed "Charney isotropy" by McWilliams *et al.* (1999)]

It is this isomorphism between QG and 2D flows which prompted Charney to conclude that an energy cascade to small scales is impossible in QG turbulence, borrowing a proof from Fjørtoft (1953) on 2D turbulence. However, we showed in Section 2 that this proof is problematic. Furthermore, because of the rather small aspect ratio of the atmosphere mentioned in section 1, QG flows in the troposphere cannot be isotropic in 3D even in the stretched coordinates $(x, y, \frac{N}{f_0}z)$, except for horizontal scales less than a few hundred kilometers. However, Charney's remarkable insight is finding application in the oceanic context, where the horizontal scales are smaller. In numerical simulations of decaying turbulence in triply periodic domains, Herring (1980) found that scales shorter than that of the peak in initial energy evolve to become Charney-isotropic, albeit rather slowly. That the QG flow, which is advected only by the horizontal velocities, can actually become isotropic in three (stretched) dimensions is surprising, but it confirms Charney's insight and his explanation for the observed equipartition of components of energy in each of the three stretched dimensions (Boer and Shepherd, 1983; Nastrom and Gage, 1985)[2]. Nevertheless it still does not follow that QG turbulence in this case should behave like 2D turbulence.

The fact that one QG equation, namely (26), takes the same form as its 2D counterpart (15), does not by itself lead to isomorphism of the two cases. There is another criterion. In triply periodic domains (or in doubly periodic domain with isothermal horizontal boundaries) and in cases where the flow is Charney-isotropic, a case can be made that the QG flow is isomorphic to 2D flow only if the dissipation operator in QG is of a particular form. That is, if the viscous term added to the right-hand side of the 2D vorticity equation (1) is $\nu\nabla^4\psi$, then the viscous term added to the right-hand side of the QG potential vorticity equation (5) should be of the rather contrived form: $\nu(\nabla^2 + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2})^2\psi$. This will be discussed further in section 4.

3.3. **Charney's proofs.** Charney (1971) also gave a separate proof of prohibition of net positive energy cascade in a semi-infinite z domain, which does not rely on the isotropic assumption. The proof involves expanding ψ in an infinite series of eigenfunctions ψ_m

$$\psi = \sum a_m \psi_m,$$

defined by the Helmholtz eigenvalue problem (with λ_m as the eigenvalue):

$$L(\psi_m) = -\lambda_m \psi_m$$

subject to the same boundary conditions for ψ in x and y and using (19) for z. Tung and Welch (2001) pointed out that the eigenfunctions $\{\psi_m\}$ are *incomplete*. In fact there is only one eigenfunction, and so the proof is not applicable. As explained in Tung and Welch (2001), the lack of completeness in the set of eigenfunctions defined by Charney is caused by the fact that the eigenvalue problem involves a singular Sturm-Liouville problem, which

[2]In QG flow, the component of energy related to the vertical wavenumber in stretched coordinates is actually the available potential energy; see Eq. (24)

is overspecified by the boundary condition (19) at the singular point $z = \infty$. If that (upper) boundary condition is replaced by one of boundedness of $\rho_0^{1/2}\psi$ as $z \to \infty$, an additional set of continuous eigenvalues would result, making the eigenfunctions complete. However, the continuous eigenvalues correspond to radiating waves, which do not conserve the total energy $\mathcal{E}$ in (18). The zero upper boundary condition is necessary for containing the energy in the domain. The problem of incompleteness it raises cannot be circumvented by altering the lower boundary condition to any other linear homogeneous condition (including the more general one we derived earlier in this section).

While this difficulty probably cannot be overcome in the semi-infinite domain while still preserving energy conservation[3], ψ in a finite domain, say only in the troposphere, $0 < z < D$, by a complete set of eigenfunctions defined in the truncated domain. This then allows Charney's proof to go forward, but that proof will ultimately fail because it is based on Fjørtoft's convergence idea. Therefore, Charney's conclusion: "All the other theorems pertaining to energy exchange among the spectral components in two-dimensional flow may now be shown to apply to three dimensional quasi-geostrophic flow as well" is difficult to justify without additional information. The key piece missing in Charney's arguments is the way energy and potential enstrophy are dissipated in QG vs 2D flows.

4. **Anomalous dissipation and direction of energy cascade.**

It is intuitive that the direction of energy flow should be from the wavenumbers of injection to the wavenumbers of dissipation (in a forced dissipative system at statistical equilibrium), although one needs to be more careful in view of the fact that the locations (in wavenumber space) of energy injection and dissipation in some cases are not known *a priori* (Tran and Shepherd, 2002).

Suppose we could, at least in a numerical model, arrange to have the energy injection occurring in a band of intermediate wavenumbers, and have two energy sinks: one at the infrared end (the small wavenumbers) due to, say, a hypodiffusion, and another at the ultra-violet end (the large wavenumbers) due to viscosity, e.g. by adding $\nu\nabla^4\psi$ to the right-hand side of Eqs. (1) and (5). Through nonlinear interactions (triad interactions, and, in 3D, vortex stretching), some of the injected energy flows upscale and some downscale. Therefore, at statistical equilibrium we would expect the direction of energy flux to be upscale on the long-wave side of energy injection and to be downscale on the short-wave side of injection. The current controversy concerns the theoretical situation as one approaches the inviscid limit. That is, what happens to the direction of energy flux when the coefficient of viscosity, ν, is taken to be zero from above?

[3] It is possible, by letting $N^2(z)$ vary, to obtain a set of trapped modes in the form of Bessel functions. See Wiin-Nelsen (1971), who assumed a constant-lapse rate atmosphere, which, however, is unrealistic above the tropopause.

Since the rate of energy dissipation is given by

$$\epsilon_D \equiv \nu \iiint |\nabla u|^2 dxdydz$$

$\epsilon_D \to 0$ as $\nu \to 0$, *unless* $|\nabla u|^2$ blows up. In the case of

$$\lim_{\nu \to 0^+} \epsilon_D = \text{finite}$$

there is said to be "anomalous dissipation," i.e. a finite dissipation even though the viscosity coefficient is vanishingly small. The phenomenon of anomalous dissipation is thus seen to be connected to the development of nonregular solutions of the inviscid equations. Onsager (1949)'s conjecture that nonzero ϵ_D is related to a lack of continuity of u was later proved by Constantin *et al.* (1994a) for the case of Euler's equation used in 3D turbulence. A review of the related mathematical problem of blow-up of nonlinear parabolic equations can be found in Galaktionov and Vazquez (2002) in this journal.

4.1. **3D turbulence.** Although not yet rigorously proved, it is very likely that singularities develop in finite time in the 3D (inviscid) Euler equation. The presence of anomalous dissipation was in fact assumed by Kolmogorov (1941) in developing his power law for 3D isotropic and homogeneous turbulence,

$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{27}$$

in an inertial subrange. Here ϵ is the energy flux from low wavenumbers to high wavenumbers and should be a constant in the inertial subrange (where there is no forcing or dissipation). It should be positive (downscale) and equal to the energy dissipation rate ϵ_D at statistical equilibrium. Kolmogorov's argument for the $-5/3$ spectrum is essentially this: For isotropic homogeneous turbulence, $E(k)$ should depend only on the local wavenumber k and the energy flux of the cascade ϵ through that wavenumber. Then using dimensional arguments, $E(k)$ must have the form (27).

4.2. **2D turbulence.** It has been proved (see Majda and Bertozzi, 2001) that the (inviscid) Euler equation in 2D is regular (no singularity in finite time). An upper bound on interior gradients of solutions to 2D Navier-Stokes equations is given by Kukavica (2001) in this journal. Thus there should be no anomalous dissipation of energy. The regularity proof is rather more difficult than a proof about the absence of anomalous dissipation of energy, which we now outline (see also Salmon, 1998). In 2D the rate of viscous dissipation of energy is (obtained by multiplying the viscous vorticity equation by the streamfunction and integrating over all space):

$$\epsilon_D = \nu \iint |\nabla^2 \psi|^2 dxdy.$$

Since the total vorticity, $\iint |\nabla^2\psi|^2 dxdy$, is a constant for 2D flows, $\epsilon_D \to 0$ as $\nu \to 0$.

Therefore in 2D turbulence, as $\nu \to 0$, there is no energy cascade (i.e. $\epsilon = 0$) in the inertial subrange on the short-wave side of energy injection. Therefore in this region, the $-5/3$ power law (27) is not present. Instead, as pointed out by Kraichnan (1967) the energy injected in the intermediate wavenumbers goes upscale, to the sink located at the largest scales.

There exists instead a forward enstrophy cascade in 2D turbulence, with a constant enstrophy flux η from low to high wavenumbers in the inertial subrange on the short-wave side of enstrophy injection:

$$\eta = \eta_D \equiv \nu \iint |\nabla \omega|^2 dx dy.$$

Since enstrophy dissipation involves a higher derivative of the streamfunction than energy dissipation, anomalous dissipation of enstrophy is possible even though the anomalous dissipation of energy is absent. Kraichnan (1967) argued for a -3 power law on the short-wave side of injection,

$$E(k) = C_1 \eta^{2/3} k^{-3}, \tag{28}$$

based on dimensional arguments and the local dependence of $E(k)$ on k and η only.[4]

4.3. **QG turbulence.** There is a misconception that QG solutions share the regularity of 2D solutions, and therefore that there is no anomalous dissipation of energy in QG turbulence. It should be pointed out that the regularity of QG solutions was proven by Bennett and Kloeden (1981, 1982) only up to a finite time ($t < T^*$ in their notation), where T^* is inversely proportional to the initial surface temperature gradients (among other factors), in the absence of forcing.

Now, the propensity of temperature fronts (discontinuities of temperature in the horizontal) to form in finite time – by rapid collapse of any initial temperature gradient at a surface where vertical motion is suppressed (e.g. at a rigid boundary or at the tropopause) – is a known feature of QG dynamics (Holton, 1979). The collapse is even more rapid in the semi-geostrophic equations (Hoskins and Bretherton, 1972). The singularity ("front") does form in finite time in the semi-geostrophic theory. No rigorous proof is available at the present time, however, about the formation of singularity in QG systems in finite time (see the numerical results of Majda and Tabak (1996), and Constantin *et al.*, 1994b), but it is known (Cordoba, 1998) that a surface QG front collapses very rapidly (as e^{e^t}). Thus, even if there is no actual singularity of the inviscid equations in finite time, it seems likely that there will be anomalous dissipation.

Unlike the 2D case, in QG one cannot show in general that $\epsilon_D \to 0$ as $\nu \to 0$. For the following case, however, it is indeed true: (i) the dissipation operator added to the right-hand side of the potential vorticity equation (5) is of the form: $\nu(\nabla^2 + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2})^2\psi$, and (ii) the vertical domain is periodic, or

[4] The -3 power law does not appear to be self-consistent with the assumption of local dependence of $E(k)$ on k. Kraichnan (1971) later introduced a log correction to (28) to ensure self-consistency.

the isothermal boundary condition is used on horizontal boundaries. Under these circumstances, we can show, by multiplying Eq. (5) by ψ and integrating over three dimensions, that the dissipation rate of total QG energy is given by:

$$\epsilon_D = \nu \iiint |(\nabla^2 + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2})^2\psi|^2 dxdydz = \nu \iiint q^2 dxdydz.$$

Since the integral, which is the total potential vorticity squared, is conserved and therefore bounded, ϵ_D vanishes as ν is taken to be zero. There is no anomalous dissipation in this case. For other cases, especially when the more commonly used form of diffusion of potential vorticity, $\nu\nabla^2 q$, is adopted as the viscous term, the above proof fails, and the possibility of downscale total energy flux, $\epsilon > 0$, must at least be considered in QG turbulence along with a downscale potential enstrophy flux, $\eta > 0$.

5. **An example of forward energy cascade in QG turbulence.** We now use a particularly simple case of QG turbulence to illustrate some of the points raised in previous sections. This example concerns surface quasi-geostrophic flows (SQG) (Blumen, 1978a,b; Pierrehumbert *et al.*, 1994; Held *et al.*, 1995).

5.1. **SQG solution.** SQG equations are the same as QG equations specialized to the case of zero relative potential vorticity, i.e.

$$L(\psi) \equiv 0 \text{ in the interior of the flow domain.} \tag{29}$$

The domain is semi-infinite in the vertical, i.e. $0 < z < \infty$, and doubly periodic in the x, y directions. The vertical variations of ρ_0 and N^2 are ignored. [The surface, $z = 0$, can be taken to be either the ground or the tropopause. In the latter case, the $z-$coordinate points downward.] The boundary condition, $w = 0$ at $z = 0$, reduces the temperature equation (7) to a conservation equation involving the advection of temperature T by the surface flow:

$$\frac{\partial}{\partial t}\Theta + J(\psi, \Theta) = 0, \quad \text{at } z = 0 \tag{30}$$

where

$$\Theta \equiv \frac{g}{NT_0}T = \frac{f_0}{N}\frac{\partial}{\partial z}\psi \tag{31}$$

from (8). The upper boundary condition is $\psi \to 0$ as $z \to \infty$.

Let $\hat{F}$ denote the horizontal Fourier transform (or series) of F, so that

$$\psi(x,y,z,t) = \frac{1}{4\pi^2}\iint \hat{\psi}(k_x,k_y,z,t)e^{i(k_x x + k_y y)}dk_x dk_y.$$

We find, from solving the elliptic equation (29), that:

$$\hat{\psi}(k_x,k_y,z,t) = \hat{\psi}_0(k_x,k_y,t)e^{-|k|(N/f_0)z} \tag{32}$$

where $k^2 \equiv k_x^2 + k_y^2$. Thus at the surface

$$\hat{\Theta} = -|k|\hat{\psi}_0. \tag{33}$$

5.2. **Energy and enstrophy conservation.** The spectral kinetic energy at the surface is

$$\frac{1}{2}(|\hat{u}|^2 + |\hat{v}|^2) = \frac{1}{2}k^2|\hat{\psi}_0|^2,$$

while the available potential energy at the surface is,

$$\frac{1}{2}\frac{g^2}{N^2}\left|\left(\frac{\hat{T}}{T_0}\right)\right|^2 \equiv \frac{1}{2}|\hat{\Theta}|^2 = \frac{1}{2}k^2|\hat{\psi}_0|^2,$$

using (33). Therefore the total energy at the surface is equally partitioned between the kinetic and available potential energies. Since the latter is the same as temperature variance, and since temperature itself is conserved at the surface from (30), available potential energy and kinetic energy are each separately conserved at the surface (when horizontally integrated). They are also conserved when integrated over z, since the solution is exponential.

The potential enstrophy conservation is satisfied identically and pointwise by the zero relative potential vorticity assumption (29):

$$\frac{1}{2}(L(\psi))^2 \equiv 0, \ z > 0.$$

This holds also at the surface, as can be seen by substituting the solution (32) into $L(\psi)$ before evaluating the expression at $z = 0$. Thus, in SQG flows, there is no flux of potential enstrophy because there is no potential vorticity. There is only the possibility of energy cascade.

5.3. **The direction of cascade.** The direction of the energy cascade in SQG turbulence is downscale. In Pierrehumbert *et al.* (1994), $\frac{1}{2}\Theta^2$ is interpreted—mathematically—as "enstrophy", since Θ in (30) plays the role of vorticity ω in 2D turbulence (1). Because presumably enstrophy cascades downscale in 2D turbulence, by analogy $\frac{1}{2}\Theta^2$ also cascades downscale in SQG. As pointed out above, $\frac{1}{2}\Theta^2$ is actually the available potential energy, and is half the total energy. Therefore the flux of the total energy, ϵ, should be positive (i.e. downscale) in SQG by this argument.

5.4. **Anomalous dissipation.** If the flux of energy is downscale, where does the energy go at statistical equilibrium? The answer must be that there is anomalous dissipation at the small scales. (In fact, it is the presence of anomalous dissipation at the small scales that gives rise to downscale flux of energy into this small-scale sink.)

If Θ and ψ were not related, and temperature advection (30) were treated as tracer advection, the gradients between temperature filaments would increase exponentially (see Holton (1979) for a simple solution involving a deformation-advecting flow field). The nonlinear case where the advecting velocity is related to the field being advected (as in (31)) was considered by Constantin *et al.* (1994b), and they found that the gradients increase even more quickly than exponentially. The phenomenon involved is that of frontogenesis discussed in earlier sections.

We therefore expect that the energy dissipation rate

$$\epsilon_D = \nu \iint |\nabla\Theta|^2 dxdy$$

will be finite in the limit $\nu \to 0^+$. Furthermore the energy flux ϵ satisfies

$$\epsilon = \epsilon_D > 0. \tag{34}$$

5.5. **The shape of the energy spectrum.** The total energy spectrum for SQG flow at the surface is

$$\begin{aligned} E(k) &= \frac{1}{2}(|\hat{u}|^2 + |\hat{v}|^2) + \frac{1}{2}|\hat{\Theta}|^2 \\ &= k^2|\hat{\psi}_0|^2 = |\hat{\Theta}|^2 \end{aligned}$$

Since $|\hat{\Theta}|^2$ is conserved, its flux from low to high wavenumbers (defined to be ϵ) should be a constant in an inertial subrange at statistical equilibrium between forcing and small scale dissipation. Using the same argument as Kolmogorov and Kraichnan, except that there is now no enstrophy flux ($\eta \equiv 0$), we assert that $E(k)$ can only depend on ϵ and k, and so must be of the form

$$E(k) = C\epsilon^{2/3}k^{-5/3} \tag{35}$$

This spectral shape is consistent with the observations in the mesoscales of the upper troposphere (Nastrom and Gage, 1985) and near the earth's surface above the boundary layer (Högström *et al.*, 1999), where frontogenesis is often observed. There is no $\eta^{2/3}k^{-3}$ part of the spectrum associated with SQG turbulence because there is no interior potential vorticity by assumption. In the atmosphere, however, it is likely that the forcing mechanism of baroclinic instability produces both a downscale potential enstrophy flux ($\eta > 0$) as well as downscale total energy flux ($\epsilon > 0$). As presented in Tung and Orlando (2002), there is a k^{-3} spectrum dominated by η on the long-wave side of the $k^{-5/3}$ part, with the change in slope occurring near a wavenumber $\sim (\eta/\epsilon)^{1/2}$. [5]

Blumen (1978a) in fact already derived the $k^{-5/3}$ behavior (where k is horizontal wavenumber) for available potential energy on horizontal surfaces based on Leith's (1968) diffusion approximation, and he concluded that its cascade is toward higher wavenumbers. Later authors tended to treat $\frac{1}{2}\Theta^2$ as an enstrophy and hence its downscale flux as an enstrophy flux consistent with 2D turbulence. We note also that Held *et al.* (1995) identified $\iint(-\psi\Theta)dxdy$ as the total energy (kinetic plus available potential energy). This would indeed have been true if Θ were a potential vorticity (see section 3), but Θ should be temperature at the surface. Thus their finding of a $k^{-8/3}$ spectrum corresponds to a quantity different from energy.

In discussing the mathematical analogies between 2D and SQG flows, Majda and Tabak (1996) correctly pointed out the redundant conservation

[5] There is no singularity in finite time in the two-level model used in that study. The sink for the downscale energy flux was instead provided in the numerical model by the subgrid hyperdiffusion.

of kinetic energy in SQG, and the fact that $\iint(-\psi\Theta)dxdy$ is not energy. They further showed that the formal analogy between 2D and SQG breaks down when one studies the development of sharp fronts.

It is not our intention to focus unduly on the rather pedagogical issue of anomalous dissipation (although the example in this section owes its downscale flux to the existence of such a sink as the viscosity coefficient vanishes). In the real atmosphere and in numerical models one cannot really take viscosity to zero, because there is always a finite dissipation at the small scales. The -5/3 spectral slope arises whenever there is a finite small-scale sink of energy (see Tung and Orlando, 2002).[6]

6. **Observations and historical development.** As reviewed above, 2D turbulence is probably not an appropriate model for atmospheric turbulence because of a few missing but crucial ingredients. Ironically, observational evidence has actually played an important role in justifying and sustaining the development of 2D turbulence theories over the years, as we will briefly outline below. Recent observational evidence now points to important discrepancies between 2D theory and data, which cannot be easily resolved.

Evidence that the atmospheric energy spectrum in the mid-latitude upper troposphere (away from the planetary boundary layer) may be in the form of simple power laws was first presented by Wiin-Nielsen (1967). Because of the stations, only results for the planetary scales (tens of thousands of kilometers) and synoptic scales (thousands of kilometers) were shown. That paper drew attention to the fact that atmospheric turbulence apparently behaved like what had just been predicted by Kraichnan (1967) using 2D turbulence inertial subrange theory. Kraichnan had argued that 2D turbulence, which conserves enstrophy in addition to energy, should possess two power laws: one k^{-3} range due to a forward enstrophy cascade and one $k^{-5/3}$ range due to an inverse energy cascade. Wiin-Nielsen's data at the time appeared to roughly fit this picture, with approximately a -3 power law for wavenumbers between 8 and 16, and a (less defined) -0.4 power law for wavenumbers smaller than 8. The break in the slopes was identified by Wiin-Nielsen (and later by Chen and Wiin-Nielsen, 1978) as the location of energy injection by baroclinic instability, which they assumed to occur in a narrow wavenumber band around 8. Numerical 2D turbulence simulations (Lilly, 1969) of energy injection near wavenumber 8 appeared to show a $-5/3$ power law for scales larger than the injection scale, and close to -3 power law for scales shorter, matching the observational data then available (but see below).

In the Note by Charney (1971) mentioned earlier, observational evidence of power law behavior was also presented from several sources, although it probably had become apparent to Charney by that time that there was no

[6]Although the same -5/3 spectral slope can also arise with upscale energy cascade, the argument in favor of the downscale cascade explanation is that of consistency with observational evidence about the flux direction and the robustness of the spectrum in regions with and without small-scale energy source, e.g. thunderstorms.

power-law behavior for the large scales (wavenumbers less than 8). Charney instead concentrated on the -3 power law presumably due to the forward potential enstrophy cascading subrange between wavenumbers 7 to 20. Interestingly, the observational data presented by Charney show a slope closer to -4 than -3. On the other hand, later global data sets analyzed by Boer and Shepherd (1983), and Straus and Ditlevsen (1999), show kinetic energy spectrum slopes which vary with height, increasing as one approaches the tropopause, but all shallower than -3.

Numerical simulation of 2D turbulence at the time mostly showed spectral slopes steeper than -3; Lilly's fortuitous -3 slope was thought to be due to too coarse a resolution (Fornberg, 1977; Basdevant *et al* 1981; McWilliams, 1984; Maltrud and Vallis 1991). The steeper-than-predicted slopes from numerical simulations were attributed to various causes, e.g. space- and time-intermittency (see additional discussions in Lilly, 1969,1972; Maltrud and Vallis, 1991; Smith and Yakhot, 1994; Bowman, 1996; and Lindborg, 1999). Recently, however, Lindborg and Alvelius (2000) produced an almost perfect k^{-3} spectrum in a 2D model with very high resolution (4096×4096), using hypodiffusion for the large-scale and hyperdiffusion for the small-scale sinks (but see comments by Tran and Shepherd, 2002).

Higher resolution observational data, collected using commercial airplane flights in the upper troposphere and lower stratosphere in the late 70's, were presented by Nastrom, Gage and Jasperson (1984) and Nastrom and Gage (1985). They showed the -3 slope in the spectra of kinetic energy and of available potential energy in the synoptic and subsynoptic scales down to 800km, and they also showed another power-law regime at *smaller* scales: a $-5/3$ slope in the spectrum of the same quantities in the mesoscales from 600km to a few kilometers. The transition from one slope to the other occurs gradually between 600km and 800km. The position of the two slopes, with the $-5/3$ slope occurring on the *short*-wave side of the -3 slope, is not consistent with the prediction of Kraichnan (1967) using 2D turbulence theory. This observational result has since been confirmed by other analyses of independent aircraft campaigns (Marenco *et al.*, 1998; Cho *et al.*, 1999a,b).

By analyzing the latter aircraft data using a 2D turbulence formulation, Lindborg (1999) first argued for an inverse energy cascade for the mesoscales in the atmosphere akin to the inverse energy cascade of 2D turbulence, and thus giving hope that the newfound $-5/3$ slope in the atmosphere could be explained by upscale energy transfer from the thunderstorms in the small scales, a theory first proposed by Lilly (1983). Later Cho and Lindborg (2001) pointed out a sign error in the third-order structure functions in Lindborg (1999). When corrected, they now conclude that the data "at mesoscales in both the upper troposphere and lower stratosphere provide no support for an inverse energy cascade [of] 2D turbulence." Furthermore, Cho and Lindborg themselves used third order structure functions and calculated a downscale energy flux in observed data.

A conceptual QG model for the observed two-slope spectral shape in the synoptic and mesoscales is presented in Tung and Orlando (2002). It involves a simultaneous downscale potential enstrophy cascade *and* a downscale total energy cascade.

7. **Concluding remarks.** We have discussed the similarities and differences between 2D turbulence and QG turbulence. Unlike 3D turbulence, which conserves only energy, there are two conserved quantities in 2D and in QG turbulence: an energy and an enstrophy. The twin conservation of energy and enstrophy by itself does not prohibit net downscale energy transfer in either 2D turbulence or QG turbulence, although it has been thought so for many years. In 2D turbulence, the prohibition turns out to be true but due to another property: the absence of "anomalous dissipation" in 2D turbulence when the viscosity coefficient vanishes. Related to this result — but not strictly dependent on it — is the absence of singularities in finite time in 2D flows. In QG turbulence, the prohibition against a downscale energy cascade appears not to be valid. Nevertheless, in idealized cases studied in numerical models of decaying turbulence, with isothermal horizontal boundaries (e.g. McWilliams, 1989), or in triply periodic domains (e.g. Herring, 1980), strong similarities with 2D turbulence are found, such as predominately upscale energy cascade and downscale potential enstrophy cascade.

QG turbulence, at least as applied to the atmosphere, differs from 2D turbulence in two important ways. It has the self-contained mechanism of baroclinic instability as the source of energy and enstrophy injection (not discussed here, but see Welch and Tung, 1998), and it has frontogenesis and the possibility of anomalous dissipation as a sink of energy at the small scales near non-isothermal surfaces. Both are possible in QG turbulence but not in 2D turbulence. Furthermore, small-scale processes in the real atmosphere are now found to provide a sink for the downscale flux of energy in the mesoscales, quite apart from the purely mathematical issue of the existence of anomalous dissipation. It is important then to point out that there is nothing in the theories for QG turbulence that prohibits such a downscale cascade of energy, and therefore one cannot automatically rule out a QG explanation.

Despite many attempts over the last 30 years, the observed atmospheric turbulence spectrum has not been explained, principally because of an overreliance on 2D turbulence theories. The $-5/3$ slope in the spectrum over the mesoscales can possibly be explained by a *forward* energy cascading inertial subrange (Tung and Orlando, 2002).

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