The Concept of Wave Overreflection and Its Application to Baroclinic Instability[1]

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ABSTRACT

The concept of wave overreflection is reviewed. The physical origin of this effect is discussed as is its relation to hydrodynamic instability. It is noted that the instabilities associated with overreflection are likely to be identical to what are commonly called critical layer instabilities.

The bulk of this paper examines baroclinic instability in terms of the overreflection of vertically propagating Rossby waves. This approach leads to rapid estimates of growth rates and phase speeds of unstable modes for arbitrary distributions of zonal velocities in models with and without lids; it also leads to efficient algorithms for calculating unstable modes "exactly."

Among our findings are the following: (i) Charney and Green modes are both essentially critical-layer instabilities. (ii) When tropospheric shear is brought to zero above some height (one scale height, for example) so that long waves may radiate to infinity (ignoring for a moment the growth rate), the growth rates are reduced somewhat, but the modes remain unstable. (iii) Baroclinic instability can be eliminated by stretching the transition region from zero shear at the ground to the interior shear sufficiently without altering the shear above this region. Explicit calculations show this depth to be about a quarter of a scale height.

Consistent with item (iii) above, we show that the potential vorticity flux of baroclinically unstable modes (a measure of their interaction with the mean flow) is confined primarily to a layer between the ground and the neighborhood of the steering level—even when the unstable eigenmodes extend to much greater heights.

1. Introduction

A wide variety of problems in dynamic meteorology (and in many other fields) are described in terms of an equation of the form

$$\psi_{yy} + Q(y, k, \ldots, \omega)\psi = 0, \qquad (1)$$

where y is some spatial coordinate (north-south distance or height), $k, \ldots$ are other parameters such as wavenumber in some direction orthogonal to y, static stability, etc., and ω is a time frequency. Typically, the same form of Q applies to both a wave propagation problem and a stability problem. Among such problem-pairs are the meridional propagation of barotropic Rossby waves and barotropic instability; the vertical propagation of internal Rossby waves and baroclinic instability; and the vertical propagation of internal gravity waves and the instability of stratified shear flows.

For the wave propagation problems we generally have forcing somewhere in the fluid at a specified frequency ω. Eq. (1) then determines the behavior of the wave away from the forcing. When Q is negative, wave fields vary exponentially and there is no wave propagation; we will refer to this as "trapping." When Q is positive and slowly varying the wave propagates away from the region of forcing. By slowly varying we mean that

$$\left|\frac{\partial Q/\partial y}{4Q^{3/2}}\right| < 1. \qquad (2)$$

Eq. (2) is the usual criterion for the WKB approximation (Schiff, 1955). It turns out that when (2) is violated, the wave may again vary exponentially (explicit examples are given in Appendix 1). This last remark has not been generally recognized. Finally, inhomogeneities in Q lead to wave reflections which may be partial, total or (as we shall describe shortly) we can have overreflection where the reflected wave is stronger than the incoming wave.

For the stability problems, homogeneous boundary conditions are applied to Eq. (1), and ω is treated as an eigenvalue. One seeks conditions under which ω is complex with the imaginary part being of such a sign as to lead to exponential growth (i.e., instability).

The wave solutions and the instabilities will be shown to be related to each other through the concept of wave overreflection. This concept has arisen

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in a number of contexts in recent years. Jones (1968), for example, found the following for stably stratified shear flow: If in a portion of the fluid where the Richardson number (Ri) is greater than 0.25 there is an internal gravity wave propagating toward a level where the horizontal phase speed of the wave equals the flow speed (i.e., a critical level where the Doppler-shifted frequency of the wave goes to zero) and if at the critical level Ri > 0.25 then there will always be a partial reflection. However, if at the critical level Ri < 0.25 then there can be over-reflection. Without a critical level there is always partial reflection. As shown by Miles (1961) and Howard (1961), Ri < 0.25 somewhere in the flow is a necessary condition for instability. There would thus appear to be some connection between over-reflection and instability. Lindzen (1974) suggested that in the case where a wave propagating upward is overreflected, the insertion of a reflecting lower boundary should give rise to instability. This suggestion was confirmed by Lindzen and Rosenthal (1976), Davis and Peltier (1976) and others. The instabilities found in these studies were, however, distinguishable from and weaker than conventional Kelvin-Helmholtz instabilities.

A more general approach to the problem was recently taken by Lindzen and Tung (1978) wherein the Kelvin-Helmholtz instability could be related to wave overreflection as well. A review of this work will be presented in Section 2 and the application of overreflection to baroclinic instability will be given in Section 3.

2. Wave overreflection in barotropic instability

In order to establish notation and preserve continuity of presentation we will briefly review some results in Lindzen and Tung (1978) (subsequently referred to as LT). LT considered barotropic flow wherein the stream function satisfies the equation

$$\frac{d^2\psi}{dy^2} + \left\{\frac{\beta - \bar{u}_{yy}}{\bar{u} - c} - k^2\right\}\psi = 0, \tag{3}$$

where $\bar{u}$ is the basic zonal flow, $\beta = df/dy$, f being the Coriolis parameter, y the northward coordinate, and where dependence on eastward direction x and time t is taken to be

$$e^{ik(x-ct)}. \tag{4}$$

Perturbation zonal and meridional velocity are approximately related to ψ by the geostrophic relations

$$u = -\frac{\partial\psi}{\partial y}, \tag{5}$$

$$v = \frac{\partial\psi}{\partial x}. \tag{6}$$

The perturbation pressure is given by

$$p = \rho_0 f\psi, \tag{7}$$

where ρ_0 is the basic pressure. Defining

$$(\overline{fg}) = \frac{1}{2\pi}\int_0^{2\pi}(\mathrm{Re}\, f)\cdot(\mathrm{Re}\, g)d(kx), \tag{8}$$

LT show that

$$\overline{uv} = \text{constant} \tag{9}$$

if c is real and $c \neq \bar{u}$, and

$$\overline{pv} = -(\bar{u} - c)\rho_0\overline{uv}, \tag{10}$$

where ageostrophic contributions are included. Now, the sign of $\overline{pv}$ yields the direction in which waves are propagating energy. This fact is exploited by LT to determine the presence of overreflection. Let

$$Q = \frac{\beta - \bar{u}_{yy}}{\bar{u} - c} - k^2 \tag{11}$$

and let

$$\beta_e = \beta - \bar{u}_{yy}. \tag{12}$$

We consider a flow where $Q > 0$ and slowly varying for $y > y_1$. (Details of the geometry are given in Fig. 1 of LT; a more detailed description will also be given later in this section.) For convenience we take $\bar{u} > c$ and $\beta_e > 0$ in the region $y > y_1$. We further assume all wave sources are to the right of the wave region. Then in the absence of a critical surface in $y < y_1$, LT use (9) to show that $\overline{pv} \leq 0$ in the wave region. This is shown to imply partial or total reflection, the latter being associated with $\overline{pv} = \overline{uv} = 0$. Thus, if overreflection exists, it is intimately associated with the presence of critical layers. The procedure followed by LT is to obtain the Frobenius expansions of the two linearly independent solutions to (3) about the critical surface and use these solutions to obtain the jump in $\overline{uv}$ across the critical layer. The solution is of the form

$$\psi(y) = Af(y - y_c) + Bg(y - y_c), \tag{13}$$

where $u(y_c) = c$; f is regular while g has a logarithmic singularity. It is this singularity which is responsible for the discontinuity of $\overline{uv}$ at $y = y_c$. LT show that

$$\overline{uv}_+ - \overline{uv}_- = \frac{k}{2}\pi\frac{\beta_e(y_c)}{\bar{u}_y(y_c)}|B|^2, \tag{14}$$

where $\overline{uv}_+$ is $\overline{uv}$ for $y > y_c$, $\overline{uv}_-$ is $\overline{uv}$ for $y < y_c$. Recall $y_c < y_1$. From (10) we see that for $y > y_c$ (where $\bar{u} > c$) $\overline{pv}_+$ will have the opposite sign of $\overline{uv}_+$; moreover, $\bar{u}_y(y_c) > 0$. For simplicity we will next restrict ourselves to problems where we either have a boundary someplace to the left of y_c where $v = 0$ and hence $\overline{uv}_- = 0$, or where in a semi-infinite region at some point to the left of y_c, Q is negative in which case $\overline{uv}_-$ again is zero. (Less restrictive cases are discussed in LT and in Section 3 of this paper.) Eq. (14) then becomes

$$\overline{uv}_+ = \frac{k}{2}\pi\frac{\beta_e(y_c)}{\bar{u}_y(y_c)}|B|^2 \qquad (15)$$

(we have adopted the convention that k is positive). Overreflection is uniquely identified with $\overline{pv}_+ > 0$ or $\overline{uv}_+ < 0$. From (15) we see that this occurs when $\beta_e(y_c) < 0$ (unless $|B|^2 = 0$). When $\beta_e(y_c) = 0$, $\overline{uv}_+ = 0$ and we have perfect reflection. The case $|B|^2 = 0$ corresponds to the regular part of (13) satisfying the boundary condition to the left of y_c. As we shall see in Section 3, this can happen, but it represents a very special case.

The above tells us that if β_e changes sign between the wave region and y_c then we will have overreflection. Incidentally (but we think importantly), the change of sign of β_e also implies a trapped region (i.e., $Q < 0$) between the wave region and y_c and a wave region to the left of y_c. Evidence is accumulating that such a distribution of wave regions, trapping regions and singularities is basic to the stability of plane-parallel flows in general. The above configuration is schematically depicted in Fig. 1 and will be discussed in detail later in this section.

LT note the obvious relations between the above conditions for wave existence and overreflection and the necessary conditions for barotropic instability. Most notably Rayleigh's inflection point theorem requires β_e to change sign within the flow domain while Fjørtoft's theorem guarantees the existence of regions where $Q > 0$. It would thus appear that the necessary conditions for instability are also sufficient conditions for the existence and overreflection of waves. This is generally true though $Q > 0$ may not always be sufficient for the existence of waves if (2) is not satisfied (*viz.* Appendix 1).

Before ending this section a number of points (not all of which are discussed in LT) should be discussed.

a. Overreflection and instability

If one has overreflected waves in a region which is bounded on the right (using the above-described configuration) by a reflecting surface (either a rigid wall or a turning point where Q becomes negative), then we expect amplification to occur when an overreflected wave is reflected back to the overreflecting region. Thus, although overreflection is a property of waves with vanishingly small imaginary frequencies, it would appear to demand growth in bounded configurations. Lindzen and Rosenthal (1976) have shown, however, that only when the phase of the overreflected waves are in phase with the reflected waves do we have normal mode (exponentially growing) instabilities. Lindzen and Rosenthal and LT refer to this condition as quantization. The above argument for growth would nonetheless appear to be true even without quantization. We discuss this case in Section 2f.

b. Critical layer instability and overreflection

Taylor (1915) and Bretherton (1966) have noted that solutions to (3) with boundary conditions corresponding to rigid walls on each side of an "inflection" point (i.e., a point where $\beta_e = 0$) have no net integrated vorticity flux. Moreover, they show that away from a critical layer such a flux exists

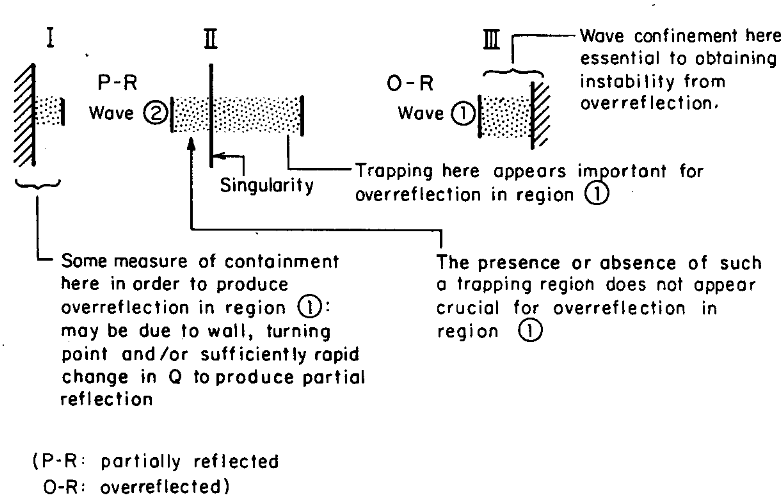

FIG. 1. Configuration of wave propagation regions, wave trapping region(s), singular surface, and boundaries associated with wave overreflection and resulting instabilities. (Dotted regions represent trapping.) Surface II refers to critical surface. Surface I contains waves in wave region ② from the left, while surface III contains waves in wave region ① from the right.

only if there is an imaginary part of the frequency. On the other hand, there exists a finite down-gradient flux of vorticity at the critical layer even in the limit of vanishing imaginary frequency (growth rate), In order to balance this flux it is concluded that one must in fact have a growing solution.

Both Taylor's result and overreflection relate properties of critical layers in the limit of vanishing growth to the need for growing solutions, and we would suggest that both approaches are equivalent. The resulting instability is often referred to as critical layer instability. In contrast to the vorticity flux approach, wave overreflection allows an immediate estimate of growth rates associated with critical layer instabilities. If, for a given wavenumber k and phase speed c_r we have overreflection, R (>1), and if the time it takes such a wave to traverse wave region (1) of Fig. 1 is τ, then (following Lindzen and Rosenthal, 1976) we expect the disturbance to amplify by a factor R in a time 2τ. As a rule, quantization will only obtain at a particular value(s) of c_r. However, as we shall note in Section 2c, this is a matter of some delicacy; quantization is not always determinable in the limit $c_i \to 0$. More generally, if for some c_r's (regardless of quantization) the above estimate of growth rate exceeds the explicitly calculated growth rate for an instability, then it is reasonable to identify that instability as a critical layer instability.

Details of calculations of the above type are presented in Section 3. In particular, for baroclinic instability, Bretherton (1966) suggested that the growth rates associated with critical layer instabilities would be less than growth rates normally calculated for baroclinic instability since the instability was essentially convective in origin. In Section 3 we will show that this is not the case; critical layer growth rate estimates for the most unstable mode overestimate actual growth rate.

c. *Critical surfaces and critical layers*

The preceding two points describe how critical surfaces (and associated wave overreflection) leads to disturbance amplification. As has already been noted, the prediction of amplification is based on wave properties in the limit $c_i \to 0$. Only in this limit is the critical surface a discrete surface. Once one has a growing disturbance, c_i is no longer zero, and the singularity associated with the former critical surface moves into the complex plane. Its projection on real y is no longer a discrete surface but a layer whose width is proportional to c_i. Thus, the identification of the steering level (where $c_r \equiv \bar{u}$) with the critical surface (where $c \equiv \bar{u}$) for overreflection is not, as a rule, appropriate. Rather, one expects that if critical layer instability is responsible for a particular instability, then there will exist, within the relevant critical layer, a critical level (or surface) whose associated overreflection is more than sufficient to account for the instability's growth rate.

Also, when c_i is sufficiently large, it will significantly affect wavelengths in wave region ① (of Fig. 1) and hence, the conditions for quantization mentioned in Section 2a. In fact, as we shall see in Section 3, the quantization condition commonly used (in the limit $c_i \to 0$) is only an approximate relation for satisfying the boundary condition at the right-hand boundary in Fig. 1. Exact quantization may depend on c_i to such an extent that approximate quantization in the limit $c_i \to 0$ may not even exist. An example of such a situation is given in Section 3c.

d. *The geometry of overreflection and the stability of stratified shear flow*

Earlier in this section, we noted that an overreflecting situation involved a particular configuration of wave regions, trapping regions and singularities, which we suspect may be of general importance. This configuration is shown in Fig. 1. Although such a configuration emerges automatically from the preceding analysis for barotropic configurations, our work on other problems suggests that it may be general. Noteworthy in this connection is the problem of stratified shear flow. The basic equation for this problem is given in Appendix 1. No simple counterpart of the inflection point theorem and Fjørtoft's theorem are currently available for this problem. Instead we have a theorem due to Miles (1961) and Howard (1961) which states that a necessary condition for instability is that the Richardson number

$$\mathrm{Ri} = \frac{N^2}{(\bar{u}_z)^2} \tag{16}$$

be less than ¼ someplace in the flow, where N is the Brunt-Väisälä frequency. Now, in Appendix 1 we show that the wave propagation occurs where $\mathrm{Ri} > ¼$. Thus, for $\mathrm{Ri} > ¼$, a wave with a critical level will be wavelike on both sides of the singularity. In terms of Fig. 1, the trapped (exponential as opposed to sinusoidal) region to the right of the singularity is missing. It is also shown in Appendix 1 that when $\mathrm{Ri} < ¼$, waves are trapped even though Q in Eq. (1) is positive. This proves suggestive for the following classical problem, namely, the stability of flow where $\bar{u}_z$ and N^2 are constant everywhere and where $\mathrm{Ri} = \text{constant} < ¼$. Such a flow is *stable* regardless of whether it is unbounded, bounded below or bounded above and below (Taylor, 1931; Case, 1960b). From our present perspective we might suspect that the absence of waves is relevant. We have therefore considered a modified flow where $\bar{u}$ = constant = u_1 for $0 \leqslant z \leqslant z_1$, and $\bar{u} = u_1 + \bar{u}_z(z - z_1)$ for $z \geqslant z_1$ where $\bar{u}_z$ = constant; N^2 remains constant and Ri $< 1/4$ above z_1. We can now have internal gravity waves in $(0,z_1)$ [as well as vorticity waves in the corner at $z = z_1$] which have critical levels above z_1. Thus we have waves separated from a critical level by a trapping (or exponential) region, but no wave region above the critical level. In calculations which are now being prepared for publication by Rosenthal and Lindzen we show that in this case waves are *not* overreflected and there are no instabilities. Fig. 1 suggests that we are still lacking wave region ②. To insert such a region, we leave the velocity profile unchanged but increase N^2 above some $z_2 > z_1$ so that in this region $(z > z_2)$ Ri $> 1/4$. There will now be waves in $(0,z)$ with critical levels in (z_1,z_2) whose continuation will be wavelike above z_2; because $\bar{u}$ is increasing with height, there will be a turning point at some $z_3 > z_2$ above which Q [*viz*., Eq. (1) will be negative]. This is now a configuration exactly like that in Fig. 1 and we find that waves in region ① are overreflected and associated with this we do find instabilities.[2]

It should be noted that the above velocity profile has no inflection point. There is no need for an inflection point because a stratified fluid can sustain *internal gravity waves* in addition to *vorticity waves*, i.e., waves whose restoring force arises from vorticity gradients. Waves can exist in regions of large $\bar{u}''$ and instabilities can arise from these waves just as in barotropic flows. Conventional Kelvin-Helmholtz instabilities are examples of this.

It should be noticed that in problems with exponential (trapping) regions on both sides of the singularity (as in the above stratified shear flow problem), overreflection is possible for waves in both wave regions ① and ② of Fig. 1. This complicates matters relative to the results in LT.

e. *Qualitative mechanism for overreflection*

The role of the inflection point in instability has always seemed a little obscure. The present work implies that the inflection point is essential to an unstable configuration but that the locus of instability is the critical layer and the associated wave overreflection. This statement hardly relieves the obscurity. Fig. 1 suggests a somewhat fuller explanation.

By explicit calculations, we find that waves in region ②, though not overreflected from surface II (at least when the exponential region between wave region ② and the singularity is missing), are partially reflected with reflection coefficients very close to unity. Thus wave region ②, as bounded by surfaces I and II, forms a slightly imperfect waveguide which exhibits enhanced response to forcing—even if the forcing is off-resonance. The additional peculiarity of the critical layer, with the trapped region on its left, is that it permits waves in region ① which are propagating toward II to force the waveguide in region ② with the resulting magnified response manifesting itself as overreflection in region ①. The presence of surface III then forces the time growth of the disturbance. The role of the singular critical layer must be emphasized. Thus when $|B|^2$ is zero [*viz*., Eq. (15)] wave region ② forms a perfect waveguide. However, $|B|^2 = 0$ means that the singular solution is zero and hence, waves in wave region ① can neither excite nor respond to resonant amplification in wave region ②.

For those familiar with the physics of lasers (and frankly, we don't number ourselves among them), it has been suggested that I, II and III could be associated with the essential three excitation levels and associated mirrors required for a laser. The mechanisms for wave amplification in both cases seem similar.

If the above is a correct explanation of plane-parallel flow instability, then the need for the two wave regions separated by the critical layer and its adjacent trapped region is relatively clear. The unquestionable existence of lasers in optics renders the existence of a similar wave amplification mechanism in fluids at least conceivable.

f. *Quantization and instability*

We have already noted (Sections 2a and 2b) that, intuitively, the existence of overreflected waves whose energy is contained (by surface III in Fig. 1) should lead to growing disturbances. Lindzen and Rosenthal (1976) have shown, however, that exponential growth requires that reflected and overreflected waves be in phase with each other (quantization condition). LT put forth the conjecture that when one has contained overreflected waves which, however, do not satisfy quantization, one might obtain algebraic growth. There is substantial reason to doubt this conjecture. For a wide variety of cases, it has been shown that in the absence of exponential growth one will *not* have long-time algebraic growth (Case, 1960a). Some of these stable cases seem to have contained overreflected waves.

At this point we can only suggest a possible solution to this situation. Apparently, without quantization, phase discrepancies will grow with successive reflections and overreflections leading to eventual cancellation and decay. This behavior would be analogous to the behavior of the function

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[2] This suggests the possibility that a necessary condition for the instability of stratified shear flows may be that the Richardson Number must be *greater* than 1/4 someplace in the flow.

$$F(T) = \frac{T}{1 + (\theta - \theta')T^2}\,.$$

For $\theta = \theta'$ (analogous to quantization), $F(T)$ increases linearly with T, but when $\theta \neq \theta'$ (regardless of how small the difference is) $F(T)_{T\to\infty} \to 0$. Nevertheless, when $\theta - \theta'$ is in some sense small, $F(T)$ will grow with T for a substantial range in T until decay eventually sets in. If this analogy holds, it suggests that long-time algebraic decay may be associated with significant initial growth. One may then question whether long-term decay is an adequate measure of stability. We are currently studying such cases.

3. Baroclinic instability

a. Basic analysis

In this section we apply the concepts of the previous section to the problem of baroclinic instability. We restrict ourselves to a pure baroclinic flow [i.e., $\bar{u} = \bar{u}(z)$] on a β-plane, and to quasi-geostrophic perturbations. The equations are derived and discussed in Charney (1947, 1973). Solutions have been obtained in these works as well as others (Kuo, 1952; Burger, 1962; Miles, 1964; Geisler and Garcia, 1977; Green, 1960). As in most of the above works, we will confine ourselves to configurations with a constant Brunt-Väisälä frequency N.

The equation for the geostropic streamfunction Ψ is

$$\psi_{zz} + \left\{ \frac{\dfrac{\beta}{\epsilon} + \dfrac{1}{H}\dfrac{d\bar{u}}{dz} - \dfrac{d^2\bar{u}}{dz^2}}{\bar{u} - c} - \frac{k^2}{\epsilon} - \frac{1}{4H^2} \right\} \psi = 0, \quad (17)$$

where solutions of the form

$$\Psi = \psi(z) e^{z/2H} e^{ik(x-ct)} \quad (18)$$

have been assumed and where

- H scale height for basic density (assumed constant) $[=RT_0/S]$
- z height
- ϵ f^2/N^2
- f Coriolis parameter
- β df/dy
- x eastward distance
- y northward distance.

Requiring the vertical velocity to vanish at the ground implies

$$\psi_z + \frac{1}{2H}\psi - \frac{\dfrac{d\bar{u}}{dz}}{\bar{u} - c}\psi = 0 \quad \text{at} \quad z = 0. \quad (19)$$

Various choices are commonly taken for an upper boundary condition

$$\psi \to 0 \quad \text{as} \quad z \to \infty, \quad (20a)$$

$$\psi = 0 \quad \text{at} \quad z = z_T, \quad (20b)$$

or

$$\psi_z + \frac{1}{2H}\psi - \frac{\dfrac{d\bar{u}}{dz}}{\bar{u} - c}\psi = 0 \quad \text{at} \quad z = z_T. \quad (20c)$$

If one chooses to use the Boussinesq approximation, it is equivalent to taking $H \to \infty$ in the above equations.

It has been noted by LT and Bretherton (1966) that (17) is mathematically identical to Eq. (3) for barotropic modes where we now set

$$\beta_e \equiv \frac{\beta}{\epsilon} + \frac{1}{H}\frac{d\bar{u}}{dz} - \frac{d^2\bar{u}}{dz^2}\,. \quad (21)$$

Since β/ϵ and $d\bar{u}/dz$ are generally positive while $|d^2\bar{u}/dz^2|$ is generally smaller than the other two terms, β_e tends to be positive everywhere in most problems. Thus, we appear to lack the "inflection" point necessary for instability. However, (19) differs from the boundary condition used in barotropic problems; Bretherton (1966) and Charney (1973) show that when $d\bar{u}/dz > 0$ (corresponding to temperature decreasing toward the pole) at $z = 0$, Eq. (19) leads to the possibility of baroclinic instability. LT and Bretherton (1966) show, furthermore, that the problem defined by (17) and (19) is equivalent to a problem where $\bar{u}_z = 0$ at $z = 0$ and increases from zero to its value just above the ground in an infinitesimal distance requiring that $d^2\bar{u}/dz^2$ be infinite in this infinitesimal region, thus rendering β_e negative in this region. β_e now changes sign, thus providing a configuration capable of instability. For disturbances with critical levels, wave region ① of Fig. 1 exists in an infinitesimal neighborhood of the ground while wave region ② exists above the critical level. The overreflected waves discussed in Section 2 are in the infinitesimal neighborhood of the ground. Although this may seem highly artificial we shall soon see that there are no real problems associated with this notion.

Before proceeding to the computational details, it is important to remark on one consequence of the above analysis. In Charney's (1947) treatment of baroclinic instability, $\bar{u}$ profiles with constant shears were considered, and it was found that such profiles were unstable regardless of the shear—provided that the shear was not zero. This has given rise to the faulty idea that neutrality with respect to baroclinic instability is achieved only by eliminating shear (or equivalently, horizontal temperature gradients). However, consistent with Charney (1973) and references therein, if we smooth the delta function transition, between zero shear at the ground and the interior shear, over a broad enough region so that the transition can be made by a profile for which $\beta_e = 0$, then we will have an "inflection"-free profile which must be stable. The details of this smoothing are given in Appendix 2 where it is shown that the required depth for this smoothing is approximately ¾H and within the smoothing region the average shear is about half of what it is in the unsmoothed profile. Thus, the elimination of about 30% of the available potential energy in a constant shear profile is *sufficient* to neutralize the flow. We will show, later in this section, that this is a substantial overestimate of the smoothing needed for neutrality.

Before computing, it is useful to nondimensionalize (17) and (19). Let

$$m = \frac{d\bar{u}}{dz} \quad \text{at} \quad z = 0, \tag{21a}$$

$$u_0 = \bar{u} \quad \text{at} \quad z = 0. \tag{21b}$$

We define

$$\tilde{u} \equiv \frac{\bar{u} - u_0}{mH}, \tag{21c}$$

$$\tilde{c} \equiv \frac{c - u_0}{mH}, \tag{21d}$$

and

$$\tilde{z} \equiv \frac{z}{H}, \tag{21e}$$

$$\alpha^2 \equiv \frac{k^2H^2}{\epsilon}. \tag{21f}$$

From (21) we have

$$\left.\begin{aligned} \frac{d\tilde{u}}{d\tilde{z}} &= 1 \quad \text{at} \quad \tilde{z} = 0 \\ \tilde{u} &= 0 \quad \text{at} \quad \tilde{z} = 0 \end{aligned}\right\}.$$

Note also from (21d) that $\tilde{c}$ is the height of the critical level in scale heights (for a constant shear profile).

Eqs. (17) and (19) become

$$\psi_{\tilde{z}\tilde{z}} + \left\{\frac{r + \dfrac{d\tilde{u}}{d\tilde{z}} - \dfrac{d^2\tilde{u}}{d\tilde{z}^2}}{\tilde{u} - \tilde{c}} - \alpha^2 - \frac{1}{4}\right\}\psi = 0, \tag{22}$$

$$\psi_{\tilde{z}} + \frac{1}{2}\psi + \frac{1}{\tilde{c}}\psi = 0 \quad \text{at} \quad \tilde{z} = 0, \tag{23}$$

where $r = \beta H/\epsilon m$.

In order to investigate overreflection and calculate unstable modes we must expand the infinitesmal region of transition from zero shear at the ground. To do this we lower the ground a small distance to $-\tilde{d}$. We now seek a continuous $\tilde{u}$ such that $\tilde{u}_{\tilde{z}} = 0$ at $\tilde{z} = -\tilde{d}$ and $\tilde{u}_{\tilde{z}} = 1$ at $\tilde{z} = 0$. Computations are greatly simplified by choosing $\tilde{u}$ in $[-\tilde{d},0]$ such that the bracketed quantity in (22) (which we will again refer to as Q) is constant. Indeed for real $\tilde{c}$ we may always find such a $\tilde{u}$ by solving

$$r + \frac{d\tilde{u}}{d\tilde{z}} - \frac{d^2\tilde{u}}{d\tilde{z}^2} = \mu^2 \times (\tilde{u} - \tilde{c}) \tag{24}$$

for $\tilde{u}$, where

$$\left.\begin{aligned} \mu^2 &= \text{positive constant} \\ \tilde{u}_{\tilde{z}} &= 0 \quad \text{at} \quad \tilde{z} = -\tilde{d} \\ \tilde{u}_{\tilde{z}} &= 1 \quad \text{at} \quad \tilde{z} = 0 \\ \tilde{u} &= 0 \quad \text{at} \quad \tilde{z} = 0. \end{aligned}\right\}. \tag{25}$$

Although (24) is only second order in $\tilde{z}$, we can satisfy all three boundary conditions since the constant in (24) is disposable. This procedure for determining $\tilde{u}$ is not without problems: first, $\tilde{u}$ must be recomputed for each $\tilde{c}$, and second, if $\tilde{c}$ is complex (as it will be in some computations) the $\tilde{u}$ will also be complex. These problems can be avoided by exploiting the smallness of $\tilde{d}$. By choosing $\tilde{d}$ always much smaller than $|\tilde{c}|$, $|\tilde{u}|$ will be much smaller than $|\tilde{c}|$ and Q can be approximated as

$$Q \approx \frac{r + \dfrac{d\tilde{u}}{d\tilde{z}} - \dfrac{d^2\tilde{u}}{d\tilde{z}^2}}{-\tilde{c}} - \alpha^2 - \frac{1}{4} \tag{26}$$

and the approximate constance of Q can be achieved by setting

$$r + \frac{d\tilde{u}}{d\tilde{z}} - \frac{d^2\tilde{u}}{d\tilde{z}^2} = -a^2, \tag{27}$$

where a^2 is a positive real number. The solution to (27) satisfying (25) is

$$\tilde{u} = -(r + a^2)\tilde{z} + (1 + r + a^2)e^{\tilde{z}} - (1 + r + a^2), \tag{28}$$

where

$$a^2 = \frac{1}{e^{\tilde{d}} - 1} - r. \tag{29}$$

For very small $\tilde{d}$,

$$a^2 \approx \frac{1}{\tilde{d}}. \tag{30}$$

In view of the approximate constancy of Q in $[-\tilde{d}, 0]$, we may write the solution to (22) in this region as

$$\psi = e^{-i\lambda\tilde{z}} + Re^{i\lambda\tilde{z}} \tag{31}$$

where $\lambda = Q^{1/2}$ and where the root of Q is chosen so that $\text{Im}\,\lambda < 0$. Initially the complexity of λ arises from the assumed presence of a small positive c_i due to linear damping. The above branch choice then guarantees that the first term on the right-hand side of (31) corresponds to an upward propagating internal Rossby wave while the second term corresponds to a downgoing wave; R is then a complex reflection coefficient.

The computation of R is straightforward. One integrates (22) subject to whatever upper boundary condition one is using. Because of the singularity at the critical level one must introduce a small amount of damping (i.e., a small positive $\tilde{c}_i$), and, if one is proceding numerically, one must have sufficient resolution to resolve the smoothed singularity. Since we are not using (23) as a lower boundary condition, we are free to use any value for $\tilde{c}_r$, since it is no longer an eigenvalue. For obvious reasons we will restrict ourselves to values of $\tilde{c}_r$ which have critical levels. The resulting solution is completely determined except for an arbitrary amplitude. Instead of (23) we require that this solution and its derivative equal (31) and its derivative, respectively, at $\tilde{z} = 0$. These conditions serve to determine the arbitrary amplitude of the solution to (22) and R in (31). From a trivial extension of Eq. (15) we know that $|R|$ will generally be greater than 1 except at isolated points where $|B|^2 = 0$. A numerical algorithm for the determination of R is given in Appendix 3. R, as an overreflection coefficient, is meaningful only in the limit of $c_i \to 0$; however, regardless of interpretation, R is calculable for any c_i.

As discussed in Section 2, the presence of a lower boundary at $\tilde{z} = -\tilde{d}$ should, in the presence of overreflection, lead to growing solutions. In order to see this better, let us study the implications of the lower boundary condition in detail. At $\tilde{z} = -\tilde{d}$, using our non-dimensionalization, Eq. (20c) becomes

$$\psi_{\tilde{z}} + \tfrac{1}{2}\psi = 0 \quad \text{at} \quad \tilde{z} = -\tilde{d}, \tag{32}$$

where ψ is formally given by (31). From (26), (30) and (31) we have

$$\lambda \sim \tilde{d}^{-1/2} \tag{33}$$

and since $\tilde{d}$ is very small, (32) can be approximated by

$$\psi_{\tilde{z}} = 0 \quad \text{at} \quad \tilde{z} = -\tilde{d}. \tag{34}$$

This approximation is not at all essential, but it significantly simplifies our discussion. In addition, we will initially allow $\tilde{c}_i$ to be arbitrary rather than vanishingly small. Substituting (31) into (34) yields

$$R\, e^{-i\lambda\tilde{d}} - e^{i\lambda\tilde{d}} = 0$$

or

$$R\, e^{-2i\lambda\tilde{d}} = 1. \tag{35}$$

We now let $|R| \equiv \mathscr{R}$ and

$$R = \mathscr{R}\, e^{i\theta}. \tag{36}$$

Moreover, since $\tilde{c}_i \neq 0$, λ is complex, i.e.,

$$\lambda = \lambda_r + i\lambda_i, \tag{37}$$

where, as we will soon see, λ_i is approximately proportional to $\tilde{c}_i$. Eq. (35) becomes

$$\mathscr{R}\, e^{2\lambda_i\tilde{d}} e^{-2i\lambda_r\tilde{d}+i\theta} = 1 \tag{38}$$

or

$$\cos(-2\lambda_r\tilde{d} + \theta) = 1 \tag{39a}$$

and

$$\mathscr{R}\, e^{2\lambda_i\tilde{d}} = 1. \tag{39b}$$

Eq. (39a) implies

$$\theta = 2\lambda_r\tilde{d} + 2n\pi, \quad n = 0, \pm 1, \pm 2, \text{etc.} \tag{40}$$

In practice we find solutions to (40) only for $n = 0$ (in the present problem). Clearly, in the limit of vanishing $\tilde{c}_i$ when λ_i vanishes and $\mathscr{R} > 1$, Eq. (39b) cannot be satisfied; there must be a finite $\tilde{c}_i$ (i.e., growth).

Eqs. (39a) and (39b) form the basis for a variety of iterative procedures which lead to the determination of the complex $\tilde{c}$'s corresponding to the unstable eigenmodes of the baroclinic instability problem. One such procedure is particularly illuminating. In this procedure our initial choice for $\tilde{c}_i$ is the smallest value allowed by the numerics for the calculation of R (essentially $\tilde{c}_i \to 0$). We then vary $\tilde{c}_r$ (which causes both θ and λ_r to vary) until (40) is satisfied. This proves possible only for $n = 0$ and leads to what we refer to as our quantization condition. Having thus determined our initial values of $\tilde{c}_r$ (and λ_r) as well as our initial estimate for $\mathscr{R}$ we turn to (39b) to estimate $\tilde{c}_i$:

$$\lambda_i = \frac{\operatorname{Im} \lambda^2}{2\lambda_r},$$

and using (31) and (26)

$$\lambda_i = \frac{-a^2}{2|\tilde{c}|^2\lambda_r}\, \tilde{c}_i \tag{41}$$

and

$$2\lambda_i\tilde{d} = \frac{-a^2\tilde{d}}{|\tilde{c}|^2\lambda_r}\, \tilde{c}_i \equiv -\Omega\tilde{c}_i.$$

Rewriting (39b)

$$e^{\Omega\tilde{c}_i} = \mathscr{R} \tag{42a}$$

or

$$c_i = \frac{\ln \mathscr{R}}{\Omega}. \tag{42b}$$

Our first estimate for $\tilde{c}_i$ is obtained by evaluating the right-hand side of (42b) on the basis of results for $\tilde{c}_i$ "vanishingly small," i.e., for the values of $\tilde{c}_r$, λ_r and $\mathscr{R}$ obtained above. With this choice for $\tilde{c}_i$ (instead of $\tilde{c}_i$ "vanishingly small") we may repeat the calculation of $\tilde{c}_r$ and R and use (42b) to obtain our next estimate $\tilde{c}_i$. The whole process is continued until convergence is obtained, leaving us with the $\tilde{c}$'s at each α corresponding to the unstable eigenmodes.

The first estimate of $\tilde{c}_i$ is independently interesting. Insofar as it is based entirely on the behavior of the internal Rossby waves in the limit of vanishing $\tilde{c}_i$, it is an estimate of the growth rate due to critical layer instability as discussed in Section 2. Thus comparisons of the initial estimates of $\tilde{c}$ with the converged values of $\tilde{c}$ indicate the importance of the role of critical layer instability in baroclinic instability. As a rule, if the initial estimate of $\tilde{c}_i$ is comparable to or greater than the converged value, one may conclude that baroclinic instability is fundamentally a critical layer instability. This matter is discussed in greater detail in Section 2b. It should be stressed that the identification with critical layer instability does not require that the initial estimate be an accurate approximation.

Before proceeding to explicit calculations, we show that this initial estimate is identical to that described in Section 2b, and in Lindzen and Rosenthal (1976). Consider Eq. (17) in the region $0 > z > -d$ (where $d = \tilde{d}H$). The bracketed quantity is the square of the dimensional vertical wavenumber (we will call this quantity n^2; $n^2 = \lambda^2/H^2$). Using (27) it is readily shown that

$$n^2 = \frac{\lambda^2}{H^2} \approx \left\{ \frac{-a^2 \dfrac{m}{H}}{u_0 - c} - \frac{k^2}{\epsilon} - \frac{1}{4H^2} \right\}, \quad (43)$$

which may be rewritten

$$n^2 \approx \left\{ \frac{-a^2 \dfrac{m}{H} k}{\hat{\omega}} - \frac{k^2}{\epsilon} - \frac{1}{4H^2} \right\}, \quad (44)$$

where $\hat{\omega} = k(u_0 - c)$ is the Doppler-shifted wave frequency. In the limit of vanishing c_i, $n = \lambda/H = n_r = \lambda_r/H$, and the vertical group velocity is given by

$$c_G = 1/(dn/d\hat{\omega}).$$

The time τ for a Rossby wave to traverse the region $0 > z > d$ is given by

$$\tau = \frac{d}{c_G} = \frac{\tilde{d}H}{c_G}.$$

Differentiating (44) with respect to $\hat{\omega}$ yields

$$2n \frac{dn}{d\hat{\omega}} = \frac{a^2 \dfrac{m}{H} k}{\hat{\omega}^2},$$

$$\frac{1}{c_G} = \frac{dn}{d\hat{\omega}} = \frac{a^2 mk}{2\lambda\hat{\omega}^2},$$

$$\tau = \frac{\tilde{d}H}{c_G} = \frac{a^2 mkdH}{2\lambda\hat{\omega}^2} = \frac{a^2 mk\tilde{d}H}{2\lambda k^2 c^2 m^2 H^2},$$

$$2kc_i\tau = \frac{a^2 \tilde{d}\tilde{c}_i}{\tilde{c}^2\lambda_r} = \Omega\tilde{c}_i \quad \text{[viz. Eq. (42a)]}.$$

Thus (42a) may be rewritten

$$\mathcal{R} = e^{2kc_i\tau}. \quad (45)$$

Eq. (45) states that c_i is the growth rate pertaining to a Rossby wave which is magnified by a factor $\mathcal{R}$ each time it travels from $z = 0$ to the lower boundary at $z = -d$ and back to $z = 0$. Eq. (45), sometimes called the laser formula, was used by Lindzen and Rosenthal (1976) and is discussed in LT. The exponential growth implied by (45) can only occur when the overreflected and reflected waves are in phase with each other. The quantization condition approximately assures this.

b. Numerical results for the Charney and Green problems

We will first apply the above to two extensively studied cases: those dealt with by Charney (1947) and Green (1960). Charney's problem assumes a basic state where

$$\frac{d\bar{u}}{d\tilde{z}} = 1; \quad (46)$$

in which case (22) reduces to

$$\psi_{\tilde{z}\tilde{z}} + \left\{ \frac{r+1}{\tilde{z} - \tilde{c}} - \alpha^2 - \frac{1}{4} \right\} \psi = 0. \quad (47)$$

Boundedness is assumed as $\tilde{z} \to \infty$. We find results are not significantly modified by assuming a rigid lid at a sufficiently great height which depends on r. A more general discussion of the role of upper boundary conditions in baroclinic instability is given in Section 3c. For the present we will take $r = 1$ (corresponding to a shear of about 2.0 m s^{-1} km^{-1}) and place a lid at $\tilde{z} = \tilde{z}_T = 4$. Fig. 2 shows plots of $\tilde{c}_r$ and $\tilde{c}_i$ vs α for both the initial estimate (labeled overreflection) and for the converged eigenvalue. The similarity of the two is striking. Both have the same qualitative behavior including the apparent node (at least to the accuracy allowed by the numerics) in $\tilde{c}_r$ and $\tilde{c}_i$ at $\alpha \approx 0.87$. One can see from Fig. 2 that the first estimate of instability (labeled OVER-REFL.) is remarkably good when compared to the finally converged value (labeled EXACT), which is the same as the eigenvalue result. This suggests that critical layer instability, as analyzed in Section 2, is the fundamental basis for baroclinic instability. This has sometimes been suggested to be the case for $\alpha \leqslant 0.87$ (sometimes referred to as Burger or Green modes) but not for $\alpha \geqslant 0.87$ (sometimes referred to as the Charney mode); no such distinction appears in the present calculations.

The nature of the neutral point at $\alpha \approx 0.87$ warrants further comment. Clearly, for $\tilde{c}_i = 0$, we must have $\mathcal{R} = 1$. But, since $\tilde{c}_r = 0$ (corresponding to a critical level at $\tilde{z} = 0$ which is essentially the inflection point in the present problem) it is not clear from Eq. (15) whether this is due to $|B|^2 = 0$

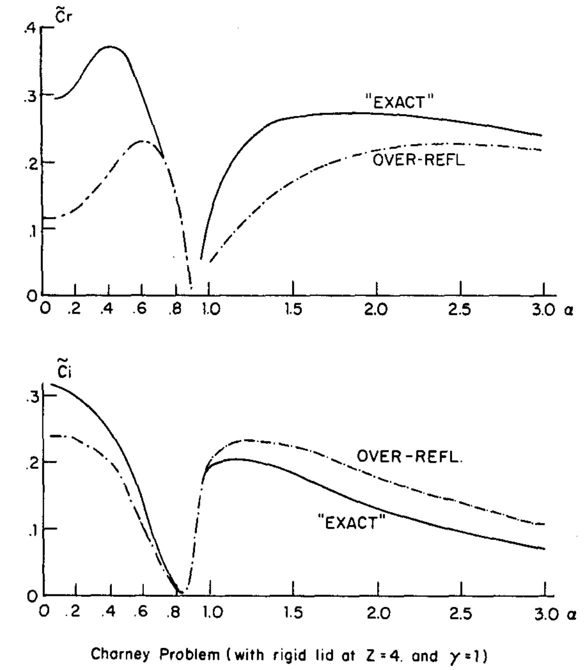


FIG. 2. $\tilde{c}_r$ and $\tilde{c}_i$ vs α for the Charney problem. Shown are "exact" results for unstable eigenmodes as well as estimates based on overreflection.

or to $\beta_e = 0$. In Fig. 3 we show $\mathcal{R} - 1$ vs $\tilde{c}_r$ (in the limit of vanishing $\tilde{c}_i$) for a sequence of α's above and below $\alpha \approx 0.87$. It is clear that $\mathcal{R} - 1 \rightarrow 0$. The approach is smooth for $\alpha \leqslant 0.87$, but seemingly non-uniform for $\alpha \geqslant 0.87$. The latter accounts for the precipitous rise in $\tilde{c}_i$ to the right of $\alpha \approx 0.87$.[3]

Some insight into the nature of this critical value of α can be obtained from a reconsideration of Eq. (47). Let

$$\left.\begin{aligned} \hat{z} &= (r+1)\tilde{z} \\ \hat{c} &= (r+1)\tilde{c} \\ \delta^2 &= \frac{\alpha^2 + 1/4}{(r+1)^2} \end{aligned}\right\} . \tag{48}$$

Eq. (47) becomes

$$\psi_{\hat{z}\hat{z}} + \left\{\frac{1}{\hat{z} - \hat{c}} - \delta^2\right\}\psi = 0. \tag{49}$$

In the limit $c_i \rightarrow 0$, we have a critical level at $\hat{z} = \hat{c}$, above which

$$Q \equiv \frac{1}{\hat{z} - \hat{c}} - \delta^2 \tag{50}$$

is positive. Wave region ② of Fig. 1 exists for $\hat{z} > \hat{c}$ and is bounded above by a turning point where $Q = 0$:

$$\hat{z} = \hat{z}_t = \hat{c} + \frac{1}{\delta^2} . \tag{51}$$

[3] If n is the zonal wavenumber at 45° latitude, $\alpha \approx 0.224n$ (*viz.* Appendix 4).

We now consider the thickness of wave region ② in terms of phase, i.e.,

$$J = \int_{\hat{c}}^{\hat{c}+1/\delta^1} Q^{1/2} d\hat{z} = \int_0^{1/\delta^2} \left(\frac{1}{x} - \delta^2\right)^{1/2} dx. \tag{52}$$

Eq. (52) is exactly integrable, yielding

$$J = \frac{\pi}{2\delta} = \pi \frac{(r+1)}{2(\alpha^2 + 1/4)^{1/2}} . \tag{53}$$

Neutral points occur whenever J is an integral multiple of π, i.e., when

$$\frac{r+1}{2(\alpha^2 + 1/4)^{1/2}} = n, \quad n = 1, 2, \ldots . \tag{54}$$

Eq. (54), as a condition for neutral points, has been found by independent means by Burger (1962) and Miles (1964). The result is discussed in Charney (1973). The above provides an interpretation of (54) in terms of the waves in region ②. Solving (54) for α yields

$$\alpha^2 = \left(\frac{r+1}{2n}\right)^2 - \frac{1}{4} . \tag{55}$$

Clearly, the critical α's depend on r (i.e., on the mean shear), and only for a finite number of n's do we have positive solutions to (55) for α^2. In partic-

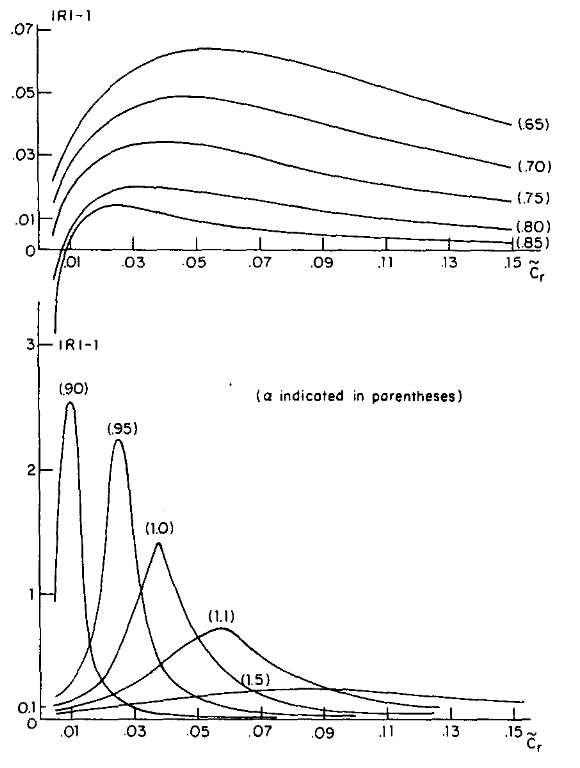


FIG. 3. $R - 1$ vs $\tilde{c}_r$ (as $\tilde{c}_i \rightarrow 0$) for α's just below α_c and just above α_c. α_c is the value of α at which the Charney problem has a neutral eigenmode.

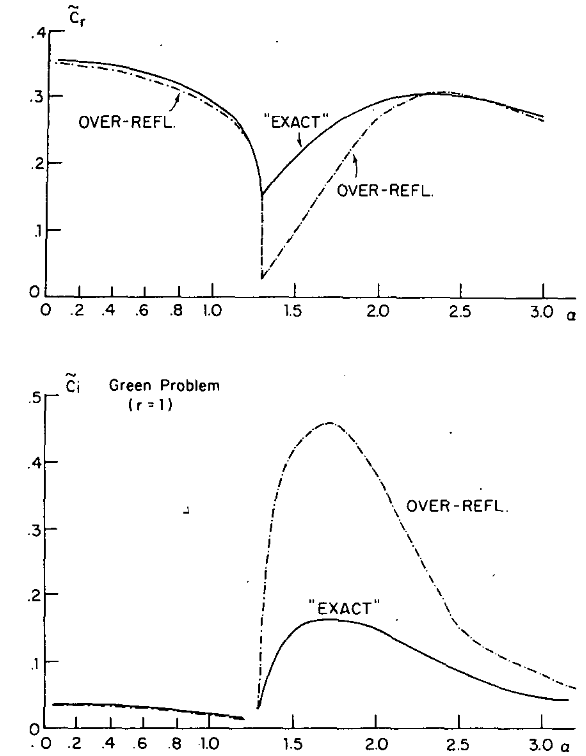


FIG. 4. $\tilde{c}_r$ and $\tilde{c}_i$ vs α for the Green problem. Shown are exact results for unstable eigenmodes as well as estimates based on overreflection.

ular, for $r = 1$, there is a neutral point only at $n \doteq 1$ where

$$\alpha = \sqrt{\frac{3}{4}} \approx 0.87.$$

We next turn to Green's (1960) problem. Green also deals with flows where $d\bar{u}/d\bar{z} = 1$, but assumes a lid at $\bar{z} = 1$ and a Boussinesq fluid. The latter requires that we drop all terms in our dimensional equations which are $O(H^{-1})$ and their counterparts in the non-dimensional equations. Eqs. (22) then becomes

$$\psi_{\bar{z}\bar{z}} + \left\{\frac{r}{\bar{z} - \bar{c}} - \alpha^2\right\}\psi = 0. \tag{56}$$

The comparisons of "overreflection" and "exact" values for $\bar{c}_r$ and $\bar{c}_i$ in this case are shown in Fig. 4, again we have taken $r = 1$. In this case we once more have a pronounced dip in $\bar{c}_r$ and $\bar{c}_i$ at a particular $\alpha \approx 1.3$. For smaller α's, overreflection estimates are accurate approximations to the exact values; however, for α's > 1.3, overreflection substantially overestimates $\bar{c}_i$. This still leads us to the conclusion that overreflection underlies the instability; however, one may ask why one has such a large overestimate in one case and not in another. In this connection we should note that when one redoes Green's problem for a non-Boussinesq fluid [i.e., (56) is replaced by (47)], the degree of overestimation is greatly diminished. We should also note that the "exact" values of $\bar{c}_i$ are of similar magnitude in all cases.

At this stage we can only speculate on what is going on. In Charney's problem, wave region ② (the waveguide region of Section 2) is distributed between $\bar{z}_c$ and a turning point [*viz*. Eqs. (50) and (51) with some bias toward $\bar{z}_c$ (*viz*. Eq. (52)]. In Green's Boussinesq problem (with $r = 1$), the upper boundary occurs before the turning point and wave region ② is concentrated at the upper boundary (where again we have to consider the effectively infinite curvature). In Green's non-Boussinesq problem, r in Eq. (56) is replaced by $r + 1$, which causes the contribution of the region, between $\bar{z}_c$ and the upper boundary, to wave region ② to be more important. Now the picture of overreflection presented at the end of Section 2 suggests that overreflection results from the establishment of a forced waveguide mode in wave region ②. Such a process takes time and if that time exceeds twice the wave travel time in wave region ① [*viz*. Eq. (45)] then one expects Eq. (45) to yield an overestimate for $\bar{c}_i$ [a very special example of this is given in McIntyre and Weissman (1978)]. The above suggests that the increased isolation of wave region ② leads to enhanced values of $\mathcal{R}$, without, however, simultaneously increasing the time available for setting up the waveguide mode. Recall, the available time is determined by wave region ①. Thus in these cases, increased $\mathcal{R}$ does not lead to notably increased $\bar{c}_i$.

A few remarks are in order for the exact solution of Green's problem. Note that, in contrast to Charney's problem, the dip in $\bar{c}_r$ at $\alpha \approx 1.3$ leads to a minimum value $\bar{c}_r \approx 0.15$ rather than zero. In view of the discussion of Fig. 3, this is not altogether surprising. The dip in $\bar{c}_i$ brings $\bar{c}_i$ very close to zero, but as Tung will show in a forthcoming paper, such dips in problems other than Charney's will not, in general, reach zero. It can also be seen in Fig. 4 that $\bar{c}_i$ for $\alpha \leqslant 1.3$ is much smaller than it is for $\alpha \geqslant 1.3$. This has suggested the possibility that the low α (Green) modes are intrinsically weak and physically different from the high α (Charney) modes. However, the small $\bar{c}_i$'s for the Green modes is a peculiar property of Green's problem. It is not found in the Charney problem (*viz*. Fig. 2); nor is it found in calculations by Geisler and Garcia (1977).

c. The radiation condition and baroclinic instability

In the Charney problem the region between the critical level (or, in the presence of $\bar{c}_i$, a region within $\bar{c}_i$ of where $\bar{u} = \bar{c}_r$) and $\bar{z}_t$ [*viz*. Eq. (51)] constitutes wave region ② in Fig. 1. If one really had a constant shear, one would expect that the choice of an upper boundary condition would not matter much as long as one applied it well above $\tilde{z}_t$. However, from Eqs. (51) and (48) we see that for small α's

$$\left.\begin{array}{l}\tilde{z}_t \approx \tilde{c} + 4(r+1) \\ \quad > 4\end{array}\right\}, \qquad (57)$$

i.e., the turning point which contains the waveguide region would occur above four scale heights. In general, tropospheric shears do not continue beyond 0.7–1.5 scale heights in which range $\tilde{u}$ reaches its maximum. It would thus appear that for small α's there might not be a turning point at some reasonable height, and the choice of an upper boundary condition could prove important. In particular, it seems possible that for α's without turning points the use of a radiation condition could substantially eliminate instability. What follows will show that this is unlikely.

We consider profiles for which

$$\frac{d\tilde{u}}{d\tilde{z}} = \frac{1}{2}\left(1 - \tanh\frac{\tilde{z}-\tilde{z}_B}{\tilde{l}}\right) \qquad (58)$$

and

$$\tilde{u} = \frac{\tilde{z}}{2} - \frac{\tilde{l}}{2}\left[\ln\left(\frac{\cosh\dfrac{\tilde{z}-\tilde{z}_B}{\tilde{l}}}{\cosh\dfrac{\tilde{z}_B}{\tilde{l}}}\right)\right], \qquad (59)$$

where $\tilde{l}$ is the thickness of the region and where $d\tilde{u}/d\tilde{z}$ makes a transition from unity below $\tilde{z}_B$ to zero above $\tilde{z}_B$. Above $\tilde{z}_B$,

$$\tilde{u} \approx \tilde{z}_B. \qquad (60)$$

An example of such a profile ($\tilde{z}_B = 0.7$, $\tilde{l} = 0.1$) is shown in Fig. 5. For such profiles, Q (the bracketed quantity) in Eq. (22) becomes, for $\tilde{z} \gg \tilde{z}_B$,

$$Q = \frac{r}{\tilde{z}_B - \tilde{c}} - \alpha^2 - \frac{1}{4}. \qquad (61)$$

Ignoring $\tilde{c}_i$, Q will be positive everywhere above $\tilde{z}_c$ for

$$\alpha^2 < \frac{r}{\tilde{z}_B - \tilde{c}} - \frac{1}{4}.$$

For realistic parameter choices, this will be the case for $\alpha^2 \leqslant 1$ or wavenumbers 4–5 on a sphere (see Appendix 4). In any event, Q will be essentially constant sufficiently above $\tilde{z}_B$.

We have examined cases with a wide variety of $\tilde{z}_B$'s and $\tilde{l}$'s. As long as $\tilde{l}$ is moderately small, its choice doesn't matter. The choice of $\tilde{z}_B$ is of some consequence, but calculations based on the $\tilde{u}$ shown in Fig. 5 are completely indicative of the general results; we therefore restrict ourselves to this case. For the following calculations our integration ceases at $\tilde{z} = 4$. In half the calculations we assume a rigid top and use Eq. (20c) (or rather its non-dimensional form). In the other half we apply a radiation condition. The latter is easily done in view of the near constancy of Q at $\tilde{z} = 4$ [*viz*. Eq. (61)]. We assume our solution to be of the form

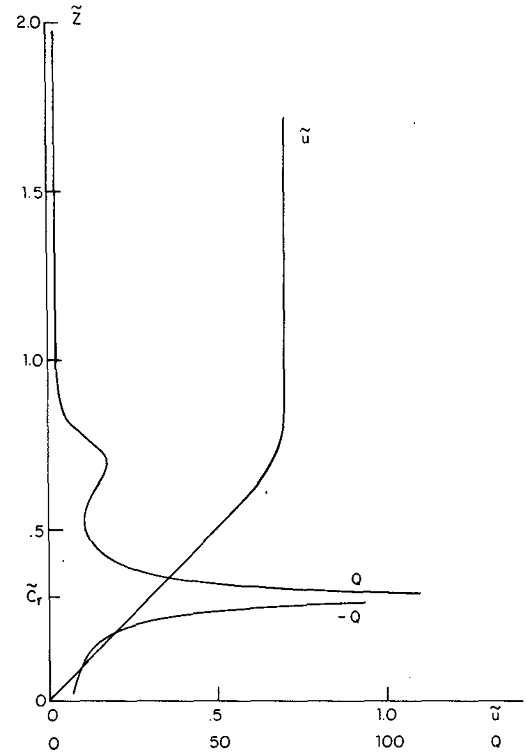


FIG. 5. $\tilde{u}(\tilde{z})$ for $\tilde{z}_B = 0.7$ and $\tilde{l} = 0.1$. This profile is used to examine the effect of a radiation upper boundary condition. Also shown is $Q(\tilde{z})$ for the choice of $\tilde{u}(\tilde{z})$. $Q(\tilde{z})$ is the bracketed quantity in Eq. (22); we have taken $r = 1$ and $\tilde{c}_r = 0.3$ in evaluating Q.

$$\tilde{\psi} = A\exp(iQ^{1/2}\tilde{z}), \qquad (62)$$

where A is an arbitrary constant, and the square root of Q is so chosen, that for positive $\tilde{c}_i$, the imaginary part of $Q^{1/2}$ is positive. With this choice of $Q^{1/2}$, the radiation upper boundary condition is obtained by differentiating (62):

$$\tilde{\psi}_{\tilde{z}} = iQ^{1/2}\tilde{\psi}. \qquad (63)$$

Equation (63) is appropriate regardless of the actual value of $\tilde{c}_i$.

Fig. 6 shows $\tilde{c}_r$ vs α for the following cases:

1) Initial quantization based on Eq. (40) for radiating solutions in the limit of $\tilde{c}_i \to 0$.
2) Initial quantization based on Eq. (40) for solutions with a rigid lid at $\tilde{z} = 4$.
3) Converged eigenmode with radiation condition.
4) Converged eigenmode with lid at $\tilde{z} = 4$.

Fig. 6 shows $\tilde{c}_r$ vs α for the above four cases; $\tilde{c}_i$ vs α is shown in Fig. 7. In the first two cases,

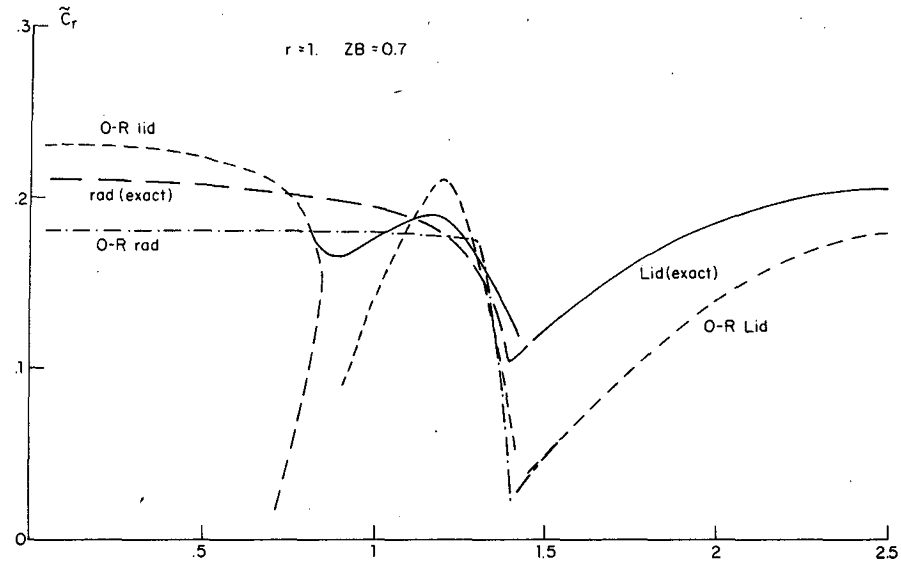


FIG. 6. $\tilde{c}_r$ vs α for baroclinic instabilities of the profile $\bar{u}(z)$ shown in Fig. 5. Results are shown for "exact" eigenmodes when a rigid lid is assumed at $\tilde{z} = 4$ and when a radiation condition is applied at $\tilde{z} = 4$. Also shown are wave overreflection estimates for each of the above cases. In all cases, we have taken $r = 1$.

c_i is an estimate [Eq. (42b)] based on wave overreflection.

There are many features in Figs. 6 and 7 which warrant comment. We will restrict ourselves to a few of the more salient ones.

• Restricting ourselves to case 1) above (i.e., overreflection estimates with radiation condition), we see from Fig. 7 that positive $\tilde{c}_i$'s (implying the existence of overreflection) are found at all α's, showing that wave radiation does not eliminate instability. The reason for this emerges from Fig. 5 where we show Q as a function of $\tilde{z}$ for $\alpha = 0.6$. Although Q remains positive as $\tilde{z} \to \infty$, it undergoes a marked change in the neighborhood of $\tilde{z} = \tilde{z}_b$. This provides sufficient partial reflection at $\tilde{z} = \tilde{z}_b$ so that the region between $\tilde{z}_c$ and $\tilde{z}_b$ still constitutes an adequate waveguide.

• A comparison of cases 1) and 2) shows an additional dip in both $\tilde{c}_r$ and $\tilde{c}_i$ near $\alpha = 0.87$ for the lidded case. The overreflection estimates also suggest that $\tilde{c}_i$ for the lidded case is much larger than with radiation for $\alpha \approx 1$. Moreover, in case 2) for

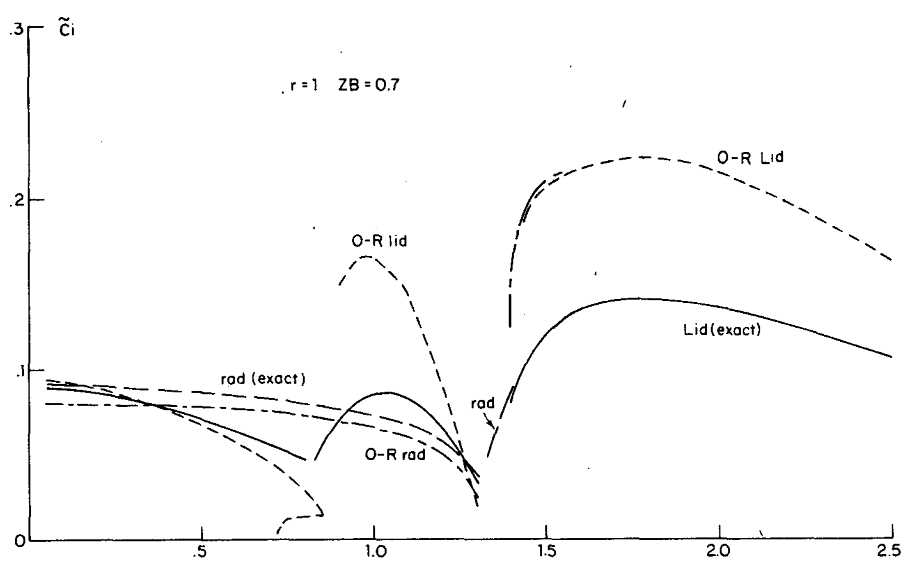


FIG. 7. As in Fig. 6 but for $\tilde{c}_i$ vs α.

$0.7 \leqslant \alpha \leqslant 0.85$ one has two c_r's where quantization occurs and for $0.85 \leqslant \alpha \leqslant 0.9$ there are no values (though there are values of $\tilde{c}_r$ where quantization nearly occurs).

• A comparison of cases 2) and 4) in Figs. 6 and 7 shows that the above described behavior for the approximate overreflection quantization is not found in the converged eigenmodes. Unique eigenmodes are found at all α's. As discussed in Section 2c, this simply means that the initial quantization is an inadequate approximation. It can still be shown that wave overreflection is the fundamental feature of the instability.

• We see from Figs. 6 and 7 that for $\alpha \gtrsim 1.4$ the presence or absence of a lid is of no consequence. For these α's, $\tilde{z}_t$ is below the lid.

• Finally, and perhaps most important, we see that results for the converged eigenmodes are similar regardless of the upper boundary condition. The results with a lid are qualitatively similar to the corresponding overreflection estimates. However, in cases 3) and 4) the differences are greatly diminished from cases 1) and 2). Clearly, the explicit inclusion of growth (i.e., $\tilde{c}_i$) reduces the impact of the upper boundary, which is not surprising since $\tilde{c}_i$ is mathematically undistinguishable from the presence of linear friction. It appears that baroclinic instability, even for long waves, is not significantly affected by the presence of lids providing, of course, that the lids are placed well above the anticipated jet maximum.

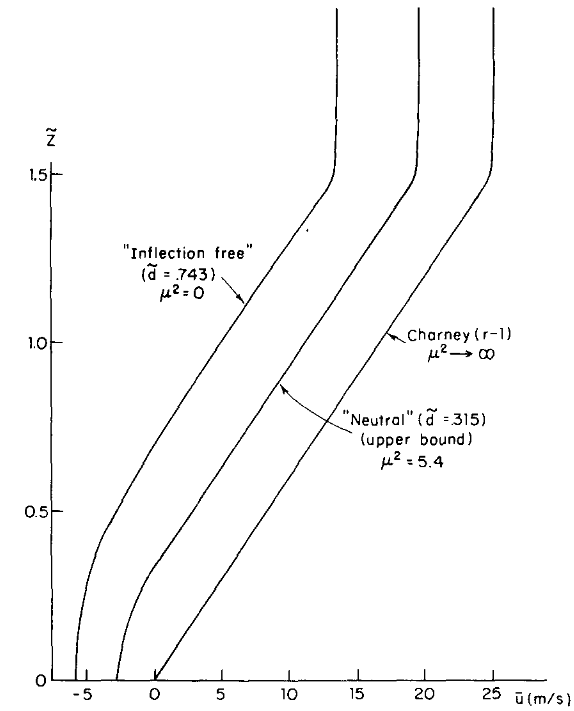
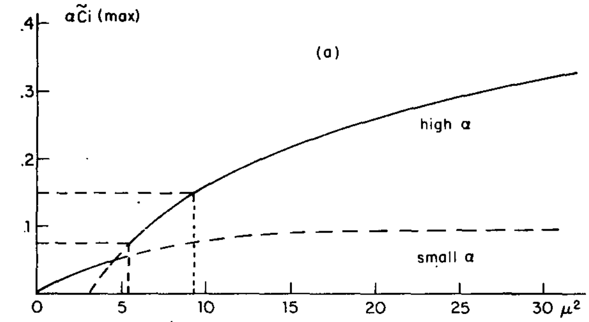


FIG. 8. $\bar{u}(\tilde{z})$ for a Charney profile where there is a transition from constant shear to no shear at $\tilde{z} = 1.5$. The neutralization of this profile is studied by spreading the transition from zero shear at the ground to the interior shear over increasingly deep regions. Also shown are profiles of $\bar{u}(\tilde{z})$ which are rigorously neutral and "effectively" neutral. These terms are explained in Section 3d.

d. Effectively neutral profiles

It was noted in Section 3a (and in Appendix 2) that if the transition from zero shear at the ground to the interior shear were effected by a $\bar{u}$ profile for which $\beta_e = 0$ [$\mu^2 = 0$ in Eq. (24)] then the resulting profile is rigorously neutral. The question arises as to whether "effective neutrality" might not be achieved with μ^2's greater than zero (i.e., with less smoothing of the transition). By effective neutrality we mean that the calculated growth times $[(kc_i)^{-1}]$ are longer than normal frictional spin down time [O(10 days)]. One may obtain a *conservative* answer to this question without explicit stability calculations by means of overreflection calculations. To do this one must note the following: If for a given α, one evaluates $\alpha\tilde{c}_i$ using Eq. (42b) for *all c_r regardless* of whether (40) is satisfied (i.e., regardless of whether we have quantization) then the maximum value one obtains for $\alpha\tilde{c}_i$ is invariably an overestimate of the $\alpha\tilde{c}_i$ for the actual eigenmode (usually a very large overestimate).

In this section we will consider $\bar{u}(\tilde{z})$ profiles where $d\bar{u}/d\tilde{z}$ goes from unity below $\tilde{z} = 1.5$ to zero above $\tilde{z} = 1.5$. The transition from $d\bar{u}/d\tilde{z} = 0$ at the ground ($\tilde{z} = 0$) to $d\bar{u}/d\tilde{z} = 1$ at $\tilde{z} = \tilde{d}$ will be made by profiles in $\bar{u}$ which satisfy Eq. (24). With Eq. (24) and "vanishingly small" $\tilde{c}_i$, Q is constant in the transition region, even when $\tilde{d}$ is not small. We consider large values of $\tilde{d}$ by allowing μ^2 to become progressively smaller, recalling that $\mu^2 = 0$ is rigorously neutral. For each choice of μ^2 we use (42b) to calculate $\alpha\tilde{c}_i$ for all c_r's and all α's. The maximum $\alpha\tilde{c}_i$, so obtained, should be an upper bound for any actual growth rate. In the calculations which follow we take $r = 1$ (corresponding to a shear of 2.05 m s^{-1} km^{-1} between $\tilde{d}$ and $\tilde{z}_b$). Profiles of $\bar{u}$ for various choices of μ^2 are shown in Fig. 8. In calculating $(\alpha\tilde{c}_i)_{\max}$ as a function of α we find that relative maxima for $(\alpha\tilde{c}_i)_{\max}$ occur at two α's until μ^2 gets smaller than about 2.5 [note that the range of α's for which waves can exist below $\tilde{z} = \tilde{d}$ diminishes as $\mu^2 \rightarrow 0$ as may be seen from Eq. (22); we will refer to the maximum α for which waves can exist as α_{cutoff}]; for smaller μ^2's there is only-one maximum. In Fig. 10a we show $(\alpha\tilde{c}_i)_{\max}$ as a function of μ^2; where relative maxima exist for two values of α, both are shown. In most cases the larger $(\alpha\tilde{c}_i)_{\max}$ is associated with the larger value of α. For $r = 1$ (parameter choices are explained in Appendix 4), a time scale for growth of 10 days corresponds to $\alpha\tilde{c}_i = 0.075$. We note in Fig. 9a where $(\alpha\tilde{c}_i)_{\max}$



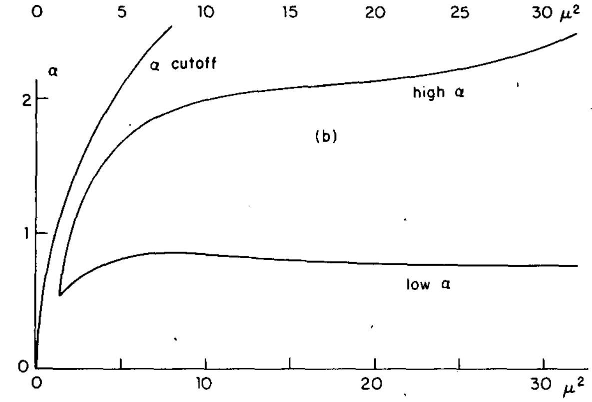
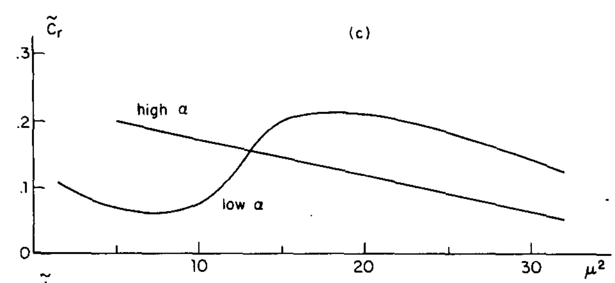


FIG. 9a. $(\alpha\tilde{c}_i)_{\max}$ vs μ^2; see Section 3d for explanation.

FIG. 9b. α vs μ^2, where α is the wavenumber corresponding to $(\alpha\tilde{c}_i)_{\max}$.



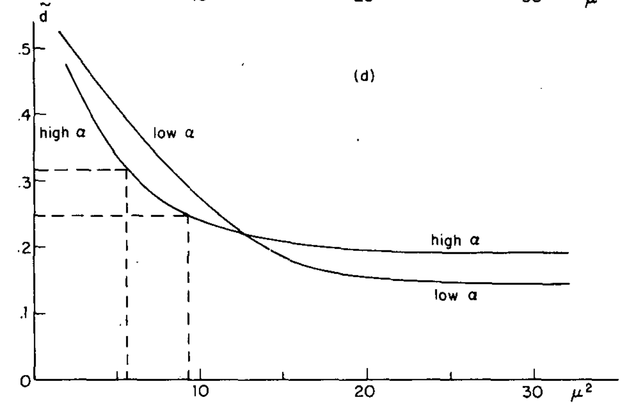


FIG. 9c. $\tilde{c}_r$ vs μ^2, where $\tilde{c}_r$ is the phase speed corresponding to $(\alpha\tilde{c}_i)_{\max}$.

FIG. 9d. $\tilde{d}$ vs μ^2, where $\tilde{d}$ is the transition depth corresponding to the $\tilde{c}_r$ associated with $(\alpha\tilde{c}_i)_{\max}$.

$= 0.075$. Since our estimates are likely to be gross over-estimates, we also indicate $(\alpha\tilde{c}_i)_{\max} = 0.15$.

In Fig. 9b we show the variation of the α's associated with maxima in $(\alpha\tilde{c}_i)$ as functions of μ^2. Also shown is α_{cutoff}. Fig. 9c shows the values of $\tilde{c}_r$ at which $(\alpha\tilde{c}_i)_{\max}$ occurs.[4] Fig. 9d shows the variation of $\tilde{d}$ with μ^2 [one can show from Eq. (24) that $\tilde{d}$ has a mild dependence on $\tilde{c}_r$]. Also shown in Fig. 9d are the values of $\tilde{d}$ associated with $(\alpha\tilde{c}_i)_{\max} = 0.075$ and $(\alpha\tilde{c}_i)_{\max} = 0.15$. The profile of $\bar{u}$ associated with the former is shown in Fig. 8. We see that $\tilde{d}$ for "effective neutralization" is substantially less than half that needed for rigorous neutralization.

Fig. 9e shows the phase discrepancy $\Delta\phi$ [i.e., $\theta - 2\lambda_r\tilde{d}$; *viz.* Eq. (40)] for $(\alpha\tilde{c}_i)_{\max}$ as a function of μ^2. We see that $(\alpha\tilde{c}_i)_{\max}$ is achieved far from quantization conditions for $\mu^2 < 20$ but approaches quantization for larger μ^2. This supports our contention that $(\alpha\tilde{c}_i)_{\max}$ is a substantial overestimate of actual growth rates.

Finally, we should note that while $r = 1$ corresponds to typical tropospheric shears, maximum zonally averaged shear is more nearly associated with $r = 0.6$. Even for such shears, "effective neutralization" is achieved for $\tilde{d} \lesssim 0.4$. Thus we may plausibly conclude that neutralization of baroclinic instability calls for only very modest reductions of mean tropospheric shears.

e. Divergence of wave action

In view of the results of the preceding section one may reasonably ask whether the fluxes due to unstable modes (in the Charney problem, for example) are such as to modify the basic state toward increasing neutrality. As noted in Section 3d, this involves, primarily, altering $\bar{u}$ below the steering

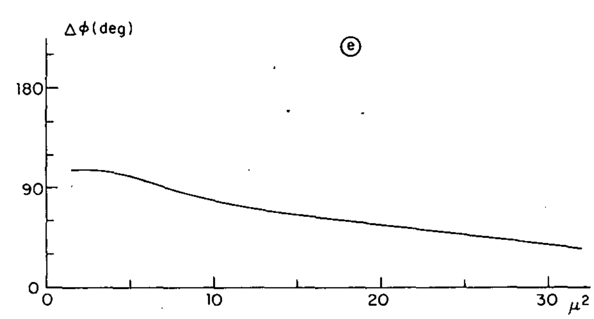


FIG. 9e. $\Delta\phi$ vs μ^2, where $\Delta\phi$ (in degrees) is the extent to which the overreflected wave associated with $(\alpha\tilde{c}_i)_{\max}$ does *not* satisfy Eq. (40).

Note that in (a) and (d) we specially indicate $(\alpha\tilde{c}_i)_{\max} = 0.075$ (which for $r = 1$, corresponds to $(\alpha\tilde{c}_i)_{\max}^{-1} = 10$ days and is hence associated with "effective" neutrality) and $(\alpha\tilde{c}_i)_{\max} = 0.15$ which may be a better indicator of "effective" neutrality since $(\alpha\tilde{c}_i)_{\max}$ should be a very substantial overestimate.

[4] $\tilde{c}_r$ is here defined as the height, in scale heights, above $\tilde{z} = \tilde{d}$ for the steering (or critical) level.

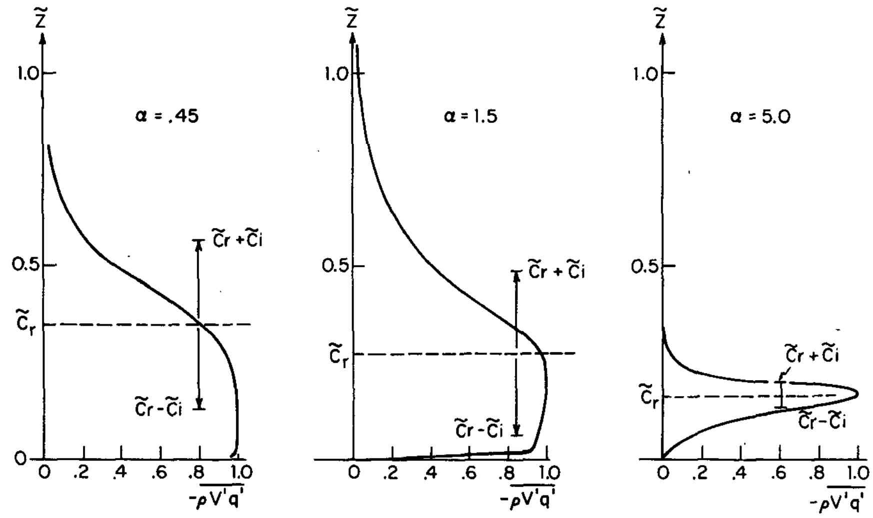


FIG. 10. The distribution (units arbitrary) of potential vorticity flux with $\tilde{z}$ for unstable eigenmodes of the usual Charney problem (with $r = 1$) at various α's.

level so as to broaden the delta-function curvature near the ground. We have not, as yet, completed the calculations relevant to this question. However, some preliminary results may prove of interest.

As noted by Andrews and McIntyre (1976), the most basic measure of wavemean flow interaction is the divergence of wave action, which for the present problem is associated with the meridional flux of potential vorticity, $\rho\overline{v'q'}$. Only where we have such a flux does the wave exchange energy with the mean flow. For wave overreflection in the limit of vanishing c_i, this flux is concentrated at the critical level. The situation is significantly different for unstable eigenmodes which can have significant c_i's. We have calculated $\rho\overline{v'q'}$ for the eigenmodes of the Charney problem (where as in Section 3b we have taken $r = 1$ and have placed a lid at $\tilde{z} = 4$). Since we are dealing with solutions of linearized equations, amplitudes are, of course, arbitrary. Distributions of $\rho\overline{v'q'}$ with altitude for various choices of α are shown in Fig. 10. The most important thing to note is that $\rho\overline{v'q'}$ is always restricted primarily to a region between the ground and a layer of thickness $\sim 2\tilde{c}_i$ centered at $\tilde{c}_r$. As noted in Section 2, the integral of $\rho\overline{v'q'}$ over the whole domain must equal zero. The negative values shown in Fig. 10 are balanced by large positive fluxes in the immediate vicinity of the ground (which, for the methodology of Section 3a is located slightly below $\tilde{z} = 0$); the large positive fluxes are not shown in Fig. 10. From Fig. 2 we see that for all modes with reasonable growth rates, $\tilde{c}_r \approx 0.2$ and more generally $\tilde{c}_r \lesssim 0.25$. This implies that the direct interaction between baroclinically unstable modes and the mean flow is restricted to a boundary layer of depth ~ 0.3 scale heights.

Three additional points should be made:

(i) The restriction of $\rho\overline{v'q'}$ to a boundary layer does not arise from a similar restriction of the unstable eigenmodes. In Fig. 11 we show the distributions of the perturbation pressure magnitudes ($|P| = |\Psi|e^{-\tilde{z}}$) of the eigenmodes whose potential vorticity fluxes are shown in Fig. 10. For $\alpha = 0.45$, $|P|$ is of large amplitude throughout the domain; even for $\alpha = 1.5$ and $\alpha = 5.0$, $|P|$ extends to much greater depths than does $\rho\overline{v'q'}$.

(ii) As a general rule, one expects potential vorticity fluxes proportional to c_i wherever $|\psi|$ is finite.

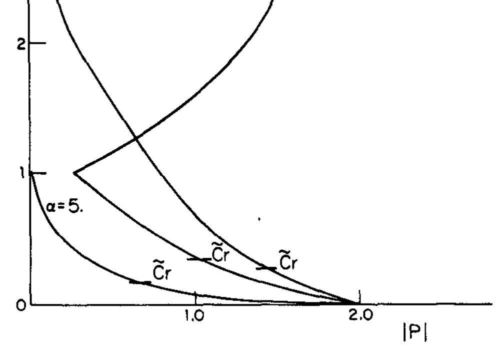


FIG. 11. The absolute value of the unstable eigenmodes as function of $\tilde{z}$ for the usual Charney problem (with $r = 1$) at those α's considered in Fig. 10.

The above results show that the coefficient of proportionality is much larger in the critical layer and below it than above it.

(iii) The fact that $\rho\overline{v'q'}$ is restricted to a boundary layer does not imply that modifications of the basic state are restricted to this layer. The interaction can force a meridional circulation which extends above the layer where we have significant potential vorticity fluxes; and this meridional circulation will in general lead to alterations of $\bar{u}$ (and $\bar{T}$). As a rule, however, these alterations will decay exponentially above the layer where we have $\rho\overline{v'q'}$. We are currently involved in calculating these alterations. The results, so far, however, do support the notion that instabilities tend to neutralize the basic state.

4. Concluding remarks

We have, in this paper, examined the nature of overreflection, and how wave overreflection can lead to instability (Section 2). In Section 3 we undertook a study of baroclinic instability from the perspective of wave overreflection. Without repeating our detailed results, we can state as our most fundamental conclusion that the underlying mechanism for baroclinic instability is, indeed, the overreflection mechanism, and that the locus of instability is the critical layer (a layer of approximate depth $2\bar{c}_i$, centered at the steering level). This conclusion is at variance with the more usual view that baroclinic instability is a convective response to available potential energy (or more simply, horizontal temperature gradients).[5] To be sure, since our equations are energetically consistent, the energy for the growing disturbance does come from the available potential energy of the basic state, but the available potential energy, per se, is not the cause of the instability. From Section 3 we see that by altering the basic state so as to prevent the overreflection mechanism we eliminate the instability even though the same alteration eliminates only a relatively small fraction of the available potential energy.

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[5] This view is not completely original. It is implicit in the identification of the Eady problem with the stability of Couette flow with free boundaries. It is also found in the identification of Green modes with critical layer instability. These matters are reviewed in Charney (1973). However, even in Charney (1973), the so-called "Charney" modes continue to be incorrectly distinguished from critical layer instabilities.

APPENDIX 1

Slow Variation and the Existence of Wave Propagation

It was noted in Section 1 that Q, in Eq. (1), being positive, does not guarantee the existence of wave propagation. This point is best introduced by considering the equation for vertical velocity perturbations in a stably stratified, Boussinesq fluid, with a plane parallel basic flow $u(z)$:

$$\frac{d^2w}{dz^2} + \left\{\frac{N^2(z)}{(u-c)^2} - \frac{u_{zz}}{(u-c)} - k^2\right\}w = 0, \quad \text{(A1.1)}$$

where the form

$$w(z)e^{ik(x-ct)}$$

is assumed for perturbations. $N^2(z) = (-g/\rho_0)d\rho_0/dz$. Consistent with the Boussinesq approximation, variations in ρ_0 have been ignored except insofar as they enter N^2. For Eq. (A1.1),

$$Q = \frac{N^2(z)}{(u-c)^2} - \frac{u_{zz}}{u-c} - k^2, \quad \text{(A1.2)}$$

where $N^2(z) > 0$.

Consider now a situation where N^2 and u_z are constant. Also let $u = c$ at $z = z_c$. Then $u - c = u_z(z - z_c)$. For simplicity, let $z_c = 0$. We then have

$$Q = \frac{N^2}{u_z^2 z^2} - k^2 = \frac{\mathrm{Ri}}{z^2} - k^2, \quad \text{(A1.3)}$$

where $\mathrm{Ri} \equiv N^2/u_z^2$, a constant, and

$$\frac{d^2w}{dz^2} + \left\{\frac{\mathrm{Ri}}{z^2} - k^2\right\}w = 0. \quad \text{(A1.4)}$$

Although (A1.4) can be solved completely, one obtains the same results more simply by setting $k^2 = 0$. This leaves

$$\frac{d^2w}{dz^2} + \frac{\mathrm{Ri}}{z^2}w = 0. \quad \text{(A1.5)}$$

Note that $Q = \mathrm{Ri}/z^2$ is *always positive*. Although (A1.5) has a pair of simple exact solutions we first consider the WKB approximations

$$w \sim Q^{-1/4}\exp\left(\pm i\int^z Q^{1/2}dz'\right) \quad \text{(A1.6a)}$$

$$\sim z^{1/2}\exp(\pm i\,\mathrm{Ri}^{1/2}\ln z) \quad \text{(A1.6b)}$$

$$\sim z^{1/2 \pm i\mathrm{Ri}^{1/2}}. \quad \text{(A1.6c)}$$

For this problem Eq. (2) yields

$$\mathrm{Ri}^{1/2} > \tfrac{1}{2}$$

or

$$\mathrm{Ri} > \tfrac{1}{4}. \quad \text{(A1.7)}$$

Interestingly, (A1.7) guarantees stability as well. (For accuracy of the WKB approximation one would really want Ri $\gg$ ¼.) To see what happens when Ri becomes less than ¼, we turn to the exact solutions of (A1.5):

$$w \sim z^{1/2 \pm i\sqrt{\mathrm{Ri} - 1/4}} \tag{A1.8a}$$

$$\sim z^{1/2} \exp(\pm i\sqrt{\mathrm{Ri} - 1/4}\ \ln z). \tag{A1.8b}$$

From (A1.8b) we see that for Ri > ¼ we do indeed have wave propagation. However, for Ri < ¼, (A1.8b) becomes

$$w \sim z^{1/2} \exp(\pm\sqrt{1/4 - \mathrm{Ri}}\ \ln z) \tag{A1.8c}$$

and we have trapping, despite the fact that $Q > 0$. The above results are fairly general for stratified shear flow (except where u'' is very large), and show that rapid variation of positive Q may lead to trapping rather than propagation. It would be nice if Eq. (2) determined a boundary between propagation and trapping. Unfortunately, this does not appear to be the case in general. The coincidence of the properties in the case of stratified shear flow seems fortuitous. It appears, for example, from exact solutions that although (2) is violated in the neighborhoods of critical levels and turning points in Eq. (3) that there is nonetheless propagation in these neighborhoods. Thus, LT may have been basically correct in ignoring rapid variation in Q.

What is probably called for is consideration of the Ricatti equation associated with Eq. (1). Let

$$\mu = -\frac{\psi_y}{\psi}. \tag{A1.9}$$

Then μ satisfies the Ricatti equation

$$\frac{d\mu}{dy} = Q(y) + \mu^2. \tag{A1.10}$$

In general, we associate propagation with μ being complex. In the case of (A1.5) we have exact solutions for μ; but normally the situation is much more difficult.

APPENDIX 2

Rigorously Neutral Baroclinic Profiles

For convenience, we consider the ground to be at $\tilde{z} = -\tilde{d}$, where $\tilde{z}$ and $\tilde{d}$ are heights z and d scaled by the scale height H. The transition region is taken to be between $\tilde{z} = -\tilde{d}$ and $\tilde{z} = 0$. Above $\tilde{z} = 0$ we have a constant shear. Rather than $\bar{u}$, we consider $\tilde{u}$ as defined in Eq. (21c). The transition profile is given by Eq. (28) with $a^2 = 0$. From Eqs. (28) and (29) we have

$$\tilde{u} = -r\tilde{z} + (1 + r)(e^{\tilde{z}} - 1), \tag{A2.1}$$

$$\tilde{d} = \ln\left(1 + \frac{1}{r}\right), \tag{A2.2}$$

where $r = \beta H/\epsilon m$ and m is the dimensional shear for $z > 0$.

In the absence of smoothing we would have $\tilde{u}(-\tilde{d}) = -\tilde{d}$. Thus the fractional reduction of average shear in the region $(-\tilde{d}, 0)$ is given by $\tilde{u}(-\tilde{d})/-\tilde{d}$, where $\tilde{u}(-\tilde{d})$ and $\tilde{d}$ are taken from (A2.1) and (A2.2). For typical atmospheric conditions (see Appendix 4), $r = 1$, and

$$\tilde{d} = 0.693,$$

$$\tilde{u}(-\tilde{d}) = -0.307,$$

$$\frac{\tilde{u}(-\tilde{d})}{-\tilde{d}} = 0.442.$$

For conditions corresponding to the zonal jet maximun $r \approx 0.6$, and

$$\tilde{d} = 0.981,$$

$$\tilde{u}(-\tilde{d}) = -0.412,$$

$$\frac{\tilde{u}(-\tilde{d})}{-\tilde{d}} = 0.419.$$

We finally turn to the question of available potential energy. In this paper we simplistically adopt the following measure for available potential energy:

$$\Phi = -\int_{p_{\text{surface}}}^{0} \frac{d\tilde{u}}{d\tilde{z}}\, dp. \tag{A2.3}$$

For the Charney problem $d\tilde{u}/d\tilde{z} = 1$ and $\Phi = p_{\text{surface}}$.

Clearly, (A2.3) is a bit peculiar in that it is independent of the dimensional shear m. However, it suffices for estimating the reduction in available potential energy resulting from altering an unstable profile to a neutral profile in the manner discussed in this appendix. Let $\tilde{z} = -\tilde{d}$ correspond to the ground where $p = p_0 = p_{\text{surface}}$. Let $\tilde{z} = 0$ be associated with a pressure $p = p_1 = p_{\text{surface}}e^{-\tilde{d}}$. (Recall $\tilde{u} = 0$ at $\tilde{z} = 0$.) Finally, assume that prior to neutralization we have a Charney profile [for which $\tilde{u}(-\tilde{d}) = -\tilde{d}$] which extends to some height $\tilde{z}_B$ (as in Fig. 5), rather than infinity, and let $p = p_2 = p_{\text{surface}}e^{-(\tilde{d}+\tilde{z}_B)}$ at $\tilde{z} = \tilde{z}_B$. The fractional reduction in Φ due to neutralization is then given by

$$\frac{\Delta\Phi}{\Phi_{0r}} = \frac{[1 - \tilde{u}_n(-\tilde{d})/-\tilde{d}](p_0 - p_1)}{(p_0 - p_2)}, \tag{A2.4}$$

where $\tilde{u}_n$ is the neutralized profile and Φ_{0r} is the unneutralized value of Φ. Recall that $\tilde{u}_n = \tilde{u}_{0r}$ for $z > 0$. For the usual Charney problem, $p_2 = 0$. For the case with $r = 1$

$$\frac{\Delta\Phi}{\Phi_{0r}} = \left(1 - \frac{0.307}{0.693}\right)(1 - e^{-0.693})$$

$$= 0.278 \text{ or a } 27.8\% \text{ reduction.}$$

For the case with $r = 0.6$

$$\frac{\Delta\Phi}{\Phi_{0r}} = \left(1 - \frac{0.412}{0.981}\right)(1 - e^{-0.981})$$

$= 0.3625$ or a 36.25% reduction.

APPENDIX 3

Numerical Algorithm

We wish to solve

$$\psi_{\tilde{z}\tilde{z}} + Q(\tilde{z})\psi = 0 \quad \text{(A3.1)}$$

subject to an arbitrary upper boundary condition

$$a\psi_{\tilde{z}} + b\psi = 0 \quad \text{at} \quad \tilde{z} = \tilde{z}_T. \quad \text{(A3.2)}$$

At $\tilde{z} = 0$ we require that the solution of (A3.1) and its derivative be continuous with (31) and its derivative, i.e.,

$$\psi(0) = 1 + R, \quad \text{(A3.3)}$$

$$\psi_{\tilde{z}}(0) = i\lambda(R - 1). \quad \text{(A3.4)}$$

This problem is readily solved by finite-difference methods. We first change variables. Let

$$x = \tilde{z}_T - \tilde{z}, \quad \text{(A3.5)}$$

where $\tilde{z} = \tilde{z}_T$ now corresponds to $x = 0$, and $\tilde{z} = 0$ corresponds to $x = z_T$. Eq. (A3.1) becomes

$$\psi_{xx} + Q(x)\psi = 0 \quad \text{(A3.6)}$$

and (A3.2) becomes

$$-a\psi_x + b\psi = 0 \quad \text{at} \quad x = 0. \quad \text{(A3.7)}$$

Eqs. (A3.3) and (A3.4) now become

$$\left.\begin{aligned} \psi(x) &= 1 + R \\ \psi_x(x) &= -i\lambda(R - 1) \end{aligned}\right\} \quad \text{at} \quad x = \tilde{z}_T. \quad \text{(A3.8)}$$

We now divide the interval $0 \leq x \leq \tilde{z}_T$ into $N + 1$ levels $x_0 = 0, x_1, x_2, \ldots, x_N = \tilde{z}_T$, with constant separation δ. In order to apply (A3.7) we also allow a fictional level x_{-1}. The finite-difference approximation to (A3.6) is

$$\psi_{n+1} + (Q_n\delta^2 - 2)\psi_n + \psi_{n-1} = 0, \quad \text{(A3.9)}$$

where the subscript n refers to evaluation at $x = x_n$. (A3.9) is solved by Gaussian elimination. This is most easily implemented by letting

$$\psi_{n-1} = \alpha_n\psi_n. \quad \text{(A3.10)}$$

Similarly

$$\psi_n = \alpha_{n+1}\psi_{n+1}. \quad \text{(A3.10a)}$$

A recursive formula for the α_n's is obtained by substitution (A3.10) into (A3.9) and comparing the result with (A3.10a). This yields

$$\alpha_{n+1} = \frac{-1}{(Q_n\delta^2 - 2 + \alpha_n)}. \quad \text{(A3.11)}$$

Thus, when we have α_0, we have all subsequent α_n's. α_0 is obtained from (A3.7) and (A3.9). The finite difference approximation to (A3.7) is

$$-a(\psi_1 - \psi_{-1}) + 2b\delta\psi_0 = 0. \quad \text{(A3.12)}$$

From (A3.9) we have

$$\psi_1 + (Q_0\delta^2 - 2)\psi_0 + \psi_{-1} = 0. \quad \text{(A3.13)}$$

We use (A3.13) to eliminate ψ_1 from (A3.12). A comparison of the result with (A3.10) yields

$$\alpha_0 = \frac{\{a(Q_0\delta^2 - 2) + 2b\delta\}}{2a}. \quad \text{(A3.14)}$$

We now turn to (A3.8) which become

$$\psi_N = 1 + R, \quad \text{(A3.15)}$$

$$\psi_N - \psi_{N-1} = -i\lambda\delta(R - 1). \quad \text{(A3.16)}$$

Now,

$$\psi_{N-1} = \alpha_N\psi_N,$$

and, using (A3.15), (A3.16) becomes

$$(1 - \alpha_N)(1 + R) = -i\lambda\delta(R - 1),$$

which may be solved for R:

$$R = \frac{-(1 - \alpha_N - i\lambda\delta)}{(1 - \alpha_N + i\lambda\delta)}. \quad \text{(A3.17)}$$

Note that $|R| = 1$ if α_N is real. The complexity of α_N arises from arbitrarily small $\tilde{c}_i$ in the neighborhood of the singularity in Q.

APPENDIX 4

Numerical Values

Our analysis has been in terms of nondimensional parameters. Characteristic values for these parameters are derived in this appendix.

Evaluating f and β at 45° latitude, we get

$$f \approx 10^{-4}\ \mathrm{s^{-1}},$$

$$\beta \approx 1.6 \times 10^{-11}\ \mathrm{s^{-1}\ m^{-1}}.$$

Typically, $N^2 \approx 1.6 \times 10^{-4}\ \mathrm{s^{-2}}$. Thus

$$\epsilon = \frac{f^2}{N^2} \approx 0.625 \times 10^{-4}. \quad \text{(A4.1)}$$

Now

$$\alpha^2 \equiv \frac{k^2H^2}{\epsilon}, \quad \text{(A4.2)}$$

where $H \sim 8 \times 10^3$ m and

$$k = \frac{n}{a\cos\phi}, \quad \text{(A4.3)}$$

where n is the zonal wavenumber at latitude a and a is the earth's radius ($\sim 6.4 \times 10^6$ m). From (A4.1), (A4.2) and (A4.3) (taking $\phi = 45°$) we get

$$\alpha \approx 0.224n \tag{A4.4a}$$

or

$$\alpha^2 \approx 0.05n^2.$$

We next turn to

$$r \equiv \frac{\beta H}{\epsilon m}. \tag{A4.5}$$

For the above choices of β, H and ϵ we have

$$r \approx \frac{2.048 \times 10^{-3}\ \mathrm{s}^{-1}}{m} \approx \frac{2.048\ \mathrm{m\ s^{-1}\ km^{-1}}}{m}.$$

Thus $r \approx 1$ corresponds to $m \approx 2.05\ \mathrm{m\ s^{-1}\ km^{-1}}$, while $r \approx 0.6$ corresponds to $m \approx 3.4\ \mathrm{m\ s^{-1}\ km^{-1}}$.

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