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ON THE DOUBLE CASCADES OF ENERGY AND ENSTROPHY IN TWO-DIMENSIONAL TURBULENCE. PART 1. THEORETICAL FORMULATION

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Abstract. The Kraichnan-Leith-Batchelor scenario of a dual cascade, consisting of an upscale pure energy cascade and a downscale pure enstrophy cascade, is an idealization valid only in an infinite domain in the limit of infinite Reynolds number. In realistic situations there are double cascades of energy and enstrophy located both upscale and downscale of injection, as long as there are cascades. We outline the statistical theory governing the double cascades and predict the form of the energy spectrum. We show that in general the twin conservation of energy and enstrophy imply the presence of two constant fluxes in each inertial range. This gives rise to a more complicated energy spectrum, which cannot be predicted using dimensional arguments as in the classical theory.

1. **Introduction.** The theory of two-dimensional turbulence is in many ways more complex than that of three-dimensional turbulence of Kolmogorov, and perhaps even richer in phenomena. Consequently, however, many fundamental issues remain. The framework proposed by Kraichnan [19], Leith [22] and Batchelor [3] (KLB) is so idealized that it is almost impossible to reproduce numerically, hence the existence of a large body of theories/explanations of why the numerical spectra are steeper than those predicted by the KLB theory [32, 35, 38]. The observed spectra in the atmosphere also have some "paradoxical" features compared with the KLB prediction. Setting aside the issue of whether two-dimensional turbulence theory is an adequate model for the atmosphere (discussed in detail in [44]), the unintuitive behavior of two-dimensional turbulence highlights the need for a deeper theoretical understanding. The goal of this paper is to take a first step by outlining the logical structure of the statistical theory that governs two-dimensional turbulence.

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Let $u_\alpha(\mathbf{x},t)$ be the Eulerian velocity field. The governing equations of two-dimensional turbulence are:

$$\begin{aligned} &\frac{\partial u_\alpha}{\partial t} + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + d_\alpha + f_\alpha, \\ &\partial_\alpha u_\alpha = 0, \end{aligned} \tag{1.1}$$

where f_α is a stochastic forcing term that injects energy and enstrophy into the system, and d_α represents a dissipation mechanism. Note that we use Einstein's summation notation whereby repeated indices are summed over the available dimensions, and ∂_α represents spatial differentiation with respect to the α component. The scope of this paper is limited to the case of forced two-dimensional turbulence in which forcing and dissipation reach a statistical equilibrium.

The unique feature of two-dimensional turbulence is that there exist two conserved quadratic invariants: energy E and enstrophy G defined as

$$\begin{aligned} E(t) &= \frac{1}{2}\int u_\alpha(\mathbf{x},t)u_\alpha(\mathbf{x},t)d\mathbf{x} \\ G(t) &= \frac{1}{2}\int \zeta^2(\mathbf{x},t)d\mathbf{x}, \end{aligned} \tag{1.2}$$

where $\zeta = \epsilon_{\alpha\beta}\partial_\alpha u_\beta$ is the scalar vorticity, and $\epsilon_{\alpha\beta}$ is the Levi-Civita tensor in two dimensions. The corresponding energy and enstrophy spectra are related by

$$G(k,t) = k^2 E(k,t). \tag{1.3}$$

where k is the wavenumber. If ε is the rate of energy injection and η is the rate of enstrophy injection, then the characteristic forcing length scale ℓ_0 is defined as $\ell_0^2 = \varepsilon/\eta$.

In physical systems modeled by two-dimensional turbulence, the dissipation d_α usually has the form

$$d_\alpha = \nu\nabla^2 u_\alpha - \beta u_\alpha. \tag{1.4}$$

The first term corresponds to molecular viscosity. It usually acts as an energy and enstrophy sink at small scales. The second term corresponds to linear damping. It models the Ekman damping in the atmosphere [34], and the energy sink in soap film experiments [37]. In some numerical experiments, hypodiffusion is used in place of linear damping for the infra-red sink. This has the advantage of concentrating the sink near a few largest wavelengths. For a similar reason, hyperdiffusion is used for the sink at small scales to concentrate the dissipation range to a small band of wavenumbers in the ultraviolet end. The general form of the dissipation term, for the combined case of hypodiffusion and hyperdiffusion, is written as

$$d_\alpha = (-1)^{\kappa+1}\nu\nabla^{2\kappa}u_\alpha + (-1)^{m+1}\beta\,\nabla^{-2m}u_\alpha. \tag{1.5}$$

The numbers κ and m are the order of hyperdiffusion and hypodiffusion correspondingly. The dependence of the results of numerical simulations on hyperdiffusion and hypodiffusion is not well understood, but there is some notable recent progress (see discussion later).

Based on Kolmogorov's [1, 17, 18] concept of an downscale energy cascade in three-dimensional turbulence, Kraichnan [19], Leith [22], and Batchelor [3] (KLB) proposed that in two-dimensional turbulence there is an upscale energy cascade and a downscale enstrophy cascade, when the stochastic forcing injects energy and enstrophy in a narrow band of intermediate length scales. Assuming that all the

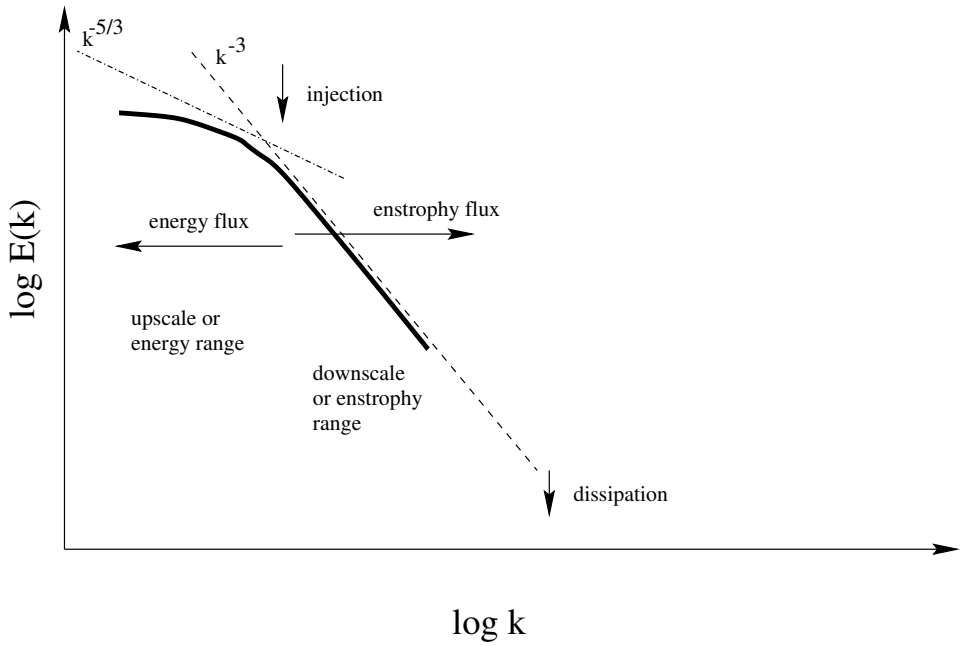

FIGURE 1. The Kraichnan-Leith-Batchelor scenario of a dual-pure cascade. There is a pure energy upscale cascade upscale of the injection and a pure downscale enstrophy cascade downscale of the injection.

energy flows upscale and all of the enstrophy flows downscale, KLB invoke a dimensional analysis argument, similar to Kolmogorov's, to show that the energy spectrum in the upscale energy range is

$$E(k) = C_{ir}\varepsilon^{2/3}k^{-5/3}, \tag{1.6}$$

and in the downscale enstrophy range is

$$E(k) = C_{uv}\eta^{2/3}k^{-3}. \tag{1.7}$$

Anticipating the objection that the dimensional analysis arguments cannot be applied to the enstrophy cascade, because of nonlocality, in a subsequent paper [21] Kraichnan proposed that the enstrophy cascade energy spectrum is given by

$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\ln(k\ell_0)]^{-1/3}, \tag{1.8}$$

and showed, using a one-loop closure model [20], that this logarithmic correction is consistent with constant enstrophy flux. Furthermore, to eliminate the singularity at $k\ell_0 = 1$, Bowman [6] showed that a constant term needs to be introduced along with the logarithmic correction, and the improved equation reads

$$E(k) = C_{uv}\eta^{2/3}k^{-3}[\chi + \ln(k\ell_0)]^{-1/3}, \tag{1.9}$$

where χ is a constant that may or may not be universal.

In the KLB idealization there is only a single flux in each inertial range: a pure energy upscale cascade on the upscale side of injection and a pure downscale enstrophy cascade on the downscale side of injection (see Fig. 1). This situation is called the *dual-pure* cascade. The dimensional analysis argument used by previous authors would fail if there is more than one flux in each inertial range. There is no

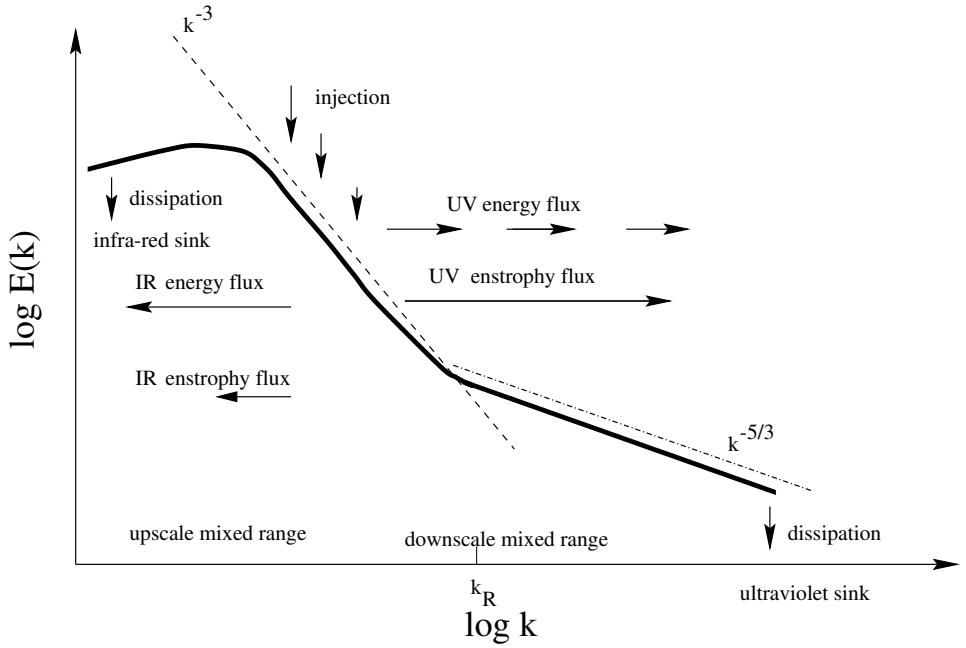

FIGURE 2. The scenario of a dual-double cascade. There is a double cascade of energy and enstrophy upscale of the injection scale and a double cascade of energy and enstrophy downscale of the injection scale.

large-scale (infra-red) sink of energy in the KLB picture, but it requires an infinite domain: all injected energy is transferred to larger and larger scales and thus escapes dissipation altogether. This picture is clearly unrealizable in numerical experiments and in nature, where the domain is finite. In reality the finite size of the numerical or natural domain contains the energy, which then piles up at the largest scales.

Eyink [8] has shown that although in the limit $\nu \to 0$, the total energy, under steady state, is infinite, for finite ν, it is bounded. It has also been shown by Tran and Shepherd [43] that in a finite domain with finite viscosity, a cascade of enstrophy on the downscale side of injection is not allowed without the presence of an infrared dissipation term that can provide an energy sink at large scales. Further considerations by Tran and Bowman [42] have argued that in the absense of a sufficient infra-red sink, most energy and enstrophy is dissipated at the forcing scale, leading to steep spectra both upscale and downscale of injection. Tran and Bowman [41] have also shown that the inverse energy cascade could be realized temporarily as a quasi-steady state even in the absense of an infrared sink, but not the enstrophy cascade. Since Kraichnan's conjecture of a dual cascade was intended for the problem of turbulence driven by the Navier-Stokes equations *without the infrared dissipation sink*, these papers highlight a number of serious problems with this conjecture.

Here, we are concerned with the case where there is a significant dissipation sink at large scales, as well as at the small scales, and the viscosity coefficients for both sinks are finite. The presence of such an infrared sink unavoidably dissipates some enstrophy, and the finite viscosity enstrophy sink at small scales will inevitably also dissipate some energy. It is therefore necessary to scrutinize, in a similar

manner, the conjecture that in turbulence driven by the infrared-damped Navier-Stokes equations there is a dual cascade that consists of an upscale double cascade of energy and enstrophy, where the energy flux is dominant, and a downscale double cascade of enstrophy and energy, where the enstrophy flux is dominant (see Fig. 2). Because there are two fluxes present in each cascade, it is not obvious that the form of the energy spectrum can be deduced from dimensional analysis. Furthermore, it is alternatively possible that non-local interactions may dominate that transfer energy and enstrophy directly to the dissipation range, or that the cascades may fail to form altogether.

The present paper introduces a theoretical framework that can address these issues. We predict the form of the energy spectrum for the case where double cascades form successfully. We also describe the possible modes of failure that may lead to a non-universal energy spectrum. Furthermore, we address the common misconception that it is not possible for an energy and enstrophy cascade to coexist in the same spectral region. In a companion paper [15], we show that unlike the case of atmospheric turbulence, in two-dimensional turbulence the contribution to the energy spectrum from the subleading cascade remains always hidden. We also review in detail the experimental evidence from numerical simulations of two-dimensional turbulence to explain what is understood so far with respect to the realizability of universal downscale and upscale double cascades. The case of the energy spectrum of atmospheric turbulence is also considered as an example from a different dynamical system where we encounter a downscale double cascade of energy and enstrophy. A more rigorous proof of the theory and its consequences will be given in future publications.

The plan of this paper is as follows. In section 2 we outline a theory that governs the double cascade scenario. We begin by reviewing, in section 2.1, the nonperturbative theory of the balance equations introduced by L'vov and Procaccia [25–29] to explain three-dimensional turbulence. In the framework of this theory, the energy conservation law implies the existence of a homogeneous solution that is responsible for the downscale energy cascade of three-dimensional turbulence. In section 2.2 we show that this theory can be extended to account for the energy and enstrophy cascades of two-dimensional turbulence. In section 2.3, we propose that the twin conservation of energy and enstrophy in two-dimensional turbulence implies the existence of two independent homogeneous solutions: one is responsible for an energy cascade, the other for an enstrophy cascade. We conclude that both upscale and downscale of the injection scale there exist a double cascade of energy and enstrophy. Because of linearity of the statistical theory, these two solutions can be linearly superimposed, giving rise to a new form for the energy spectrum involving both energy and enstrophy cascades. In section 3, we address the common misconception that it is not possible to have an inertial range in which both the energy and the enstrophy flux are constant. In section 3.1, we show that in fact a double cascade of energy and enstrophy is permitted both in the downscale direction and the upscale direction. In section 3.2, we review the argument given by Kraichnan against the coexistence of constant energy flux and constant enstrophy flux in the same range, and show that it is consistent with our point of view when the role of the similarity assumption is clarified. Section 4 concludes the paper. Appendix A lists the relationship between the energy spectrum and the generalized structure functions. Appendix B shows the terms of the balance equations in detail.

2. **Theoretical framework.** Both the K41 theory for three-dimensional turbulence, and the KLB theory for two-dimensional turbulence are based on a dimensional analysis argument. However, Frisch [11, 12] has suggested that Kolmogorov's second paper [17] leads to the following more rigorous reformulation of the dimensional analysis argument. First, one postulates local homogeneity, local isotropy, self-similarity, and the existence of an anomalous energy sink. An anomalous sink is defined as one which remains finite in the limit as the viscous coefficient ν approaches zero from above. Then, one uses these assumptions to derive the 4/5 law, and from the self-similarity assumption the scaling for all structure functions and thus the energy spectrum can be deduced. It is possible to formulate a theory for two-dimensional turbulence in the same way. The problem is that the assumptions involved are not obviously true in two-dimensional turbulence. For example, it is not obvious that an anomalous enstrophy sink exists at small scales, and an anomalous energy sink at large scales. Furthermore, it turns out that the assumption of self-similarity is too restricting.

We propose that one way to make progress is to adapt the theoretical work of L'vov and Procaccia [25–29] to two-dimensional turbulence. This amounts to replacing the assumption of the anomalous sinks with the weaker assumption that the energy and enstrophy cascades satisfy the fusion rules. We continue to postulate local homogeneity and local isotropy, and we also postulate a weaker assumption of self-similarity than the one used by Frisch.

From the fusion rules, it is possible to show the locality of interactions in the inertial range. Furthermore, it is also possible to calculate the location of the boundary between the inertial range in the dissipation range, as well as the boundary between the inertial range and the forcing range. In the first case, we obtain the location of the dissipation length scale, and from that we may calculate whether the anomaly of the enstrophy or energy sink is recovered. In the second case, we find whether the proposed inertial range is stable with respect to perturbations to the statistical forcing or the boundary conditions. These two calculations amount to necessary and sufficient conditions for the formation of an observable inertial range.

In future publications, we will present these developments in detail. In the present paper we restrict ourselves to the conclusions that are suggested from the structure of the theory, as long as we assume that the aforementioned conditions are satisfied.

2.1. **Theoretical background for three-dimensional turbulence.** We begin with a brief overview of the corresponding theory [25] for three-dimensional turbulence, before concentrating on two-dimensional turbulence. This review stresses those aspects of the theory that we find relevant for the problem of two-dimensional turbulence, and it reflects, to some extent, our point of view. Let w_α represent the Eulerian velocity difference, defined as

$$w_\alpha(\mathbf{x}, \mathbf{x}', t) = u_\alpha(\mathbf{x}, t) - u_\alpha(\mathbf{x}', t). \tag{2.1}$$

and define the generalized structure function in terms of the following product

$$F_n^{\alpha_1\alpha_2\cdots\alpha_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \left[\prod_{k=1}^n w_{\alpha_k}(\mathbf{x}_k, \mathbf{x'}_k, t) \right] \right\rangle, \tag{2.2}$$

where $\langle\cdot\rangle$ denotes ensemble average. For convenience, we omit the tensorial superscripts, unless they are needed for clarity. The inertial range, for the case of

three-dimensional turbulence, can be characterized as the region where the generalized structure functions satisfy approximately

$$F_n(\{\lambda \mathbf{x}_k, \lambda \mathbf{x'}_k\}_{k=1}^n, t) = \lambda^{\zeta_n} F_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t), \tag{2.3}$$

where ζ_n are the corresponding scaling exponents and $\lambda \in (1-\epsilon, 1+\epsilon)$ with ϵ small. The energy spectrum $E(k)$ is related to F_2 via a linear transformation (see appendix A for details). If $0 < \zeta_2 < 2$, then the energy spectrum has scaling $E(k) \sim k^{-1-\zeta_2}$ [12]. If there is a logarithmic correction, then the result also holds for $\zeta_2 = 2$. Kolmogorov's theory [17, 18] predicts that, for the energy range of three-dimensional turbulence, the scaling exponents satisfy $\zeta_n = n/3$. In reality, there are non-trivial corrections, called "intermittency corrections"[12].

Differentiating F_n with respect to time, and substituting into the Navier-Stokes equations, gives a sequence of equations of the form

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + \beta H_n + Q_n, \tag{2.4}$$

where D_n is the combined contribution of the pressure and the nonlinear term, Q_n is the contribution of the forcing term, J_n accounts for diffusion or hyperdiffusion, and H_n accounts for damping or hypodiffussion (see appendix B for the detailed form of the terms). Equations of this type have been introduced by L'vov and Procaccia [26, 28]. It has been shown, in section IV-B and appendix B of reference [28], that D_n can be rewritten in the form

$$D_n = \mho_n F_{n+1} + I_n, \tag{2.5}$$

where $\mho$ is a linear operator, and I_n represents the interaction of velocity differences with the mean flow. Under global homogeneity, where the mean flow is removable by a Gallilean transform, we have $I_n = 0$. For a stationary system, the generalized structure functions will satisfy an infinite chain of balance equations of the form

$$\mho_n F_{n+1} + I_n = \nu J_n + \beta H_n + Q_n. \tag{2.6}$$

The essential point is that these equations are a *linear* system with respect to the generalized structure functions F_n. They become nonlinear only when they are truncated by some nonlinear closure schemes, such as the Leith [22] and Pouquet [36] schemes used by Lilly [23]. It follows that they have a homogeneous solution that satisfies the equation $\mho_n F_{n+1} = 0$, and a particular solution driven by Q_n and I_n. The viscous terms act to modify the homogeneous solution by introducing a dissipation range, as discussed in the next section.

The distinguishing features of the direct energy cascade are that it exhibits universal scaling, and that it involves a process by which energy is transferred progressively from large scales towards small scales by triad interactions. The fusion rules are intended to describe this cascade process mathematically. The locality of interactions in the integrals of $\mho_n F_{n+1}$ can be proven from the fusion rules. However, it is still possible for the non-local interactions to contribute sufficient influence in the inertial range and destroy universal scaling. The extent of this influence is characterized by the magnitude of *the other terms* of the balance equations.

Universal features, such as the energy cascade, are represented by the homogeneous solution, as long as dissipative corrections can be neglected, whereas non-universal features, such as the forcing range, are represented by the particular solution. The extent of the energy cascade is therefore the region where the homogeneous solution dominates the particular solution[1]. This leads to a calculation that enables us to examine the stability of the energy cascade with respect to perturbations to the forcing statistics by comparing the order of magnitude of Q_n against $\mathcal{O}_n F_{n+1}$. Similarly, it is also possible to derive the location of the dissipation length scales [29] by comparing the order of magnitude of νJ_n and βH_n against the typical term in $\mathcal{O}_n F_{n+1}$. The fusion rules are used to find the scaling exponents of $\mathcal{O}_n F_{n+1}$, so that it can be compared with the other terms.

It should be stressed that it is not obvious that a non-trivial homogeneous solution should exist in the first place. It can be shown, however, that a consequence of the conservation of energy is that the equation $\mathcal{O}_2 F_3 = 0$ has a non-trivial solution if $\zeta_3 = 1$ [29]. To see this explicitly, note that after some calculations, we find

$$\begin{aligned}\mathcal{O}_2 F_3(\mathbf{x}_1, \mathbf{x'}_1, \mathbf{x}_2, \mathbf{x'}_2) &= \frac{1}{2}\frac{d[S_3(r_{12}) - S_3(r_{12'})]}{dr_1} + \frac{1}{2}\frac{d[S_3(r_{1'2'}) - S_3(r_{1'2})]}{dr_{1'}} \\ &= A[r_{12}^{\zeta_3 - 1} - r_{12'}^{\zeta_3 - 1} + r_{1'2'}^{\zeta_3 - 1} - r_{1'2}^{\zeta_3 - 1}].\end{aligned} \tag{2.7}$$

where $r_{12} = \|\mathbf{x}_1 - \mathbf{x}_2\|$, etc, and

$$S_3(r) = \langle [u_\alpha(\mathbf{x}_0 + r\mathbf{e}, t) - u_\alpha(\mathbf{x}_0, t)]^3 \rangle \tag{2.8}$$

is the standard 3rd order structure function (assuming local homogeneity and local isotropy) with $\mathbf{e}$ an arbitrary unit vector. Equation (2.7) is zero (the homogeneous equation $\mathcal{O}_2 F_3 = 0$) for all configurations of velocity differences when either the numerical constant $A = 0$ (the trivial solution), or when $\zeta_3 = 1$ (the non-trivial solution). Given $\zeta_3 = 1$, one may show that it leads to a constant energy flux proportional to A.

Further development [5, 30] of this theory has given theoretical grounds in support of the self-similarity property of the generalized structure functions F_n. The argument can be summarized as follows: first, it is observed that the homogeneous equations are invariant with respect to the following group of transformations

$$\mathbf{r} \mapsto \lambda \mathbf{r}, \quad F_n \mapsto \lambda^{nh + \mathcal{Z}(h)} F_n. \tag{2.9}$$

This means that in an inertial range, where the homogeneous equations are a valid approximation, solutions $F_{n,h}$ that satisfy the self-similarity property

$$F_{n,h}(\{\lambda \mathbf{x}_k, \lambda \mathbf{x'}_k\}_{k=1}^n, t) = \lambda^{nh + \mathcal{Z}(h)} F_{n,h}(\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t), \tag{2.10}$$

are admissible. What actually happens in the inertial range of three-dimensional turbulence is that the correct solution is the linear combination of these solutions, given by

$$F_n = \int d\mu(h) F_{n,h}. \tag{2.11}$$

This conclusion also follows from the multifractal model of Frisch [12]. In the case here, however, it is established on theoretical grounds without the multifractal

[1]This is in fact the more rigorous definition of an inertial range with a cascade of a conserved quantity in the traditional sense. The self-similarity property of the generalized structure functions in an inertial range is only approximately true in two-dimensional turbulence, and, appearently, even for the case of three-dimensional turbulence.

assumption. For each n the observable scaling exponent ζ_n is given by

$$\zeta_n = \min_h [nh + \mathcal{Z}(h)]. \tag{2.12}$$

All the other contributions are hidden. In other words, for each F_n, the observable scaling exponent ζ_n is caused by a $F_{n,h}$ with distinct h.

According to L'vov and Procaccia, $\mathcal{Z}(h)$ can be evaluated from a solvability condition applied on the homogeneous equations. This can be used, in principle, to calculate the scaling exponents [4]. Unfortunately, this is a very difficult calculation. The only scaling exponent that is not difficult to evaluate is ζ_3. Recently L'vov and Procaccia [31], proposed an alternative perturbative argument that can calculate all the scaling exponents accessible to experiment, given ζ_2 and ζ_4. An independent calculation by Giles [14] closes the argument by showing how the scaling exponents can be evaluated diagrammatically without any experimental input. For our present purpose, only ζ_2 is needed. Based on the assumption that intermittency corrections are negligeble for ζ_2, we will use $\zeta_2 \approx 2\zeta_3/3$. This yields the well-known $k^{-5/3}$ spectrum of three-dimensional turbulence.

2.2. **The case of two-dimensional turbulence.** These arguments can be repeated for the case of two-dimensional turbulence. The balance equations have the same form, and the relevant locality and stability arguments can be extended to the two-dimensional case as well. The homogeneous equation $\mathcal{O}_n F_{n+1} = 0$ still has a solution that corresponds to an energy cascade like before. However, two-dimensional turbulence also conserves enstrophy, and it is anticipated that there is a corresponding enstrophy cascade. A homogeneous solution corresponding to the enstrophy cascade cannot be obtained from $\mathcal{O}_n F_{n+1} = 0$. To show how the enstrophy cascade can arise, it is necessary to introduce the generalized structure functions of the *vorticity* differences

$$G_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left\langle \prod_{k=1}^n [\zeta(\mathbf{x}_k, t) - \zeta(\mathbf{x'}_k, t)] \right\rangle. \tag{2.13}$$

These are related with F_n through differentiation as follows:

$$G_n(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) = \left[\prod_{k=1}^n \varepsilon_{\alpha_k\beta_k} (\partial_{\alpha_k, \mathbf{x}_k} + \partial_{\alpha_k, \mathbf{x'}_k}) \right] F_n^{\beta_1\beta_2\ldots\beta_n}(\{\mathbf{x}_k, \mathbf{x'}_k\}_{k=1}^n, t) \tag{2.14}$$

The relationship between F_n and G_n is a linear transformation, written in abbreviated form as $G_n = \mathcal{T}_n F_n$. For G_n a similar infinite chain of equations can be written by applying $\mathcal{T}_n$ on the balance equations for F_n

$$\frac{\partial G_n}{\partial t} + \mathcal{T}_n \mathcal{O}_n F_{n+1} + \mathcal{T}_n I_n = \nu \mathcal{T}_n J_n + \beta \mathcal{T}_n H_n + \mathcal{T}_n Q_n. \tag{2.15}$$

Since $\mathcal{T}_n$ is a local differential operator, the relative scaling between the terms remains the same. It follows that the locality and stability proofs don't need to be repeated for this equation. The corresponding homogenous equation

$$\mathcal{T}_n \mathcal{O}_n F_{n+1} = 0 \tag{2.16}$$

admits two non-trivial solutions: one corresponding to the energy cascade (the solution of $\mathcal{O}_n F_{n+1} = 0$), and one corresponding to the enstrophy cascade, which

is exclusive to the operator $\mathcal{T}_n\mathcal{O}_n$. Specifically, using the conservation of enstrophy, one may show that

$$\mathcal{T}_2 O_2 F_3(\mathbf{x}_1, \mathbf{x'}_1, \mathbf{x}_2, \mathbf{x'}_2) = B[r_{12}^{\zeta_3-3} - r_{12'}^{\zeta_3-3} + r_{1'2'}^{\zeta_3-3} - r_{1'2}^{\zeta_3-3}]. \tag{2.17}$$

and the solvability condition for $\mathcal{T}_2\mathcal{O}_2F_3 = 0$ is $\zeta_3 = 3$. It can be shown that this corresponds to constant enstrophy flux proportional to B. Using the axiomatic assumption $\zeta_2 = 2\zeta_3/3$, we obtain the k^{-3} spectrum of the enstrophy cascade. However, since $\mathcal{O}_2F_3 = 0$ is satisfied when $\zeta_3 = 1$, the necessary and sufficient solvability condition for $\mathcal{T}_2\mathcal{O}_2F_3 = 0$ is $\zeta_3 \in \{1, 3\}$. We conclude that two homogeneous solutions are admissible to eq. (2.16): an energy cascade with constant energy flux and no enstrophy flux, or an enstrophy cascade with constant enstrophy flux and no energy flux. These two solutions were shown to exist by Kraichnan [21], who however did not know that they could be superimposed (see discussion in section 3.2). Furthermore, Kraichnan's argument shows that the reason why both solutions, by themselves, correspond to pure cascades is because they are individually self-similar.

It is appropriate to use equation (2.15) because we expect the vorticity differences statistics to reach stationarity. It is the stationarity condition that legitimizes using the balance equations, as modified by the $\mathcal{T}_n$ operator. The enstrophy cascade solution itself is extracted from the conservation of enstrophy. Furthermore, unlike the case of three-dimensional turbulence, in two-dimensional turbulence the generalized structure functions F_n have logarithmic corrections. It is unclear whether these corrections can be accounted for as the effective scaling of the combination of the individual $F_{n,h}$ solutions as a whole, or whether they should be treated as subleading contributions to $F_{n,h}$ themselves. In any event, as discussed in [15], we are satisfied with adopting the theory of Falkovich and Lebedev [9, 10] to derive the logarithmic corrections, as long as the existence of separation of scales in the enstrophy range can be shown.

Another point that requires careful clarification for the case of two-dimensional turbulence is the effect of the dissipative terms in controlling the location of the dissipation scales. The dissipation terms in the balance equation can be written as

$$\begin{aligned} \nu J_n + \beta H_n &= \nu \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{2\kappa} + \nabla_{\mathbf{x'}_k}^{2\kappa}) F_n + \beta \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{-2m} + \nabla_{\mathbf{x'}_k}^{-2m}) F_n. \\ &\equiv \mathcal{D}_n F_n, \end{aligned} \tag{2.18}$$

which is a linear transformation $\mathcal{D}_n$ on F_n. It follows that the contribution of the inertial range combined with the dissipation range to F_n is in fact a homogeneous solution to the following modified equation:

$$\mathcal{T}_n\mathcal{O}_nF_{n+1} - \mathcal{T}_n\mathcal{D}_nF_n = 0. \tag{2.19}$$

In other words, the presence of the dissipative terms modifies the linear operator that generates the balance equations, and in doing so *it modifies the homogeneous solutions* responsible both for the leading and subleading cascades both downscale and upscale. The modification amounts to truncating the inertial range with the dissipation range.

To find the length scale where transition to dissipation range takes place, it is sufficient to substitute the inertial range solution F_n for one of the four cascade possibilities, and compare the interaction term with the dissipative term. The point where the dissipative term becomes important is where the dissipation terms modify

the homogeneous solution and where the dissipation range begins. It follows from this that the location of the dissipation scale corresponding to one of the homogeneous solutions present is independent of the energy or enstrophy flux corresponding to the other homogeneous solutions. Without this fact it is not possible to estimate the dissipation scales using dimensional analysis, or the method employed by Frisch [12], which Smith [39] attempted to apply to the problem at hand.

2.3. **The double cascade energy spectrum.** The overall structure of the proposed theory has interesting consequences. We have two distinct homogeneous solutions that correspond to the two conservation laws. Furthermore, because the homogeneous equations (2.16) are an *infinite linear system*, a linear combination of the two solutions is also a solution. We also have an upscale and a downscale spectral region where universal behavior can be situated.

In the limiting case of infinite Reynolds numbers in an infinite domain, we are expecting a pure upscale energy cascade and a pure downscale enstrophy cascade. The pure cascades correspond to the individual homogeneous solutions discussed earlier. In the more realistic case of finite domain and finite viscosities, we anticipate instead a downscale and upscale double cascade of energy and enstrophy. From the linearity of the statistical theory it is clear that such double cascades are realized as *linear combinations* of the two homogeneous solutions that represent the energy cascade and the enstrophy cascade. As one approaches the limit of infinite Reynolds number, for each double cascade, one solution will be dominant and responsible for the observable scaling of the structure functions and the energy spectrum, whereas the other solution will be hidden.

It should be remembered that each homogeneous solution is individually modified by the dissipative terms independently of the other homogeneous solution. Furthermore, each individual homogeneous solution is itself a bundle of homogeneous solutions $F_{n,h}$ combined in a specific way to guarantee the solvability condition. In two-dimensional turbulence we have two such bundles of solutions coexisting side by side; one for the energy cascade and one for the enstrophy cascade.

For the downscale double cascade, the linear structure of the theory implies the following expression for the structure functions $S_n(r)$:

$$S_n(r) = S^{(\varepsilon)}_{n,uv}(r) + S^{(\eta)}_{n,uv}(r) + S^{(p)}_{n,uv}(r), \forall r \in (0, \ell_0), \tag{2.20}$$

where $S^{(p)}_{n,uv}(r)$ is the particular solution driven by the forcing and the boundary term, $S^{(\varepsilon)}_{n,uv}(r)$ is the contribution of the homogeneous solution corresponding to the energy cascade, and $S^{(\eta)}_{n,uv}(r)$, similarly, the contribution corresponding to the enstrophy cascade. They are given by

$$\begin{aligned} S^{(\varepsilon)}_{n,uv} &= a_{n,1}(\varepsilon_{uv} r)^{n/3} \mathcal{D}^{(\varepsilon)}_{n,uv}(r/\ell^{(\varepsilon)}_{n,uv}) \\ S^{(\eta)}_{n,uv} &= a_{n,2}(\eta_{uv} r^3)^{n/3} (\chi_n + \ln(\ell_0/r))^{2n/3} \mathcal{D}^{(\eta)}_{n,uv}(r/\ell^{(\eta)}_{n,uv}), \end{aligned} \tag{2.21}$$

where ε_{uv} and η_{uv} are the downscale energy and enstrophy fluxes, and $\mathcal{D}^{(\eta)}_{n,uv}$, $\mathcal{D}^{(\varepsilon)}_{n,uv}$ are the dissipative corrections, expected to satisfy $\mathcal{D}^{(\varepsilon)}_{n,uv}(x) \approx 1$ and $\mathcal{D}^{(\eta)}_{n,uv}(x) \approx 1$ when $0 < x < 1$. The energy spectrum is a linear transform of $S_2(r)$ (see appendix A) and it is given by

$$E(k) = E^{(\varepsilon)}_{uv}(k) + E^{(\eta)}_{uv}(k) + E^{(p)}_{uv}(k), \ \forall k\ell_0 \gg 1, \tag{2.22}$$

where $E_{uv}^{(p)}(k)$ is the contribution of the particular solution , and $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi+\ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \tag{2.23}$$

with $\mathcal{D}_{uv}^{(\varepsilon)}$ and $\mathcal{D}_{uv}^{(\eta)}$ describing the dissipative corrections. The scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$ are the dissipation length scales for the downscale energy and enstrophy cascade. The absence of intermittency corrections, which would introduce additional factors involving the scale ℓ_0, is assumed. Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrun will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi+\ln(k\ell_0)]^{-1/3}. \tag{2.24}$$

It should be stressed however that without a more detailed development of this theory, one cannot take it for granted that there will exist a spectral region – the inertial range – where this approximation can be justified. Furthermore, the mixed form of the energy spectrum $E(k)$ is obviously not self-similar. This point is relevant to the discussion in section 3.2.

Similar expressions can be written for the upscale range. Of particular interest is the energy spectrum which is written as

$$E(k) = E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k) + E_{ir}^{(p)}(k),\ \forall k\ell_0 \ll 1, \tag{2.25}$$

with the constituent terms defined similarly, where

$$\begin{aligned} E_{ir}^{(\varepsilon)}(k) &= a_{ir}\varepsilon_{ir}^{2/3}k^{-5/3}\mathcal{D}_{ir}^{(\varepsilon)}(k\ell_{ir}^{(\varepsilon)}) \\ E_{ir}^{(\eta)}(k) &= b_{ir}\eta_{ir}^{2/3}k^{-3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}). \end{aligned} \tag{2.26}$$

It is unclear at this point whether or not the upscale enstrophy cascade should admit a logarithmic correction. Again, $\ell_{ir}^{(\varepsilon)}$, $\ell_{ir}^{(\eta)}$ are the dissipation scales of the upscale energy and enstrophy cascades.

If we postulate that the fusion rules are satisfied by all four cascades, then it is possible to calculate the location of all four dissipation scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$, $\ell_{ir}^{(\varepsilon)}$, $\ell_{ir}^{(\eta)}$. The theoretical basis for such a calculation lies in observing that the dissipation terms act by modifying the homogeneous solutions, as discussed in the previous section. A detailed account of the calculation of all the relevant scales will be given elsewhere. It is based on an upgraded version of the method outlined in [28]. To find the dissipation length scales of the energy spectrum, it is sufficient to find the corresponding dissipation scales of F_2, with which they have the same order of magnitude. For the dominant cascades, we find that the dissipation scales are given by

$$\begin{aligned} \ell_{uv}^{(\eta)} &= \ell_0\left[\frac{\mathcal{R}_{uv}^{(\eta)}}{\mathcal{R}_{0,uv}^{(\eta)}}\right]^{-1/(2\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}}\frac{\eta_{uv}^{1/3}}{\nu}\right]^{-1/(2\kappa)} \\ \ell_{ir}^{(\varepsilon)} &= \ell_0\left[\frac{\mathcal{R}_{ir}^{(\varepsilon)}}{\mathcal{R}_{0,ir}^{(\varepsilon)}}\right]^{3/(2+6m)} = \left[\frac{1}{\mathcal{R}_{0,ir}^{(\varepsilon)}}\frac{\varepsilon_{ir}^{1/3}}{\beta}\right]^{3/(2+6m)}, \end{aligned} \tag{2.27}$$

where, $\mathcal{R}_{uv}^{(\eta)}$ and $\mathcal{R}_{ir}^{(\varepsilon)}$, are the Reynolds numbers corresponding to the downscale enstrophy cascade and upscale energy cascade, correspondingly, defined as

$$\begin{aligned} \mathcal{R}_{uv}^{(\eta)} &= \frac{\eta_{uv}^{1/3}\ell_0^{2\kappa}}{\nu} \\ \mathcal{R}_{ir}^{(\varepsilon)} &= \frac{\varepsilon_{ir}^{1/3}\ell_0^{-2m-2/3}}{\beta}. \end{aligned} \tag{2.28}$$

The numerical constants $\mathcal{R}_{0,uv}^{(\varepsilon)}, \mathcal{R}_{0,uv}^{(\eta)}$ can be interpreted as critical Reynolds numbers for the corresponding cascades, and they may have some dependence on κ and m. They represent the minimum required Reynolds number for separating the dissipation scale from the injection scale.

These relations are consistent with what is anticipated from dimensional analysis. However, for the relevant case of molecular diffusion $\kappa = 1$, we find that the dissipation scale $\ell_{uv}^{(\eta)}$ is instead obtained by solving the following transcendental equation

$$\left(\frac{\ell_0}{\ell_{uv}^{(\eta)}}\right)^{3/2} \ln\left(\frac{\ell_0}{\ell_{uv}^{(\eta)}}\right) = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\eta)}}\frac{\eta_{uv}^{1/3}\ell_0^2}{\nu}\right]^{3/4}. \tag{2.29}$$

We see that in most cases the result is consistent with dimensional analysis and with "Hypothesis 3" of Eyink [8]. The interesting exception is the case $\kappa = 1$ which is consistent with "Hypothesis 2" of Eyink [8] but not "Hypothesis 3". These results should be modified if there are intermittency corrections to the cascades or if it should turn out that the fusion rules fail. A similar calculation for the subleading cascades gives the other two dissipation scales

$$\begin{aligned} \ell_{uv}^{(\varepsilon)} &= \ell_0\left[\frac{\mathcal{R}_{uv}^{(\varepsilon)}}{\mathcal{R}_{0,uv}^{(\varepsilon)}}\right]^{3/(2-6\kappa)} = \left[\frac{1}{\mathcal{R}_{0,uv}^{(\varepsilon)}}\frac{\varepsilon_{uv}^{1/3}}{\nu}\right]^{3/(2-6\kappa)} \\ \ell_{ir}^{(\eta)} &= \ell_0\left[\frac{\mathcal{R}_{ir}^{(\eta)}}{\mathcal{R}_{0,ir}^{(\eta)}}\right]^{1/(2m)} = \left[\frac{1}{\mathcal{R}_{0,ir}^{(\eta)}}\frac{\eta_{ir}^{1/3}}{\beta}\right]^{1/(2m)}, \end{aligned} \tag{2.30}$$

where, $\mathcal{R}_{uv}^{(\varepsilon)}$ and $\mathcal{R}_{ir}^{(\eta)}$, are defined as

$$\begin{aligned} \mathcal{R}_{uv}^{(\varepsilon)} &= \frac{\varepsilon_{uv}^{1/3}\ell_0^{2\kappa-2/3}}{\nu} \\ \mathcal{R}_{ir}^{(\eta)} &= \frac{\eta_{ir}^{1/3}\ell_0^{-2m}}{\beta}. \end{aligned} \tag{2.31}$$

For the case of Ekman damping $m = 0$, the expression given for $\ell_{ir}^{(\eta)}$ is invalid. If we allow the logarithmic correction, then the expression for $\ell_{ir}^{(\eta)}$ should be replaced by an exponential function. However it is far more plausible that this result means that a constant inverse enstrophy flux may not be possible for this case. That would imply that the energy flux cannot be constant either, as we will show in the next section. It should be stressed again that these results hinge on the validity of the fusion rules.

For the case where there is the dominant downscale enstrophy flux and the dominant upscale energy flux, the interesting dissipation scales are defined as $\ell_{ir} \equiv \ell_{ir}^{(\varepsilon)}$ and $\ell_{uv} \equiv \ell_{uv}^{(\eta)}$. That is, ℓ_{ir} is the leading upscale energy dissipation scale, and ℓ_{uv} is the leading downscale enstrophy dissipation scale.

To put this theory on solid ground, it is necessary to consider a number of subtle questions, such as locality and stability. The crucial condition that needs to be established to show that energy and enstrophy cascades will form both on the upscale and downscale side of injection is threefold: First, the homogeneous solution shouldn't be hidden by the particular solution. Second, the dissipative corrections to the homogeneous solution should not destroy the inertial range but allow ample room for a cascade to form. Third, the dissipative scales must be positioned so that the incoming energy and enstrophy can be dissipated. Further development of this theory promises to show under what circumstances this condition is satisfied.

3. On the coexistence of constant energy and enstrophy flux.

The notion of an inertial range where constant (in wavenumber) energy flux and a constant enstrophy flux coexist runs against a common misconception. It is widely believed that Kraichnan [19] showed, in his original paper, that it is not possible for constant energy flux and constant enstrophy flux to coexist in the same inertial range. As a matter of fact, the conclusion itself is false, and a careful examination of the actual argument that Kraichnan made will show that it can be adopted to corroborate the opposite conclusion: that it is possible for constant energy flux and constant enstrophy flux to coexist. We will begin with a presentation of our own simpler proof. Then, we will discuss Kraichnan's argument in detail.

3.1. Proof that fluxes can coexist.

There are several objections that can be raised against the claim that constant energy flux and constant enstrophy flux cannot coexist. First, we can argue that the claim is inconsistent with numerical simulations. Consider for example the case of the direct enstrophy cascade. Lindborg and Alvelius [24] report that they were able to produce a direct enstrophy cascade where the enstrophy flux is constant. Because the simulation takes place in a finite domain and the dissipation wave number is itself finite, there is inevitably a certain amount of energy dissipated with the enstrophy. This energy has to find its way from the forcing range to the dissipation range. Because the enstrophy flux is observably constant in the inertial range, the energy flux is constrained to be constant as well, from the relationship (3.4), which we will prove in a moment.

In our opinion, nothing more needs to be said to settle the matter. However, because this claim is considered, by many, controversial, it is still useful and interesting to support it with a theoretical proof. We begin with showing that when one of the two fluxes is constant, the other flux is also required to be constant. In other words, the possibility where either the energy or the enstrophy flux is constant and the other flux is not constant is not allowed. Then, we will show that the scenario where both fluxes are constant and non-zero is allowed.

Let $\varepsilon(k)$ be the amount of energy and $\eta(k)$ be the amount of enstrophy transferred by triad interactions from the $(0,k)$ interval to the $(k,+\infty)$ interval. The conservation of energy and enstrophy by the triad interactions implies the following boundary conditions for $\varepsilon(k)$ and $\eta(k)$

$$\begin{aligned}\varepsilon(0) &= \lim_{k\to+\infty}\varepsilon(k) = 0\\ \eta(0) &= \lim_{k\to+\infty}\eta(k) = 0.\end{aligned} \tag{3.1}$$

Recall that the evolution of the energy and enstrophy spectrum is governed by

$$\begin{aligned} \frac{\partial E(k)}{\partial t} + \frac{\partial \varepsilon(k)}{\partial k} &= -\mathcal{D}(k)E(k) + F(k) \\ \frac{\partial G(k)}{\partial t} + \frac{\partial \eta(k)}{\partial k} &= -\mathcal{D}(k)G(k) + k^2 F(k), \end{aligned} \tag{3.2}$$

where $\mathcal{D}(k)$ is the dissipation operator given by

$$\mathcal{D}(k) = 2\nu k^{2\kappa} + 2\beta k^{-2m}, \tag{3.3}$$

and $F(k)$ is the forcing spectrum. The second equation can be obtained from the first equation by multiplying it with k^2. It follows that the energy flux $\varepsilon(k)$ and enstrophy flux $\eta(k)$ satisfy the following constraint

$$\frac{\partial \eta(k)}{\partial k} = k^2 \frac{\partial \varepsilon(k)}{\partial k}. \tag{3.4}$$

Note that this relationship, originally proven by Leith [22], is exact, and it is an immediate consequence of the relationship $G(k) = k^2 E(k)$ between the energy spectrum and the enstrophy spectrum. This immediately implies that it is not permitted for only one of the two fluxes $\varepsilon(k)$ and $\eta(k)$ to be constant in wavenumber.

To show that it is possible for a constant energy and enstrophy flux to coexist we proceed with the following argument. It can be shown that $\varepsilon(k)$ and $\eta(k)$ are related with the third order standard structure function $S_3(r)$ by linear transformations written as

$$\begin{aligned} \varepsilon(k) &= \int_0^{+\infty} \mathcal{K}_1(k,r) S_3(r) dr \\ \eta(k) &= \int_0^{+\infty} \mathcal{K}_2(k,r) S_3(r) dr, \end{aligned} \tag{3.5}$$

where $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$ are the kernels of appropriate integro-differential operators. A similar relationship of this form for the energy flux $\varepsilon(k)$ was given by Frisch [12] for the case of three-dimensional turbulence. In a double cascade scenario, we anticipate from the linearity of the statistical theory that

$$\begin{aligned} S_{3,uv}(r) &= -a_1 \varepsilon_{uv} r + a_2 \eta_{uv} r^3 \\ S_{3,ir}(r) &= a_3 \varepsilon_{ir} r - a_4 \eta_{ir} r^3. \end{aligned} \tag{3.6}$$

These relationships are expected on the same grounds as the form of the energy spectrum given by (2.22) as long as the corresponding cascades form successfully. Without having detailed knowledge of the kernels $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$, it is natural to suggest that the contributions of the energy and enstrophy homogeneous solutions are mixed together in the evaluation of the energy flux and enstrophy flux. This would have implied that the presence of the enstrophy solution interferes with the energy flux $\varepsilon(k)$, and the presence of the energy solution interferes with the enstrophy flux $\eta(k)$. If that is the case, then it is plausible to expect that it may be forbidden for both fluxes to be constant simultaneously as a result of this interference effect.

We will show that these interferences do not take place, and in fact we will demonstrate that this result can be established without having full knowledge of the mathematical form of the kernels $\mathcal{K}_1(k,r)$ and $\mathcal{K}_2(k,r)$. It is a direct consequence of the conservation of enstrophy and the relationship between the energy and enstrophy spectrum.

Begin with introducing a potential function $P(k)$, defined from the energy flux $\varepsilon(k)$ as

$$P(k) = \int_0^k 2q\varepsilon(q)\, dq. \tag{3.7}$$

It follows that there is a kernel $\mathcal{K}(k,r)$, that relates the function $P(k)$ with $S_3(r)$ by a linear transformation

$$P(k) = \int_0^{+\infty} \mathcal{K}(k,r) S_3(r) dr. \tag{3.8}$$

The energy flux can be calculated from the following relationship

$$\varepsilon(k) = \frac{1}{2k}\frac{\partial P(k)}{\partial k}. \tag{3.9}$$

Likewise, the enstrophy flux is given by

$$\begin{aligned} \eta(k) &= \int_0^k q^2 \frac{\partial \varepsilon(q)}{\partial q}\, dq = k^2\varepsilon(k) - \int_0^k 2q\varepsilon(q)\, dq \\ &= \frac{k}{2}\frac{\partial P(k)}{\partial k} - P(k). \end{aligned} \tag{3.10}$$

These relationships imply the following corresponding relationships between the kernels $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{K}$.

$$\begin{aligned} \mathcal{K}_1(k,r) &= \frac{1}{2k}\frac{\partial \mathcal{K}(k,r)}{\partial k} \\ \mathcal{K}_2(k,r) &= \frac{k}{2}\frac{\partial \mathcal{K}(k,r)}{\partial k} - \mathcal{K}(k,r). \end{aligned} \tag{3.11}$$

For an inertial range (k_1, k_2) with a double cascade of energy and enstrophy we require that

$$\begin{aligned} \varepsilon(k) &= \varepsilon,\ \forall k \in (k_1,k_2) \\ \eta(k) &= \eta,\ \forall k \in (k_1,k_2). \end{aligned} \tag{3.12}$$

It is sufficient to show that there is a unique function $P(k)$ that can satisfy both conditions. Indeed, the requirement that the energy flux be constant gives

$$\frac{1}{2k}\frac{\partial P(k)}{\partial k} = \varepsilon \Longleftrightarrow P(k) = c_1 + \varepsilon k^2. \tag{3.13}$$

where c_1 is an integration constant. Likewise, the requirement that the enstrophy flux be constant gives

$$\frac{k}{2}\frac{\partial P(k)}{\partial k} - P(k) = \eta \Longleftrightarrow P(k) = -\eta + c_2 k^2. \tag{3.14}$$

where c_2, likewise, is an integration constant. It follows that both conditions can be satisfied by $P(k)$ if $c_1 = -\eta$ and $c_2 = \varepsilon$, yielding

$$P(k) = \varepsilon k^2 - \eta,\ \forall k \in (k_1,k_2). \tag{3.15}$$

The fact that the two forms of the function $P(k)$ are consistent so that adjusting the integration constants c_1 and c_2 can yield a function $P(k)$ consistent with double energy and enstrophy fluxes, provides the evidence that the double cascade is indeed allowed. The permission is also implicit in the constraint given by equation (3.4), however this argument establishes the necessary consistency between equations (3.12) and (3.6). In realistic situations $P(k)$ has an additional term for

dissipative adjustments, which can be safely neglected when universal cascades form successfully.

The well-known derivations of the scaling of $S_3(r)$ for the cases of a pure energy cascade and a pure enstrophy cascade, suggest the following implications

$$\begin{aligned}\int_0^{+\infty} \mathcal{K}(k,r)S_3(r)dr = \varepsilon k^2 &\Longleftrightarrow S_3(r) = a\varepsilon r \\ \int_0^{+\infty} \mathcal{K}(k,r)S_3(r)dr = -\eta &\Longleftrightarrow S_3(r) = b\eta r^3.\end{aligned} \tag{3.16}$$

Since our theory already predicts the form of $S_3(r)$, as a linear combination it is only sufficient to show that

$$\begin{aligned}S_3(r) = a\varepsilon r + b\eta r^3 &\Longrightarrow P(k) = \varepsilon k^2 - \eta \\ &\Longrightarrow [\varepsilon(k) = \varepsilon] \wedge [\eta(k) = \eta].\end{aligned} \tag{3.17}$$

The first step follows from the converse of (3.16), and the second step from (3.13), (3.14). Thus, both steps can be justified without detailed knowledge of the kernels $\mathcal{K}_1, \mathcal{K}_2$.

3.2. **A review of Kraichnan's argument.** There is a widely believed folklore argument, sometimes attributed to Kraichnan. According to this argument, one begins by subdividing the inertial range wavenumber interval into logarithmically spaced little intervals. Then, one argues that, because of locality, there exists a cascade of adjacent pair interactions such that both energy and enstrophy flow from one interval into the next. Consequently, from the point of view of a particular interval, it is being forced only by the preceding interval, and therefore the energy and enstrophy flux into the interval has to satisfy

$$\eta(k) = \varepsilon(k)k^2. \tag{3.18}$$

It follows, according to this (incorrect) argument, that it is not possible for constant energy flux and constant enstrophy flux to coexist, because (3.18) cannot be satisfied by $\varepsilon(k)$ and $\eta(k)$ if they are both constant.

As a matter of fact, Kraichnan has explicitly argued against the foundation of this argument in the fifth paragraph of section 1 of his paper. He explained that the concept of the dominant interactions as adjacent pair interactions between logarithmically spaced intervals may be valid in three-dimensional turbulence but is not applicable in two-dimensional turbulence, where transfers can only occur between the middle member of a triad and the two triad members, one on the long-wave side and one on the short-wave side.

However, Kraichnan continues in the seventh paragraph, with the following argument. He claims that for "similarity cascades", for the case where the energy flux is constant, for example, the enstrophy flux is given by

$$\eta(k) = A\varepsilon k^2, \tag{3.19}$$

using a "similarity argument". It follows that when the constant A is non-zero, there is a violation of the conservation of enstrophy, therefore it must be that $A = 0$. That would imply that the enstrophy flux itself is zero. A similar argument can be made for the case where the enstrophy flux is constant. This argument appears to prove that constant energy flux and constant enstrophy flux cannot coexist.

Kraichnan takes it for granted that inertial ranges are self-similar, so he does not consider it necessary to define his notion of self-similarity explicitly in section 1 of his paper. The definition he gives in section 2 is

$$\begin{aligned} E(\lambda k) &= \lambda^{-n} E(k) \\ T(\lambda k, \lambda p, \lambda q) &= \lambda^{\zeta} T(k,p,q), \end{aligned} \tag{3.20}$$

where $T(k,p,q)$ are the spherically integrated transfer rates of the triad interactions. As long as we assume that there are no intermittency corrections, the scaling exponent ζ is given by

$$\zeta = -\frac{1+3n}{2} = -\frac{7+3\zeta_2}{2}, \tag{3.21}$$

where $n = 2+\zeta_2$ is the absolute value of the slope of $E(k) \sim k^{-n}$ and the scaling exponent actually used by Kraichnan. The energy cascade corresponds to $\zeta = -3$ and the enstrophy cascade to $\zeta = -5$.

There is no physical justification behind asserting this similarity condition, and this is precisely where the flaw in Kraichnan's argument is located. We agree with Kraichnan that if one *assumes* this similarity condition, then the cascades have to be pure cascades. However, it is obvious that by asserting a similarity condition on the energy spectrum one axiomatically excludes the energy spectrum of the double cascade, which has two spectral slopes in linear combination. The same problem arises if one imposes a self-similarity condition on the transfer rates $T(k,p,q)$ or the structure functions $S_3(r)$. In our theoretical framework, the self-similarity of the generalized structure functions is applicable only to the individual homogeneous solutions but not to a linear combination of those solutions. The same principle applies to the triad interactions transfer rates used in Kraichnan's argument.

The rest of Kraichnan's argument proceeds as follows. The fluxes can be obtained from the triad interactions transfer rates by the following integrals

$$\begin{aligned} \varepsilon(k) &= \frac{1}{2}\int_k^\infty d\kappa \int_0^k \int_0^k T(\kappa,p,q)\, dp\, dq - \frac{1}{2}\int_0^k d\kappa \int_k^\infty \int_k^\infty T(\kappa,p,q)\, dp\, dq \\ \eta(k) &= \frac{1}{2}\int_k^\infty d\kappa \int_0^k \int_0^k \kappa^2 T(\kappa,p,q)\, dp\, dq - \frac{1}{2}\int_0^k d\kappa \int_k^\infty \int_k^\infty \kappa^2 T(\kappa,p,q)\, dp\, dq. \end{aligned} \tag{3.22}$$

These relations are exact. Using his similarity condition, and the constraint between energy and enstrophy transfer rates, and some algebra, Kraichnan shows that the fluxes are given by

$$\begin{aligned} \varepsilon(k) &= k^{\zeta+3} \int_0^1 dv \int_1^{+\infty} dw\, W_\varepsilon(v,w,\zeta) T(1,v,w) \\ \eta(k) &= k^{\zeta+5} \int_0^1 dv \int_1^{+\infty} dw\, W_\eta(v,w,\zeta) T(1,v,w). \end{aligned} \tag{3.23}$$

This relation is similar, in spirit, to the relation (3.5) between $\varepsilon(k)$, $\eta(k)$ and $S_3(r)$. The relation $\eta(k) = A\varepsilon(k)k^2$, used by Kraichnan in section 1, follows immediately from (3.23), which is correct only when the self-similarity condition on $T(k,p,q)$

holds. The functions W_ε and W_η read

$$W_\varepsilon(v,w,\zeta) = -\frac{1}{w^2-v^2}\left[(1-v^2)\int_1^w u^{-\zeta-4}du - (w^2-1)\int_v^1 u^{-\zeta-4}du\right]$$
$$W_\eta(v,w,\zeta) = -\frac{1}{w^2-v^2}\left[(1-v^2)w^2\int_1^w u^{-\zeta-6}du - (w^2-1)v^2\int_v^1 u^{-\zeta-6}du\right]. \tag{3.24}$$

A simple evaluation of the integrals shows that

$$\begin{aligned} W_\eta(v,w,-3) &= 0 \\ W_\varepsilon(v,w,-5) &= 0. \end{aligned} \tag{3.25}$$

The other two relevant combinations $W_\varepsilon(v,w,-3)$ and $W_\eta(v,w,-5)$ are non-zero. From these Kraichnan derives, implicitly, the following equivalences:

$$\begin{aligned} \zeta = -3 &\Longleftrightarrow [\varepsilon(k) = \varepsilon] \wedge [\eta(k) = 0] \\ \zeta = -5 &\Longleftrightarrow [\varepsilon(k) = 0] \wedge [\eta(k) = \eta]. \end{aligned} \tag{3.26}$$

Therefore it follows that, as long as the similarity condition[2] holds, double cascades are not allowed.

If we assume that Kraichnan's self-similarity condition applies *to the homogeneous solutions only*, then it follows that the energy cascade solution transfers no enstrophy and the enstrophy cascade solution transfers no energy. This is a desirable conclusion since in the case of a pure cascade there will be only one homogeneous solution present. In a double cascade of energy and enstrophy, the transfer rate can be decomposed into two parts

$$T(k,p,q) = T_\varepsilon(k,p,q) + T_\eta(k,p,q), \tag{3.27}$$

with each part being individually self-similar with different scaling exponents as follows

$$\begin{aligned} T_\varepsilon(\lambda k, \lambda p, \lambda q) &= \lambda^{-3} T_\varepsilon(k,p,q) \\ T_\eta(\lambda k, \lambda p, \lambda q) &= \lambda^{-5} T_\eta(k,p,q). \end{aligned} \tag{3.28}$$

The first part $T_\varepsilon(k,p,q)$ is responsible for the energy transfer and transfers no enstrophy. The second part $T_\eta(k,p,q)$ is responsible for the enstrophy transfer and transfers no energy. Therefore, Kraichnan's argument can the leveraged to show that the interactions responsible for the enstrophy transfer do not interfere with the energy transfer and vice versa. We have given a simpler proof of the same result in the preceding section, but some readers might find this argument more convincing.

4. **Conclusions.** Unlike the very idealized and clearly unrealizable scenario of pure upscale energy cascade and pure downscale enstrophy cascade envisaged by Kraichnan, Leith and Batchelor for two-dimensional turbulence, in realistic situations there are double cascades, as long as universality does not fail. Upscale of energy and enstrophy injection, there are upscale fluxes of both energy and enstrophy. Downscale of injection there are downscale fluxes of both energy and enstrophy. We establish theoretically that constant energy flux and constant enstrophy flux can co-exist in the same inertial range. When both are present, dimensional analysis which was

[2]It should be noted that, the self-similarity of the energy spectrum has not been used in this argument, and it is in fact violated for the case of a pure enstrophy cascade by the logarithmic correction. Only the self-similarity of the triad interactions has been invoked.

used by Kraichnan to yield the energy spectrum is no longer valid. A correct form for the energy spectrum is derived here, which turns out to be a linear superposition of the $k^{-5/3}$ spectrum associated with the energy flux (regardless of the direction), the k^{-3} spectrum associated with the enstrophy flux, and a particular solution contributed by forcing and the boundary conditions. This follows from the linearity of the infinite system governing the inertial range behavior for the structure functions. These two results apply for finite viscosities as well as in the limit of small viscosities. In the second part of this series, we show that the subleading contribution to the energy spectrum remains hidden in two-dimensional turbulence, although it is exposed in atmospheric turbulence. The successful formation of a double cascade that can be observed in the energy spectrum requires that the contributions of the homogeneous solution dominate the particular solution, and that the the dissipative corrections to the homogeneous solution allow a spectral region where it remains unmodified. A detailed development of this theory promises to tell us under what conditions these requirements are satisfied.

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Appendix A. **Relationship between energy spectrum and F_2.** The relationship between the energy spectrum and the second order structure function has been part of the folklore of the statistical theory of turbulence for a long time. The one-dimensional version of this result was discovered independently by Wiener [45], Khinchin [16], and Einstein (for reprint and interesting commentary see [7, 13, 46]), and its relevence to turbulence was first highlighted by Taylor [40]. The three-dimensional version was used by Batchelor [2] in his famous monograph, as well as his papers, and a version for any dimension can be found in Panchev [33], without proof. The one dimensional and three-dimensional version have also been stated without proof by Frisch [12].

The version shown here has been derived independently while referring to Panchev [33], and using the definition of the energy spectrum given by Frisch [12]. The localized energy spectrum is defined as

$$E_{\alpha\beta}(k,\mathbf{x},t) = \frac{1}{2}\frac{d}{dk}\langle u_\alpha^{<k}(\mathbf{x},t)u_\beta^{<k}(\mathbf{x},t)\rangle, \tag{A.1}$$

where $u_\alpha^{<k_0}$ is the filtered velocity field defined as

$$u_\alpha^{<k_0}(\mathbf{x},t) = \frac{1}{(2\pi)^d}\int d\mathbf{k}\int d\mathbf{l}\; u_\alpha(\mathbf{l},t)\exp(i\mathbf{k}\cdot(\mathbf{l}-\mathbf{x}))H(k_0-\|\mathbf{k}\|), \tag{A.2}$$

where H is the Heaviside function

$$H(x) = \begin{cases} 1 & ,x>0 \\ 1/2 & ,x=0 \\ 0 & ,x<0 \end{cases}. \tag{A.3}$$

Integrating over the spatial dependence gives the standard energy spectrum

$$E(k,t) = \int d\mathbf{x} E_{\alpha\alpha}(k,\mathbf{x},t). \tag{A.4}$$

Likewise, we define a 2nd order structure function $S_2(\rho)$ in terms of F_2 as follows

$$S_2(\rho) \equiv \frac{1}{\gamma_d} \int d\mathbf{x} \int_{SO(d)} d\Omega(A) F_2(\mathbf{x}, \mathbf{x} + \rho A\mathbf{e}, \mathbf{x}, \mathbf{x} + \rho A\mathbf{e}), \tag{A.5}$$

where $\mathbf{e}$ is a fixed unit vector, and $d\Omega(A)$ represents spherical integration defined as

$$\begin{aligned} d\mathbf{x} &= r^{d-1} dr d\Omega(A) \\ d\Omega(A) &= \prod_{\alpha=1}^{d-1} (\sin\phi_\alpha)^{d-1-\alpha} d\phi_\alpha, \end{aligned} \tag{A.6}$$

with $\phi \in [0,\pi]^{d-2} \times [0, 2\pi)$ and

$$\gamma_d \equiv \int_{SO(d)} d\Omega(A) = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{A.7}$$

With these definitions in place, the following relationship between the energy spectrum and the second order structure function can be derived rigorously

$$S_2(\rho, t) = 4 \int_0^{+\infty} dk E(k,t)[1 - \Phi_d(k\rho)], \tag{A.8}$$

where, $\Phi_d(x)$ is the kernel of the spherical Fourier transform given by

$$\begin{aligned} \Phi_d(x) &\equiv \frac{1}{\gamma_d} \int_{SO(d)} d\Omega(A) \exp[ix\mathbf{e} \cdot (A\mathbf{e})] \\ &= \frac{(2\pi)^{d/2}}{\gamma_d} \frac{J_{(d-2)/2}(x)}{x^{(d-2)/2}}. \end{aligned} \tag{A.9}$$

For $d = 2$ we have $\Phi_2(x) = J_0(x)$, where J_0 is the Bessel function.

Using this relationship, it is then possible to prove the relationship between the scaling exponent of F_2 and the scaling exponent of the energy spectrum. In particular, it can be shown that for $1 < m < 3$

$$E(k) = ak^{-m}, \quad \forall k \gg k_0 \Longleftrightarrow S_2(r) = aC(d) r^{m-1}, \quad \forall r \ll k_0^{-1}, \tag{A.10}$$

where the constant $C(d)$ depends on the dimension d and reads

$$C(d) = 4 \int_0^{+\infty} \frac{1 - \Phi_d(x)}{x^m}\, dx. \tag{A.11}$$

Appendix B. **The terms of the balance equations.** Recall that the balance equations for the generalized structure functions read

$$\frac{\partial F_n}{\partial t} + D_n = \nu J_n + \beta H_n + Q_n, \tag{B.1}$$

To write equations concisely, we introduce the following notation to represent aggregates of position vectors

$$\begin{aligned} \mathbf{X} &= (\mathbf{x}, \mathbf{x}') \\ \{\mathbf{X}\}_n &= \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\} \\ \{\mathbf{X}_\alpha\}_n^k &= \{\mathbf{X}_1, \ldots, \mathbf{X}_{k-1}, \mathbf{X}_{k+1}, \ldots, \mathbf{X}_n\}. \end{aligned} \tag{B.2}$$

We use the notation $\{\mathbf{X}\}_n + \Delta\mathbf{r}$ as a shorthand to represent shifting all the constituent vectors of $\{\mathbf{X}\}_n$ by the same displacement $\Delta\mathbf{r}$.

We now describe the terms in detail. The dissipation terms are given by

$$\begin{aligned} H_n &= \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{-2m} + \nabla_{\mathbf{x}'_k}^{-2m}) F_n \\ J_n &= \sum_{k=1}^{n} (\nabla_{\mathbf{x}_k}^{2\kappa} + \nabla_{\mathbf{x}'_k}^{2\kappa}) F_n, \end{aligned} \tag{B.3}$$

where $\nabla_{\mathbf{x}_k}^{2k}$ differentiates with respect to $\mathbf{x}_k$ and similarly with $\nabla_{\mathbf{x}'_k}^{2k}$, $\nabla_{\mathbf{x}_k}^{-2m}$ and $\nabla_{\mathbf{x}'_k}^{-2m}$. The forcing contribution is given by

$$\begin{aligned} Q_n(\{\mathbf{X}\}_n, t) &= \sum_{k=1}^{n} Q_{kn}(\{\mathbf{X}\}_n^k, \mathbf{X}_k, t) \\ Q_{kn}(\{\mathbf{X}\}_{n-1}, \mathbf{Y}, t) &= \left\langle \left[\prod_{k=1}^{n-1} w_{\alpha_k}(\mathbf{X}_k, t) \right] \varphi_\beta(\mathbf{Y}, t) \right\rangle, \end{aligned} \tag{B.4}$$

where

$$\varphi_\alpha(\mathbf{X}, t) = f_\alpha(\mathbf{x}, t) - f_\alpha(\mathbf{x}', t). \tag{B.5}$$

If we assume that the velocity field is globally homogeneous, it can be shown that D_n can be rewritten exclusively in terms of velocity differences. Specifically, D_n can be obtained by applying a linear operator $\mathcal{O}$ on F_{n+1}, and we write

$$\begin{aligned} D_n(\{\mathbf{X}\}_n, t) &= \sum_{k=1}^{n} D_{kn}(\{\mathbf{X}\}_n, t) \\ D_{kn}(\{\mathbf{X}\}_n, t) &= \int \mathcal{O}(\mathbf{X}_k, \mathbf{Y}_1, \mathbf{Y}_2)\ F_{n+1}(\{\mathbf{X}\}_n^k, , \mathbf{Y}_1, \mathbf{Y}_2, t)\ d\mathbf{Y}_1 d\mathbf{Y}_2 \end{aligned} \tag{B.6}$$

In the more general inhomogeneous case, the correct expression for D_n is

$$D_n(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} D_{kn}(\{\mathbf{X}\}_n, t) + I_n(\{\mathbf{X}\}_n, t), \tag{B.7}$$

where I_n represents the effect of inhomogeneities given by

$$I_n(\{\mathbf{X}\}_n, t) = \sum_{k=1}^{n} (\partial_{\beta,\mathbf{x}_k} + \partial_{\beta,\mathbf{x}'_k}) \left\langle \mathcal{U}_\beta(\{\mathbf{X}\}_n, t) \left[\prod_{k=1}^{n} w_{\alpha_k}(\mathbf{X}_k, t) \right] \right\rangle, \tag{B.8}$$

where $\mathcal{U}_\beta(\{\mathbf{X}\}_n, t)$ is defined as

$$\mathcal{U}_\alpha(\{\mathbf{X}\}_n, t) = \frac{1}{2n} \sum_{k=1}^{n} \left(u_\alpha(\mathbf{x}_k, t) + u_\alpha(\mathbf{x}'_k, t) \right), \tag{B.9}$$

The $\mathcal{O}$ operator is too cumbersome to write explicitly. However, D_{kn} can be written as

$$\begin{aligned} D_{kn}(\{\mathbf{X}_n\}) &= \int d\mathbf{x} P_{\alpha_k \beta}(\mathbf{x}) \left\langle \left[\prod_{j=1, j\neq k}^{n} w_{\alpha_j}(\mathbf{X}_j) \right] \mathcal{L}_\beta(\mathbf{X}_k - \mathbf{x}, \{\mathbf{X}\}_n) \right\rangle \\ \mathcal{L}_\beta(\mathbf{Y}, \{\mathbf{X}\}_n) &= \frac{1}{n} \sum_{k=1}^{n} [w_\gamma(\mathbf{y}, \mathbf{x}_k) \partial_{\gamma,\mathbf{y}} + w_\gamma(\mathbf{y}, \mathbf{x}'_k) \partial_{\gamma,\mathbf{y}'}] w_\beta(\mathbf{Y}), \end{aligned} \tag{B.10}$$

where $P_{\alpha\beta}$ is the projection operator defined as

$$(\delta_{\alpha\beta} - \partial_\alpha \partial_\beta \nabla^{-2}) v_\beta(\mathbf{x}) = \int d\mathbf{y}\ P_{\alpha\beta}(\mathbf{x} - \mathbf{y}) v_\beta(\mathbf{y}). \tag{B.11}$$

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