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Does the subdominant part of the energy spectrum due to downscale energy cascade remain hidden in quasi-geostrophic turbulence?

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In systems governing two-dimensional turbulence, surface quasi-geostrophic turbulence, (more generally α-turbulence), two-layer quasi-geostrophic turbulence, etc., there often exist two conservative quadratic quantities, one "energy"-like and one "enstrophy"-like. In a finite inertial range there are in general two spectral fluxes, one associated with each conserved quantity. The energy spectrum in general has a contribution from each of the fluxes, and our previous work showed that these two contributions to the energy spectrum can be linearly superimposed despite the highly nonlinear nature of the problem. Often, one of the fluxes is dominant and the energy spectrum then has the visual shape of the case with a single flux; the contribution from the subdominant flux is effectively hidden. The relative magnitudes of the spectral fluxes depend on the dissipative sinks in the system, and varies according to the physical/mathematical system under consideration. We derive an important inequality involving the "energy" and enstrophy" fluxes for each representative system. This result then allows us to determine the effective energy spectral shape in the general case of double cascades.

1. INTRODUCTION

The characteristic feature of two-dimensional turbulence is that there are two conserved quantities, kinetic energy and enstrophy. This led Kraichnan (1967), Leith (1968), and Batchelor (1969) to conjecture that there will exist two inertial ranges, one located upscale of the spectral region of injection and another on the downscale side of injection. In the upscale side, it is assumed that there is only an upscale flux of energy, and no flux of enstrophy. On the downscale side, likewise, there is only a downscale flux of enstrophy, and no flux of energy. One then uses a dimensional analysis argument to calculate the energy spectrum $E(k)$ where it is assumed that in each inertial range it depends only on the corresponding single flux and the wavenumber k. The same type of argument was used in the energy cascade of three-dimensional turbulence (Batchelor, 1947; Kolmogorov, 1941a,b). Although three-dimensional turbulence also has two conserved quantities, energy and helicity, one has the option to inject energy without injecting helicity. In two-dimensional turbulence it is not possible to inject energy without injecting enstrophy and vice versa, because the two quantities are related

Initial efforts to simulate the enstrophy cascade yielded confusing reports of various numerical slopes. Consequently, alternative theories have been proposed over the past 30 years to explain them (Moffatt, 1986; Polyakov, 1993; Saffman, 1971). Recently, in carefully set up experiments, it was shown that it is possible to obtain the enstrophy cascade in agreement with the KLB theory (Ishihira and Kaneda, 2001; Lindborg and Alvelius, 2000; Pasquero and Falkovich, 2002) . A numerical simulation with very good diagnostics has shown that the inverse energy cascade can be obtained accordingly (Boffetta et al., 2000). There are also however many papers that question the universality of these results (Danilov, 2005; Danilov and Gurarie, 2001a,b; Tran and Bowman, 2003, 2004). A review can be found in Gkioulekas and Tung (2005c) and Tabeling (2002).

Further confusion has resulted from efforts to explain the observed energy spectrum of the atmosphere with the KLB theory. Observations show that there is a robust energy spectrum with slope -3 which transitions at large wavenumbers into slope $-5/3$ (Gage, 1979; Gage and Nastrom, 1986; Nastrom and Gage, 1984; Nastrom et al., 1984). In the KLB theory, on the other hand, one expects that at small wavenumbers the energy spectrum will have slope $-5/3$ from the inverse energy cascade, which will then transition at the forcing wavenumber, into a -3 slope from the direct enstrophy cascade. The apparent contradiction between these two predictions has led to various explanations and debate (Dewan, 1979; Lilly, 1989; Lindborg, 1999; VanZadt, 1982).

It was conjectured by Tung and Orlando (2003a) that the observed atmospheric energy spectrum results from the downscale cascade of enstrophy and energy injected at the large scales by baroclinic instability and dissipated at the smallest length scales. If η_{uv} is the downscale enstrophy flux and ε_{uv} is the downscale energy flux, it was suggested that they would coexist on the downscale side of injection and that their separate contributions to the energy spectrum would give the latter a compound spectral shape, with a -3 slope transitioning to a shallower $-5/3$ slope as the wavenumber increases. The transition from -3 slope to $-5/3$ slope occurs at the transition wavenumber k_t with order of magnitude estimated by $k_t \approx \sqrt{\eta_{uv}/\varepsilon_{uv}}$. Tung and Orlando (2003a) have also demonstrated numerically that a two-layer quasi-geostrophic channel model with thermal forcing, Ekman damping, and hyperdiffusion can reproduce this energy spectrum. The diagnostic shown in figure 7 of (Tung and Orlando, 2003a), shows both the constant downscale energy and enstrophy fluxes coexisting in the same inertial range. Furthermore, recent measurements and data analysis by (Cho and Lindborg, 2001) have confirmed the existence of a downscale energy flux and estimate $\eta_{uv} \approx 2 \times 10^{-15}\text{s}^{-3}$ and $\varepsilon_{uv} \approx 6 \times 10^{-11}\text{km}^2\text{s}^{-3}$. From these estimates we find the mean value of the transition scale $k_t = \sqrt{\eta_{uv}/\varepsilon_{uv}} \approx 0.57 \times 10^{-2}\text{km}^{-1}$ and $\lambda_t = 2\pi/k_t \approx 1 \times 10^3\text{km}$ which has the correct order of

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magnitude.

This theory is contrary to the widely accepted misconception that the argument by (Fjørtøft, 1953) forbids a downscale energy flux in two-dimensional turbulence, and through the isomorphism theorem of Charney (1971) also in quasi-geostrophic turbulence. This misconception has been clarified by Merilees and Warn (1975), Tung and Welch (2001) and Gkioulekas and Tung (2005a).

As has been pointed out by previous authors (Borue, 1994; Eyink, 1996), as long as the dissipation terms at large-scale and small scales have finite viscosity coefficients and the inertial ranges exist, the downscale enstrophy flux will be accompanied by a small downscale energy flux, and the upscale energy flux will be accompanied by a small upscale enstrophy flux. Dimensional analysis arguments are premised on the assumption that these additional fluxes can be ignored, consequently the energy spectrum predictions obtained by such arguments are valid only to leading order. While this assumption can be justified for strictly two-dimensional turbulence, we will argue in this paper that it cannot obviously be justified for models of quasi-geostrophic turbulence, such as the two-layer model, where the subleading contributions can be important in the inertial range. Predicting the form of these subleading corrections requires a subtle mathematical argument, given by Gkioulekas and Tung (2005b), that goes beyond dimensional analysis.

In particular, Gkioulekas and Tung (2005b) have shown that the subleading fluxes are associated with a subleading downscale energy cascade and a subleading inverse enstrophy cascade that contribute *linearly* to the total energy spectrum in addition to the dominant contributions. As a result, in the downscale inertial range, the total energy spectrum $E(k)$ has the following three contributions:

$$E(k) = E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k) + E_{uv}^{(p)}(k), \ \forall k\ell_0 \gg 1, \quad (1)$$

where $E_{uv}^{(\varepsilon)}(k)$, $E_{uv}^{(\eta)}(k)$ are the contributions of the downscale energy and enstrophy cascade, given by

$$\begin{aligned} E_{uv}^{(\varepsilon)}(k) &= a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3}\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)}) \\ E_{uv}^{(\eta)}(k) &= b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi+\ln(k\ell_0)]^{-1/3}\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)}), \end{aligned} \quad (2)$$

with $\mathcal{D}_{uv}^{(\varepsilon)}$ and $\mathcal{D}_{uv}^{(\eta)}$ describing the dissipative corrections. The scales $\ell_{uv}^{(\varepsilon)}$,$\ell_{uv}^{(\eta)}$ are the dissipation length scales for the downscale energy and enstrophy cascade. Finally, $E_{uv}^{(p)}(k)$ is the contribution from the effect of forcing and the sweeping interactions. The latter can become significant via the violation of statistical homogeneity caused by the boundary conditions (see Gkioulekas (2005) for details). Thus, in the inertial range where the effect of forcing and dissipation can be ignored, the energy spectrun will take the simple form

$$E(k) \approx a_{uv}\varepsilon_{uv}^{2/3}k^{-5/3} + b_{uv}\eta_{uv}^{2/3}k^{-3}[\chi+\ln(k\ell_0)]^{-1/3}. \quad (3)$$

It should be emphasized that the formation of cascades observable in the energy spectrum is by no means guaranteed. There are two prerequisites that need to be satisfied: first, the contribution of the particular solution $E_{uv}^{(p)}(k)$ has to be negligible both downscale and upscale of the injection scale, i.e.

$$\begin{aligned} E_{uv}^{(p)}(k) &\ll E_{uv}^{(\varepsilon)}(k) + E_{uv}^{(\eta)}(k), \ \forall k\ell_0 \gg 1 \\ E_{ir}^{(p)}(k) &\ll E_{ir}^{(\varepsilon)}(k) + E_{ir}^{(\eta)}(k), \ \forall k\ell_0 \ll 1. \end{aligned} \quad (4)$$

Second, the dissipative adjustment $\mathcal{D}_{uv}^{(\eta)}(k\ell_{uv}^{(\eta)})$ and $\mathcal{D}_{uv}^{(\varepsilon)}(k\ell_{uv}^{(\varepsilon)})$ of the homogeneous solution has to be such that it does not destroy the power law scaling in the inertial range. Furthermore, the dissipation scales $\ell_{uv}^{(\eta)}$ and $\ell_{uv}^{(\varepsilon)}$ have to be positioned so that the incoming energy and enstrophy can be dissipated.

This principle of linear superposition of the enstrophy cascade and the energy cascade is similar to the superposition of isotropic and anisotropic contributions to the generalized structure functions (Arad et al., 1999; Biferale and Procaccia, 2005) and the principle of $\mathcal{Z}(h)$ covariance in the direct energy cascade of three-dimensional turbulence (Belinicher et al., 1998a,b; L'vov and Procaccia, 1998); the same idea is implicit in the multifractal model of Frisch (1995). It has been obtained by exploiting the mathematical structure of the exact statistical theory of two-dimensional turbulence (i.e. the complete infinite system of equations governing the relevant structure functions). Nonlinear results, such as the one that was proposed by Lilly (1989), follow from closure models instead of the exact theory.

In two-dimensional turbulence, the fluxes ε_{uv} and η_{uv} are constrained by an inequality, that was communicated to us by Danilov (Gkioulekas and Tung, 2005c) . This constraint implies that the contribution of the downscale energy cascade to the energy spectrum is overwhelmed by the contribution of the downscale enstrophy cascade and cannot be seen visually on a plot. This result was conjectured earlier by Smith (2004) who claimed that the downscale energy cascade can never have enough flux to move the transition wavenumber k_t into the inertial range. The two-layer model is a different dynamical system than the two-dimensional Navier-Stokes equations, and although the superposition principle is a deep mathematical result that is valid in both cases, the validity of the Danilov inequality in the two-layer model is not obvious (Gkioulekas and Tung, 2005c; Tung, 2004).

In the present paper we will show that in the two-layer model when the Ekman dissipation coefficient ν_E is below a critical value, then the Danilov inequality will be satisfied. We will also argue that the asymmetric presence of Ekman damping on the bottom layer but not the top layer may cause the violation of the Danilov inequality for larger values of the Ekman dissipation coefficient. In this case, the top layer has more energy than the bottom layer, as is realistic in the atmosphere, and provided that the difference in energy between the two layers is large enough, the downscale energy cascade will be made observable in the energy spectrum. The simulation of Tung and Orlando (2003a) has shown that it is possible to have an observable downscale energy cascade, which implies a violation of the Danilov inequality. The role of the argument in this paper is to explain how and why this can happen, given that it is a surprising and very unexpected result.

An immediate implication of our argument is that the existence of an extensive observable $k^{-5/3}$ in the energy spectrum of the atmosphere has the physical interpretation that the atmosphere is very baroclinic. We will also show that in the surface quasi-geostrophic model, which represents the most extreme case of baroclinicity, the downscale energy cascade becomes completely dominant.

The paper is organized as follows. The Danilov inequality is reviewed in section 2 where we make some simple generalizations. Its implications for two-dimensional turbulence, α-turbulence, and SQG turbulence are discussed in section 3.The surface quasi-geostrophic model is discussed in section 4 and that the two-layer model in section 5. Conclusions and some further remarks are given in section 6.

2. THE DANILOV INEQUALITY IN ONE-LAYER MODELS

The governing equation for a wide range of one-layer hydrodynamic models takes the form:

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = \mathcal{D} + \mathcal{F}, \tag{5}$$

where $\mathcal{F}$ is the forcing and $\mathcal{D}$ is the dissipation and $\zeta = -\mathcal{L}\psi$. Here, $\mathcal{L}$ is a linear isotropic operator involving the derivatives with respect to the horizontal coordinates. For a general combination of hyper- and hypo-diffusion:

$$\mathcal{D} = -\nu_0(-\Delta)^p \zeta - \nu_1(-\Delta)^{-h}\zeta, \tag{6}$$

with p, h, positive integers. $p = 1, h = 0$ yields the combination of molecular viscosity and Ekman damping.

For 2D turbulence, $\mathcal{L}$ is given by $\mathcal{L} = -\Delta$, where Δ is the Laplacian operator and the streamfunction ψ is related to the 2D nondivergent velocity as

$$(u, v) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) \tag{7}$$

For barotropic QG turbulence, also known as Charney-Hasegawa-Nima (CHM) turbulence (Charney, 1948; Hasegawa et al., 1979; Hasegawa and Mima, 1978), $\mathcal{L}$ is given instead by $\mathcal{L} = -\Delta - \lambda^2$, where λ^2 is a given positive constant. Another interesting family of one-layer models are the α-turbulence models where $\mathcal{L} = \Lambda^\alpha$ with $\Lambda \equiv (-\Delta)^{1/2}$. The case $\alpha = 1$ corresponds to surface quasi-geostrophic turbulence (SQG) which is an extreme baroclinic model, and not a barotropic model like 2D turbulence or CHM turbulence.

2.1. Conservation laws

Let $\|f\|$ be the norm of $f(x, y)$ defined as

$$\|f\| \equiv \iint (f(x, y))dxdy. \tag{8}$$

There are two inviscid quadratic invariants for (5), which are:

$$A = (1/2)\|(-\psi\zeta)\|, \tag{9}$$

$$B = (1/2)\|\zeta^2\|. \tag{10}$$

Note that B is always conserved, whereas the conservation law of A requires that $\mathcal{L}$ be self-adjoint, i.e.

$$\|f(\mathcal{L}g)\| = \|(\mathcal{L}f)g\|. \tag{11}$$

For example in 2D turbulence it is seen that,

$$E \equiv (1/2)\|(u^2 + v^2)\| = (1/2)\||\nabla\psi|^2\| \tag{12}$$

$$= (1/2)\|(-\psi\zeta)\| \equiv A, \tag{13}$$

is the kinetic energy of the 2D fluid, and

$$G \equiv (1/2)\|\zeta^2\| \equiv B \tag{14}$$

is the enstrophy. The energy spectrum $E(k)$ and enstrophy spectrum $G(k)$ are defined so that $E = \int_0^\infty E(k)dk$ and $G = \int_0^\infty G(k)dk$, where $k = |\mathbf{k}|$ is the isotropic wavenumber magnitude. In the more general case, $A(k)$ is similarly defined as the spectrum of A, and $B(k)$ the spectrum of B. In the general case $A(k)$ may not necessarily be the energy spectrum, as will be demonstrated.

The relationship, $\zeta = -\mathcal{L}\psi$, translates into the spectral relationships in the Fourier space

$$\hat{\zeta}(\mathbf{k}) = L(|\mathbf{k}|)\hat{\psi}(\mathbf{k}), \quad B(k) = L(k)A(k), \tag{15}$$

where

$$\hat{\psi}(\mathbf{k}, t) = \iint \psi(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}. \tag{16}$$

We will assume that $L(k) > 0$, so that both $A(k)$ and $B(k)$ are positive. Furthermore, we will assume that $L(k)$ is a monotonically increasing function of k. In 2D turbulence, $L(k) = k^2$; in CHM turbulence, $L(k) = k^2 + \lambda^2$; in α-turbulence, $L(k) = k^\alpha$, and in SQG, $L(k) = k$.

Furthermore, $A(k)$ and $B(k)$ satisfy the following spectral equations:

$$\frac{\partial A(k)}{\partial t} + \frac{\partial \Pi_A(k)}{\partial k} = -D_A(k) + F_A(k) \tag{17}$$

$$\frac{\partial B(k)}{\partial t} + \frac{\partial \Pi_B(k)}{\partial k} = -D_B(k) + F_B(k). \tag{18}$$

Here, $D_A(k)$ and $D_B(k)$ are the spectral dissipations rate of $A(k)$ and $B(k)$, respectively, with

$$D_B(k) = L(k)D_A(k), \tag{19}$$

$$D_A(k) = [\nu_0 k^{2p} + \nu_1 k^{-2h}]A(k) > 0, \tag{20}$$

for a combination of hyper- and hypo-viscosities. Furthermore, $F_A(k)$ and $F_B(k)$ are the spectra of forcing also related by $F_B(k) = L(k)F_A(k)$, and, $\Pi_A(k)$ and $\Pi_B(k)$ are the spectral fluxes of A and B. Ensemble average is taken in

(17) and (18), but will not be denoted with different symbols here. The Leith (1968) constraint on the fluxes generalizes to

$$\frac{\partial \Pi_B(k)}{\partial k} = L(k) \frac{\partial \Pi_A(k)}{\partial k}, \tag{21}$$

and it shows that if $\Pi_B(k)$ is constant, then $\Pi_A(k)$ is also constant. The conservation laws for A and B come out as the following boundary conditions on $\Pi_A(k)$ and $\Pi_B(k)$:

$$\Pi_A(0) = \lim_{k \to +\infty} \Pi_A(k) = 0 \tag{22}$$

$$\Pi_B(0) = \lim_{k \to +\infty} \Pi_B(k) = 0. \tag{23}$$

2.2. The Danilov inequality

Assuming that the injection (forcing) of A and B occurs in $[k_1, k_2]$, then at statistical equilibrium, we have, from (17) and (18):

$$\Pi_A(k) = \int_k^{+\infty} D_A(q)\, dq, \text{ for } k > k_2 \tag{24}$$

$$\Pi_B(k) = \int_k^{+\infty} D_B(q)\, dq, \text{ for } k > k_2 \tag{25}$$

$$\Pi_A(k) = -\int_0^{k} D_A(q)\, dq, \text{ for } 0 < k < k_1 \tag{26}$$

$$\Pi_B(k) = -\int_0^{k} D_B(q)\, dq, \text{ for } 0 < k < k_1 \tag{27}$$

since $F_A(k) = 0$ and $F_B(k) = 0$ for $0 < k < k_1$ and $k > k_2$. For wavenumbers $k > k_2$, we have therefore

$$L(k)\Pi_A(k) - \Pi_B(k) = \int_k^{+\infty} [L(k) - L(q)] D_A(q)\, dq < 0 \tag{28}$$

Similarly, for wavenumbers $0 < k < k_1$, we have:

$$L(k)\Pi_A(k) - \Pi_B(k) = -\int_0^{k} [L(k) - L(q)] D_A(q)\, dq < 0 \tag{29}$$

Consequently, for all wavenumbers $k \in (0, k_1) \cup (k_2, +\infty)$ not in the forcing range, we have:

$$L(k)\Pi_A(k) - \Pi_B(k) < 0 \tag{30}$$

This inequality was brought to our attention by Danilov (2004, personal communication) for the case of 2D turbulence.

Previously, Fjørtøft (1953) and Eyink (1996) derived a similar, but looser, bound, for the downscale energy flux Π_E: $\Pi_E(k) < \eta_0/k^2$, involving the total rate of enstrophy injection η_0. This looser inequality is often used to show (Salmon, 1998) that in two-dimensional turbulence with an infinite downscale range, the energy flux $\Pi_E(k)$, vanishes. This is not true for the case of small but finite viscosity where the downscale spectral range is finite. It is also not true for the kind of quasi-geostrophic turbulence which is not 2D-like.

3. IMPLICATIONS FOR THE ENERGY SPECTRUM

The significance of the inequality (30) is that it decides whether the transition wavenumber k_t is within the inertial range, thus making a transition from the leading cascade to the subleading cascade observable in the energy spectrum $E(k)$. Whether this happens depends on the baroclinicity of the system, as we will show below by considering different cases.

It should be noted that in the following arguments it is *assumed* that an inertial range exists either upscale or downscale of injection. Unlike the case of 3D turbulence, where the downscale energy cascade is very robust, it is well known that in 2D turbulence there are circumstances where the leading inverse energy cascade (Danilov, 2005; Danilov and Gurarie, 2001a,b; Gkioulekas and Tung, 2005c) or the leading downscale enstrophy cascade (Tran and Bowman, 2003, 2004; Tran and Shepherd, 2002) may fail to appear as expected. Some of these issues are also relevant to the case of α-turbulence (Tran, 2004).

In general, the failure of cascades is to be attributed to the absense of a sufficiently strong large-scale dissipation sink. Since the observational evidence suggests that cascades exist in atmospheric turbulence, we will simply assume that without further discussion.

3.1. Two-dimensional turbulence

We begin with the classic case of 2D turbulence in finite domain with finite viscosity for the infrared and ultraviolet dissipations. In the inertial range on the downscale side of injection, $\Pi_A(k) = \epsilon_{uv}$, and $\Pi_B(k) = \eta_{uv}$. The inequality (30) implies that $\epsilon_{uv} k^2 < \eta_{uv}$ for all k in this inertial range. The energy spectrum in (1), valid in the inertial range, can be rewritten to leading order, omitting the logarithmic correction:

$$E(k) \sim C_1 \varepsilon_{uv}^{2/3} k^{-5/3} + C_2 \eta_{uv}^{2/3} k^{-3} \tag{31}$$

$$\sim C_2 \eta_{uv}^{2/3} k^{-3} \left(1 + \frac{C_1}{C_2} \left(\frac{\varepsilon_{uv} k^2}{\eta_{uv}} \right)^{2/3} \right) \tag{32}$$

$$\approx C_2 \eta_{uv}^{2/3} k^{-3} \tag{33}$$

where we use $\epsilon_{uv} k^2 < \eta_{uv}$. This sequence of steps is valid asymptotically in the limit of large separation between the forcing scale and the dissipation scale, for wavenumbers k in the inertial range. A similar argument can be made for the inertial range upscale of injection.

As shown previously by Gkioulekas and Tung (2005b,c) on the downscale side of injection the dominant cascade is the enstrophy cascade with $E(k) \sim k^{-3}$, and on the upscale side of injection the dominant cascade is the inverse energy cascade with $E(k) \sim k^{-5/3}$. By "dominant" we mean that even for finite Reynolds numbers the contributions of the subleading downscale energy cascade and the subleading inverse enstrophy cascade are hidden for *all* the wavenumbers k in the inertial range.

This argument proves more rigorously a conjecture by Smith (2004) that in 2D turbulence, on the downscale side of injection, we have no transition to shallower scaling $E(k) \sim k^{-5/3}$. His other conjecture, that the same result also holds for the two-layer QG model, can be justified only when the Ekman dissipation coefficient ν_E is below a critical value.

3.2. α-turbulence and SQG turbulence

This argument can be extended to the case of α-turbulence, which includes the case of SQG ($\alpha = 1$). Here $L(k) = k^\alpha$ and we assume $\alpha > 0$. Using the linear superposition principle discussed in Gkioulekas and Tung (2005b) the spectrum of $A(k)$ and $B(k)$ are, in the downscale inertial range:

$$A(k) = C_1(\Pi_A)^{2/3} k^{-\frac{7}{3}+\frac{1}{3}\alpha} + C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha} \quad (34)$$

$$B(k) = |k|^\alpha A(k). \quad (35)$$

Here, Π_A and Π_B are the constant fluxes on the downscale side of the forcing range. The inequality (30) becomes $k^\alpha \Pi_A < \Pi_B$ for all k in the inertial ranges. Consequently, for wavenumbers k in the inertial range, the spectrum $A(k)$ is given by

$$A(k) \sim C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha} \left(1 + \frac{C_1}{C_2}\left(\frac{\Pi_A k^\alpha}{\Pi_B}\right)^{2/3}\right) \quad (36)$$

$$\sim C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha}, \quad (37)$$

again in the limit of large separation between the forcing scale and the dissipation scale. Thus, in the downscale range, there is no observable transition and therefore:

$$A(k) \cong C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha}, \quad (38)$$

$$B(k) \cong C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}+\frac{2}{3}\alpha}. \quad (39)$$

In the upscale range, the fluxes in the inequality (30) become negative. The spectra become then

$$A(k) \cong C_1(\Pi_A)^{2/3} k^{-\frac{7}{3}+\frac{1}{3}\alpha}, \quad (40)$$

$$B(k) = k^\alpha A(k). \quad (41)$$

For the SQG model, we show in the next section that the visible energy spectrum actually has the shallower $-5/3$ slope even though the Danilov inequality is also satisfied, because actually it is $B(k)$ which is the 2D energy spectrum and $A(k)$ is instead the 3D energy spectrum; there is no enstrophy cascade.

4. PHYSICAL INTERPRETATION OF $A(k)$ AND $B(k)$ IN SQG TURBULENCE

There has been considerable confusion over the physical interpretation of the surface quasi-geostrophic model. Although its mathematical formulation is in the form of a one-layer model, it represents a three-dimensional system that corresponds to the baroclinic limit of the three-dimensional quasi-geostrophic model. Once that is taken into account, the physical interpretation of the spectra $A(k)$ and $B(k)$ and the physical implications of the Danilov inequality have to be revised.

As derived by Charney (1971), 3D QG flow conserves the 3D potential vorticity ξ, which is advected horizontally by the streamfunction ψ. Here, both ψ and ξ are 3D fields. For constant Coriolis parameter f, the governing conservation law for ξ takes the form:

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = 0, \quad (42)$$

with ξ given by

$$\xi = \Delta\psi + \frac{f^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial \psi}{\partial z}\right) \equiv \mathcal{P}\psi, \quad (43)$$

where $\rho_0(z)$ is the ambient air density, and $N^2(z)$ the Brunt-Väisälä frequency. Here we have omitted the forcing and dissipation terms.

In SQG this potential vorticity ξ is assumed, a priori, to be identically zero for $z > 0$. The streamfunction ψ is solved from $\xi = \mathcal{P}\psi = 0$. With ρ_0 and N^2 taken to be constants, the horizontal Fourier transform of $\psi(x, y, z, t)$ is obtained as

$$\hat{\psi}(\mathbf{k}, z, t) = \hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z}, \quad (44)$$

using the boundedness boundary condition as $z \to \infty$.

Most of the dynamics in this model are occurring at the surface, where the boundary condition of vanishing vertical velocity, w, applied to the potential temperature (Θ) equation leads to:

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = \mathcal{D} + \mathcal{F}, \quad (45)$$

where

$$\Theta = \frac{g}{N}\frac{T}{T_0} = \frac{f}{N}\frac{\partial \psi}{\partial z}, \quad (46)$$

now plays the role of the conserved quantity ζ in (5), $\mathcal{D} = \nu\Delta\Theta$ is the thermal diffusion, and

$$\mathcal{F} = Q = \alpha_E(\Theta_0 - \Theta) \quad (47)$$

is the thermal heating in the commonly used form of Newtonian cooling (see Tung and Orlando (2003b)) which includes a forcing term $\alpha_E\Theta_0$ and the Ekman damping term $-\alpha_E\Theta$. This equation is to be solved on a 2D surface $z = 0$. It has the same form as the vorticity equation for 2D turbulence (e.g. (5)), except that the spectral relationship between the advected quantity Θ and the advecting field ψ is given instead by

$$\hat{\Theta}(\mathbf{k}, z, t) = \frac{f}{N}\frac{\partial}{\partial z}\left(\hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z}\right) \quad (48)$$

$$= -|\mathbf{k}|[\hat{\psi}_0(\mathbf{k}, t) e^{-|\mathbf{k}|(N/f)z}] \quad (49)$$

$$= -|\mathbf{k}|\hat{\psi}_0(\mathbf{k}, z, t) \quad (50)$$

which reduces to $\hat{\Theta}(\mathbf{k}, t) = -|\mathbf{k}|\hat{\psi}(\mathbf{k}, t)$ at $z = 0$. Thus SQG corresponds to $L(k) = k$.

It can be shown (Charney, 1971; Tung and Orlando, 2003b) that the 3D QG energy density

$$\mathcal{E} \equiv \frac{1}{2}\rho_0\left[|\nabla\psi|^2 + \frac{f^2}{N^2}\left(\frac{\partial \psi}{\partial z}\right)^2\right], \quad (51)$$

is an invariant (i.e. independent of time), *when integrated over the 3D domain.* $\mathcal{E}$ is the sum of the kinetic energy density $\mathcal{E}_K$ and the potential energy density $\mathcal{E}_P$ which are given by

$$\mathcal{E}_K = (1/2)\rho_0(u^2+v^2) = (1/2)\rho_0|\nabla\psi|^2 \tag{52}$$

$$\mathcal{E}_P = \frac{1}{2}\rho_0\left(\frac{f}{N}\right)^2\left(\frac{\partial\psi}{\partial z}\right)^2 = \frac{1}{2}\rho_0\Theta^2 \tag{53}$$

For SQG, using Parseval's identity, the energies integrated over the horizontal surface are given by

$$E_P = \|\mathcal{E}_P\| = \frac{1}{2}\rho_0\|\Theta^2\| \tag{54}$$

$$E_K = \|\mathcal{E}_K\| = \frac{1}{2}\rho_0\||\nabla\psi|^2\| \tag{55}$$

$$= \frac{1}{2}\rho_0\int(i\mathbf{k}\hat{\psi}(\mathbf{k},t))\cdot(-i\mathbf{k}\hat{\psi}^*(\mathbf{k},t))\;d\mathbf{k} \tag{56}$$

$$= \frac{1}{2}\rho_0\int(|-k\hat{\psi}(\mathbf{k},t)|)^2\;d\mathbf{k} = \frac{1}{2}\rho_0\|\Theta^2\| = E_P \tag{57}$$

It is thus seen that the kinetic energy density and the available potential energy density, when integrated horizontally, are equipartitioned in flows such as SQG where potential vorticity is zero, and that

$$2B \equiv \|\Theta^2\| = (E_P+E_K)/\rho_0 = E/\rho_0 \tag{58}$$

is the total energy at the lower surface. The 3D energy is, instead.

$$\begin{aligned} E_{3D} &\equiv \int_0^\infty \|\mathcal{E}\|dz = \int_0^\infty \rho_0 dz\|\Theta^2\| \\ &= \int_0^\infty \rho_0 dz \iint dk_x dk_y \left|\hat{\Theta}|_{z=0}\right|^2 e^{-2|\mathbf{k}|(N/f)z} \\ &= \frac{1}{2}\rho_0 \iint dk_x dk_y \frac{f}{N|\mathbf{k}|}\hat{\Theta}\hat{\Theta}^*\Big|_{z=0} \\ &= \frac{1}{2}\rho_0 \iint \frac{f}{N}\left(-\hat{\psi}\hat{\Theta}^*|_{z=0}\right)dk_x dk_y \\ &= \frac{1}{2}\rho_0\frac{f}{N}\;\|(-\psi\Theta|_{z=0})\|. \end{aligned}$$

This expression for 3D energy turns out to be $\rho_0(f/N)A$, with A defined earlier as $A \equiv (1/2)\|(-\psi\Theta)\|$. Previous authors have made use of the similarity between the form of vorticity equation (5) in 2D turbulence and the temperature equation (45) in SQG turbulence to identify, by analogy, A as the "energy" and B as the "enstrophy" (Held et al., 1995; Pierrehumbert et al., 1994). As pointed out in Tung and Orlando (2003b) and also here, $2B$ is the total energy integrated over the lower surface, and includes kinetic plus available potential energy. The physical interpretation for A was not given, but can now be seen to be the total energy integrated over the 3D domain. There is no potential enstrophy $(\xi^2/2)$ per se in SQG turbulence, because potential vorticity ξ has been taken to be zero identically. There is also no flux of potential enstrophy in SQG.

In an inertial range where both fluxes of A and B are simultaneously present, the energy spectrum is the same as that for $2B(k)$ in α-turbulence with $\alpha = 1$:

$$E(k) = 2B(k) = C_1(\Pi_A)^{2/3}k^{-1} + C_2(\Pi_B)^{2/3}k^{-5/3}. \tag{59}$$

From the Danilov inequality we learn that the visible energy spectrum in the inertial range downscale from the injection scale is given by

$$E(k) \cong C_2\epsilon^{2/3}k^{-5/3}, \tag{60}$$

where $\Pi_B(k) = \epsilon$, a positive constant in the downscale range. This $k^{-5/3}$ energy spectrum is now predicted by our theory. The flux ϵ is not the "enstrophy" flux, nor is it the flux of 3D energy A, but is the 2D flux of 2D energy $2B = E$. This $k^{-5/3}$ shape is also seen in numerical simulations (Hoyer and Sadourny, 1982).

For the spectral region upscale from that of injection the energy spectrum should have the form:

$$E(k) \cong C_1(\Pi_A)^{2/3}k^{-1}, \tag{61}$$

where Π_A is the inverse flux of the 3D total energy in the horizontal spectral direction.

5. TWO-LAYER QG MODELS

The results in the previous sections demonstrate that QG turbulence can exhibit a variety of behaviors. Barotropic models possess an energy spectrum with -3 spectral slope, as in 2D turbulence, while SQG turbulence, which is baroclinic (with its exponential decay with height), has a spectrum with a $-5/3$ slope, in the downscale inertial range. Two-layer QG models have both a barotropic and a baroclinic component (see Salmon (1978, 1980, 1998)), and we therefore expect a mixture of -3 and $-5/3$ slopes depending on the degree of baroclinicity. The governing equations can be rewritten in the form of the conservation law with $\zeta = -\mathcal{L}\psi$, if we make $\mathcal{L}$ into a matrix and ψ into a column vector. The results obtained so far for scalar relations do not hold in this case. We will discuss the more general theory of multi-layers in a future paper. Here we only wish to explain why and how the Danilov inequality can fail.

A relatively realistic version of two-layer model applicable to studying atmospheric turbulence in the troposphere was adopted in Tung and Orlando (2003a). In this model forcing is due to thermal heating, which injects energy directly into the baroclinic part of the total energy. The two-layer fluid sits atop of an Ekman boundary layer near the ground, which introduces Ekman pumping in the lower layer (Holton, 1979) but *not* in the upper layer. If one artificially adds an identical Ekman damping in the upper layer it can be easily shown that Danilov's inequality applies, and we leave the proof to the interested reader.

Two-layer QG models conserve potential vorticity in each layer in the absence of forcing and damping. In the forced-dissipative case, the governing equations read:

$$\text{Top layer:}\qquad \frac{\partial\zeta_1}{\partial t} + J(\psi_1,\zeta_1) = \mathcal{D}_1 + \mathcal{F}_1 \tag{62}$$

$$\text{Bottom layer:}\qquad \frac{\partial\zeta_2}{\partial t} + J(\psi_2,\zeta_2) = \mathcal{D}_2 + \mathcal{F}_2, \tag{63}$$

where

$$\zeta_1 = \Delta\psi_1 - \frac{k_R^2}{2}(\psi_1 - \psi_2),\ \zeta_2 = \Delta\psi_2 + \frac{k_R^2}{2}(\psi_1 - \psi_2),$$

are the potential vorticity in each layer. $k_R \equiv (2\sqrt{2}f)/(hN)$ is the Rossby radius of deformation wavenumber and is taken as a given constant (h is the height). The dissipation terms, $\mathcal{D}_i$, include momentum dissipation of relative vorticity, $\Delta\psi_i$, in each layer, and Ekman damping from the lower boundary layer:

$$\mathcal{D}_1 = \nu(-\Delta)^{p+1}\psi_1,\ \ \mathcal{D}_2 = \nu(-\Delta)^{p+1}\psi_2 - \nu_E\Delta\psi_2. \quad (64)$$

The forcing terms can be shown to satisfy

$$\mathcal{F}_1 = -\frac{k_R^2}{2f}Q,\ \ \mathcal{F}_2 = -\frac{k_R^2}{2f}Q \quad (65)$$

where Q is the radiative heating term to the temperature equation.

The two inviscid quadratic invariants are the total energy $A = E$ and potential enstrophy $B = G$, defined as

$$E \equiv \frac{1}{2}\| - (\psi_1\zeta_1 + \psi_2\zeta_2)\| \quad (66)$$

$$= \frac{1}{2}\|\{|\nabla\psi_1|^2 + |\nabla\psi_2|^2 + \frac{k_R^2}{2}(\psi_1 - \psi_2)^2\}\|, \quad (67)$$

where the first two terms represent the kinetic energy in each layer, and the last term is the so-called available potential energy; and

$$G \equiv \frac{1}{2}\|(\zeta_1^2 + \zeta_2^2)\| = \frac{1}{2}\|\{(\Delta\psi_1)^2 + (\Delta\psi_2)^2 + \frac{k_R^2}{4}[k_R^2(\psi_1 - \psi_2)^2 + 2|\nabla\psi_1 - \nabla\psi_2|^2]\}\|. \quad (68)$$

In the above defined conserved quantities, the contributions from the two layers are summed, which allows us to discuss the spectral fluxes of E and G through the horizontal wavenumber domain $\mathbf{k}$. The governing spectral equations now have the same form as (17) and (18), but the appropriate spectral dissipation rates are different. They are given by $D_E(k) = 2\pi k d_E(k)$ and $D_G(k) = 2\pi k d_G(k)$, where $d_E(k)$ and $d_G(k)$ are the one-dimensional dissipation rates which read

$$d_E(k) = \nu k^{2p}k^2|\hat{\psi}_1|^2 + (\nu k^{2p} + \nu_E)k^2|\hat{\psi}_2|^2 > 0$$

$$d_G(k) = (k^2 + k_R^2)d_E(k) - k^2k_R^2[\nu k^{2p} + \frac{1}{2}\nu_E]Re\{\hat{\psi}_1\hat{\psi}_2^*\}$$

$$= k^2d_E(k) + (2\nu k^{2p} + \nu_E)\frac{k_R^2}{2}k^2[|\hat{\psi}_1|^2 + |\hat{\psi}_2|^2 - Re(\hat{\psi}_1\hat{\psi}_2^*)] + \nu_E\frac{k_R^2}{2}k^2[|\hat{\psi}_2|^2 - |\hat{\psi}_1|^2].$$

It is seen that the the first two terms in $d_G(k)$ are positive but the sign of $d_G(k)$ is indeterminant, depending on the difference in kinetic energy in each layer ($(1/2)k^2|\hat{\psi}_1|^2$ vs $(1/2)k^2|\hat{\psi}_2|^2$). So

$$k^2\Pi_E(k) - \Pi_G(k) = \int_k^\infty 2\pi q[k^2d_E(q) - d_G(q)]dq$$

$$= \int_k^\infty 2\pi q\left\{(k^2 - q^2)d_E(q) + \nu_E\frac{k_R^2}{2}q^2[|\hat{\psi}_1(q)|^2 - |\hat{\psi}_2(q)|^2]\right.$$

$$\left. -(2\nu q^{2p} + \nu_E)\frac{k_R^2}{2}q^2[|\hat{\psi}_1(q)|^2 + |\hat{\psi}_2(q)|^2 - Re(\hat{\psi}_1(q)\hat{\psi}_2^*(q))]\right\} dq. \quad (69)$$

The first and third terms in the integrand are negative. It follows that Danilov's inequality holds, i.e. $k^2\Pi_E(k) - \Pi_G(k) < 0$, *if* either the top layer has no more kinetic energy than the lower layer (i.e. $|\hat{\psi}_1(q)|^2 - |\hat{\psi}_2(q)|^2 < 0$) or if $k_R = 0$, which is the barotropic limit where the two layers do not interact. There is a wider sufficient condition in terms of ν_E, which can be derived by requiring that the sum $A_{12}(k, q)$ of the first two terms in (69) should be negative. Using the identity

$$|\hat{\psi}_1(q)|^2 + |\hat{\psi}_2(q)|^2 - \Re(\hat{\psi}_1(q)\hat{\psi}_2^*(q)) > 0, \quad (70)$$

it is easy to see that

$$k^2\Pi_E(k) - \Pi_G(k) \le \int_k^\infty 2\pi q A_{12}(k, q)\ dq \quad (71)$$

thus it is sufficient to show that $A_{12}(k, q) < 0$ for all $k < q < k_{max}$.

Note that $A_{12}(k,q)$ can be bounded from above as:

$$A_{12}(k,q) = (k^2-q^2)d_E(q) + (1/2)\nu_E k_R^2 q^2[|\hat{\psi}_1(q)|^2 - |\hat{\psi}_2(q)|^2] \tag{72}$$
$$= (k^2-q^2)[\nu q^{2p} q^2|\hat{\psi}_1(q)|^2 + (\nu q^{2p}+\nu_E)q^2|\hat{\psi}_2(q)|^2] + (1/2)\nu_E k_R^2 q^2[|\hat{\psi}_1(q)|^2 - |\hat{\psi}_2(q)|^2] \tag{73}$$
$$= [\nu q^{2p} q^2(k^2-q^2) + (1/2)\nu_E k_R^2 q^2]|\hat{\psi}_1(q)|^2 + [(k^2-q^2)(\nu q^{2p}+\nu_E)q^2 - (1/2)\nu_E k_R^2 q^2]|\hat{\psi}_2(q)|^2 \tag{74}$$
$$\leq [\nu q^{2p} q^2(k^2-q^2) + (1/2)\nu_E k_R^2 q^2]|\hat{\psi}_1(q)|^2 \tag{75}$$

This leads to the following *sufficient* condition to *satisfy* Danilov's inequality:

$$\nu_E < 2\nu k_{\max}^{2p}\left(\frac{k_{\max}}{k_R}\right)^2 \tag{76}$$

where $k_{\max}$ is either the truncation wavenumber in the numerical model, or, in the theoretical case of infinite resolutions, is the hyperviscosity dissipation wavenumber, beyond which the spectral enstrophy dissipation rate becomes negligible. Therefore a *necessary* condition to *violate* Danilov's inequality is

$$\nu_E > 2\nu k_{\max}^{2p}\left(\frac{k_{\max}}{k_R}\right)^2 \tag{77}$$

.

It is interesting to note that in the numerical simulation of the two-layer model the algorithm adopted by Tung and Orlando (2003a) for determining the magnitude of the hyperviscosity coefficient is $\nu_E \gg \nu k_{\max}^{2p}$, for all but the last twenty wavenumbers k in the dissipation range. Tung and Orlando (2003a) obtained an energy spectrum with the compound slope configuration and the transition wavenumber k_t occured in the inertial range downscale from injection in agreement with the condition $k_t \approx \sqrt{\eta_{uv}/\epsilon_{uv}}$, thus implying a violation of Danilov's inequality.

6. CONCLUSIONS AND DISCUSSION

The classical KLB theory of 2D turbulence relies for its mathematical simplicity and elegance on two unrealistic assumptions: that the domain is infinite, and that the Reynolds number approaches infinity. When these two assumptions are relaxed, the situation becomes more complicated. The downscale enstrophy cascade is accompanied with a hidden downscale energy cascade, and similarly the inverse energy cascade is accompanied with a hidden inverse enstrophy cascade. This is true as long as the leading cascades themselves exist, which requires the presence of a sufficiently strong dissipation sink at small wavenumbers. The fluxes associated with the subleading cascades are constrained by the Danilov inequality, and as a result the subleading cascades cannot contribute large enough terms to the energy spectrum to create an observable effect. This situation changes, however, in baroclinic models of quasi-geostrophic turbulence.

The surface quasi-geostrophic model represents an extreme baroclinic case where the entire behavior in the three-dimensional domain is constrained by the behavior of the system at the $z=0$ layer. In this model there is no enstrophy, and the dominant feature is the downscale energy cascade.

We have shown that in the two-layer quasi-geostrophic model, the violation of the Danilov inequality is possible only as a result of asymmetric Ekman damping operating on only one of the two-layers. This creates an imbalance between the amount of energy accumulated in one layer versus the amount accumulated in the other layer, and the downscale energy cascade will become observable on the condition that this imbalance is sufficiently large. We have derived in the present paper a sufficient condition for *not* violating the Danilov inequality which explains why the $k^{-5/3}$ spectrum has not been observed in previous simulations of the two-layer model. Deriving a theoretical sufficient condition for violating the Danilov inequality remains an open question. However, the numerical simulation by Tung and Orlando (2003a) has confirmed that a double cascade with the transition wavenumber located in the inertial range can be realized. This can only occur when the Danilov inequality is violated for some wavenumbers k in the inertial range. The parameterization of the Ekman damping in that simulation does in fact satisfy the necessary condition derived in this paper.

As long as we operate within the framework of multiple-layer models with a finite number of layers, one cannot rule out the alternative theory that the atmospheric energy spectrum might reflect a double downscale cascade of helicity and energy instead of enstrophy and energy (see discussion in section 6.5 of Branover et al. (1999), and figure 3 of Bershadskii et al. (1993)). However, most of the current debate has been focused on the somewhat mysterious nature of the very extensive and robust $k^{-5/3}$ spectrum.

Our work in the present paper explains why it can be reproduced in numerical simulations that use baroclinic models, while the same effect cannot be realized in simulations of two-dimensional turbulence. On the other hand our work here does not rule out the possibility that the shallower part of the spectrum observed by Nastrom and Gage (1984) over the mesoscales can be due to dynamics other than QG, whether it is barotropic or baroclinic, especially for scales of 100 km or less (see e.g. Lindborg (2005) with Bousinesq dynamics). Our present work serves to point out that over the larger scales ($\gtrsim 600$km), where the transition to shallower spectrum occurs, baroclinic QG theory is a viable mechanism for explaining the transition from -3 to $-5/3$ slopes.

Furthermore, as proposed first by Tung and Orlando (2003a), the downscale energy flux, which is important in explaining the $k^{-5/3}$ energy spectrum over the mesoscales in most theories, originates at larger scales (the synoptic scales).

Its contribution to the energy spectrum is hidden for smaller wavenumbers under the k^{-3} part of the spectrum, and then emerges for larger k past the transition scale. It remains an open question, one that is beyond the scope of this paper, to explain how this downscale energy flux can be continued into length scales too small for QG theory to describe, and how it is eventually dissipated.

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