An Inequality between Fluxes of Energy and Enstrophy in 2D and QG Turbulence

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In systems governing two-dimensional turbulence, surface quasi-geostrophic turbulence, (more generally α-turbulence), two-layer quasi-geostrophic turbulence, etc., there often exist two conservative quadratic quantities, one "energy"-like and one "enstrophy"-like. In a finite inertial range there are in general two spectral fluxes, one associated with each conserved quantity. The energy spectrum in general has a contribution from each of the fluxes, and our previous work showed that these two contributions to the energy spectrum can be linearly superimposed despite the highly nonlinear nature of the problem. Often, one of the fluxes is dominant and the energy spectrum then has the visual shape of the case with a single flux; the contribution from the subdominant flux is effectively hidden. The relative magnitudes of the spectral fluxes depend on the dissipative sinks in the system, and varies according to the physical/mathematical system under consideration. We derive an important inequality involving the "energy" and enstrophy" fluxes for each representative system. This result then allows us to determine the effective energy spectral shape in the general case of double cascades.

1. Introduction

In the theory of three-dimensional (3D) turbulence of incompressible fluids (Batchelor 1947; Kolmogorov 1941*a*,*b*), there is one conserved quantity, the kinetic energy of the flow. In the inertial range, where forcing and dissipation can be ignored, this conserved quantity "cascades" from low to high wavenumbers through triad interactions. In two-dimensional (2D) turbulence, there are two conserved quantities, kinetic energy and enstrophy, the latter being half of the square of vorticity. Conserved quantities are still transferred in wavenumber space through triad interactions, but the situation is now more complicated. In the classical theory of Kraichnan (1967), Leith (1968), and Batchelor (1969), a simpler picture was adopted under, however, rather unattainable assumptions. In the KLB theory, there are two inertial ranges, one located upscale of the spectral region of injection and another on the downscale side of injection. In the upscale side, there is only an upscale flux of energy, and no flux of enstrophy. On the downscale side, there is only a downscale flux of enstrophy, and no flux of energy. There are also theoretical extensions of the KLB model (Falkovich & Lebedev 1994*a*,*b*; Yakhot 1999) that explain the absense of intermittency corrections.

The theory is very elegant, but it requires both inertial spectral ranges to be infinite. This has proved difficult to implement numerically, with the consequence that reports of various numerical slopes and alternative theories abound over the past 30 years (Moffatt 1986; Polyakov 1993; Saffman 1971). Recently, in carefully set up experiments, the prediction of the KLB theory appears to have been verified under some conditions in one or the other inertial range (Boffetta, Celani & Vergassola 2000; Ishihira & Kaneda

2001; Lindborg & Alvelius 2000). There are also many more cases however where different slopes are found numerically (Danilov 2005; Danilov & Gurarie 2001*a*,*b*; Tran & Bowman 2004). A review can be found in Gkioulekas & Tung (2005*b*) and Tabeling (2002).

It has been shown in Gkioulekas & Tung (2005*a*,*b*) that in 2D as well as in quasi-geostrophic (QG) turbulence, energy flux (ϵ) and enstrophy flux (η) in general coexist in a *finite* spectral inertial range. Finite spectral ranges apply to the more realistic cases of finite domains and small but finite viscosities. The energy spectrum associated with each flux can be linearly superimposed to form a compound power law, with components $\epsilon^{2/3}k^{-5/3}$ and $\eta^{2/3}k^{-3}$, where k is the wavenumber. Thus in an inertial range on the short-wave side of injection, where both energy and enstrophy cascade downscale ($\epsilon > 0, \eta > 0$), the energy spectrum takes the form

$$E(k) \sim C_1\epsilon^{2/3}k^{-5/3} + C_2\eta^{2/3}k^{-3}, \tag{1.1}$$

where C_1 and C_2 are order-one universal, dimensionless constants. This energy spectrum in principle has the compound shape of -3 slope transitioning to $-5/3$ slope as k increases; asymptotically the first term dominates for small k and the second term for large k. The transition wavenumber occurs at $k_t \sim (\eta/\epsilon)^{1/2}$, which is where the first and second terms on the right-hand side of (1.1) are of the same order of magnitude. Whether or not such a double-slope shape can be realized depends on the relative magnitudes of ϵ and η. Since, at least in the case of 2D turbulence, both energy and enstrophy are dissipated by the same molecular viscosity (or hyperviscosity), one may anticipate a relationship between ϵ and η based on their respective dissipation rates at the ultraviolet end of the spectrum.

A very elegant inequality between energy and enstrophy fluxes was suggested to us by Danilov (2004, personal communication). A general derivation of what we called the "Danilov inequality" was given in Gkioulekas & Tung (2005*b*) for the case of 2D turbulence. We speculated that it probably does not apply to the two-layer QG model considered by Tung & Orlando (2003*a*), whose numerical result on the energy spectrum shows the anticipated compound shape consistent with atmospheric observation(Gage 1979; Gage & Nastrom 1986; Nastrom & Gage 1984; Nastrom, Gage & Jasperson 1984). Smith (2004), on the other hand, thought the compound slope could be due to the finite resolution of the numerical model. The present article is the first of two papers extending the Danilov inequality to QG turbulence. Here we deal with the general case where the conserved quantity, called potential vorticity in QG theory, is a linear differential operator of the streamfunction. Multi-layer baroclinic models require the modification involving turning the linear operator into a matrix operator, and will be considered in a separate paper. We will nevertheless point out towards the end of this paper how Danilov's inequality is violated in QG models which are more baroclinic than barotropic.

2. Common mathematics for QG turbulences which are 2D-like

2.1. *Conservation laws*

In QG turbulence, potential vorticity, $\zeta = -\mathcal{L}\psi$, is conserved (in the absence of forcing and dissipation). $\mathcal{L}$ is a linear isotropic operator involving the derivatives with respect to the horizontal coordinates. For 2D turbulence, $\mathcal{L} = -\Delta$, where Δ is the Laplacian operator. For barotropic QG turbulence, also know as Charney-Hasegawa-Nima (CHM) turbulence ((Charney 1948; Hasegawa, Maclennan & Kodama 1979; Hasegawa & Mima 1978), $\mathcal{L} = -\Delta - \lambda^2$, where λ^2 is a given positive constant. In α-turbulence, $\mathcal{L} = \Lambda^\alpha$ with $\Lambda \equiv (-\Delta)^{1/2}$. SQG is the special case of $\alpha = 1$.

The governing potential-vorticity conservation equation takes the following form:

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = \mathcal{D} + \mathcal{F}, \tag{2.1}$$

where $\mathcal{F}$ is the forcing and $\mathcal{D}$ is the dissipation. The stream-function ψ is related to the 2D nondivergent velocity as $(u, v) = (-\partial\psi/\partial y, \partial\psi/\partial x)$. For a general combination of hyper- and hypo-diffusion: $\mathcal{D} = -\nu_0(-\Delta)^p\zeta - \nu_1(-\Delta)^{-h}\zeta$, with p, h, positive integers. $p = 1, h = 0$ yields molecular viscosity.

There are two inviscid quadratic invariants for (2.1), which are, using the notation $\|f\| \equiv \iint (f(x, y))dxdy$:

$$A = (1/2)\|(-\psi\zeta)\|, \quad \text{and} \quad B = (1/2)\|\zeta^2\|,$$

under a certain condition, which is essentially that the Fourier transform of $\mathcal{L}$, denoted as $L(k)$, is either real-valued or imaginary-valued. For example in 2D turbulence it is seen that, $E \equiv (1/2)\|(u^2 + v^2)\| = (1/2)\||\nabla\psi|^2\| = (1/2)\|(-\psi\zeta)\| \equiv A$ is the kinetic energy of the 2D fluid, and $G \equiv (1/2)\|\zeta^2\| \equiv B$ is the enstrophy. The energy spectrum $E(k)$ and enstrophy spectrum $G(k)$ are defined so that $E = \int_0^\infty E(k)dk$ and $G = \int_0^\infty G(k)dk$, where $k = |\mathbf{k}|$ is the isotropic wavenumber magnitude. In the more general case, $A(k)$ is similarly defined as the spectrum of A, and $B(k)$ the spectrum of B. In the general case $A(k)$ may not necessarily be the energy spectrum, as will be demonstrated.

We will assume that $L(k) > 0$, so that both $A(k)$ and $B(k)$ are positive. Furthermore, we will assume that $L(k)$ is a monotonically increasing function of k.

2.2. *The spectral equations*

The relationship, $\zeta = -\mathcal{L}\psi$, translates into the spectral relationships in the Fourier space

$$\hat{\zeta}(\mathbf{k}) = L(|\mathbf{k}|)\hat{\psi}(\mathbf{k}), \quad B(k) = L(k)A(k),$$

where $\hat{\psi}(\mathbf{k}, t) = \iint \psi(\mathbf{x}, t)e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$. For example, in 2D turbulence, $L(k) = k^2$, and so $G(k) = k^2E(k)$. In CHM turbulence, $L(k) = k^2 + \lambda^2$. So $G(k) = (k^2 + \lambda^2)E(k)$ is the relationship between potential enstrophy and total energy in CHM turbulence, where the total QG energy, E, is A and the QG potential enstrophy, G, is B.

Furthermore, $A(k)$ and $B(k)$ satisfy the following spectral equations:

$$\frac{\partial A(k)}{\partial t} + \frac{\partial \Pi_A(k)}{\partial k} = -D_A(k) + F_A(k) \tag{2.2}$$

$$\frac{\partial B(k)}{\partial t} + \frac{\partial \Pi_B(k)}{\partial k} = -D_B(k) + F_B(k). \tag{2.3}$$

$D_A(k)$ and $D_B(k)$ are the spectral dissipations rate of $A(k)$ and $B(k)$, respectively, with

$$D_B(k) = L(k)D_A(k), \quad \text{and} \quad D_A(k) = [\nu_0 k^{2p} + \nu_1 k^{-2h}]A(k) > 0,$$

for a combination of hyper- and hypo-viscosities. $F_A(k)$ and $F_B(k)$ are the spectra of forcing, and, $\Pi_A(k)$ and $\Pi_B(k)$ are the spectral fluxes of A and B. Ensemble average is taken in (2.2) and (2.3), but will not be denoted with different symbols here.

2.3. *The Danilov inequality*

Assuming that the injection (forcing) of A and B occurs in $[k_1, k_2]$, then at statistical equilibrium, we have, from (2.2) and (2.3):

$$\Pi_A(k) = \int_k^\infty D_A(q)dq, \quad \Pi_B(k) = \int_k^\infty D_B(q)dq, \quad k > k_2. \tag{2.4}$$

Since

$$L(k)\Pi_A(k) - \Pi_B(k) = \int_k^\infty [L(k) - L(q)] D_A(q) dq < 0, \tag{2.5}$$

we have

$$L(k)\Pi_A(k) < \Pi_B(k) \quad \text{for} \quad k > k_2. \tag{2.6}$$

Similarly, we can show that

$$L(k)\Pi_A(k) > \Pi_B(k) \quad \text{for} \quad k < k_1. \tag{2.7}$$

These inequalities were brought to our attention by Danilov (2004, personal communication) for the case of 2D turbulence. Previously, Eyink (1996) derived a similar, but looser, bound, for the downscale energy flux Π_E: $\Pi_E(k) < \eta_0/k^2$, involving the total rate of enstrophy injection η_0.

2.4. *The energy spectrum*

The significance of the inequalities (2.6) and (2.7) is that, combined with the usual assumption of the existence of universal local cascades, we can obtain a prediction about the energy spectrum $E(k)$ in the inertial range. For example, for the classic case of 2D turbulence in a finite domain with finite viscosity for the infrared and ultraviolet dissipations, following the argument given in Gkioulekas & Tung (2005*a*,*b*), we learn that on the downscale side of injection the dominant cascade is the enstrophy cascade with $E(k) \sim k^{-3}$, and on the upscale side of injection the dominant cascade is the inverse energy cascade with $E(k) \sim k^{-5/3}$. According to our theory, in both cases, the energy spectrum is a superposition of the contributions to $E(k)$ from both cascades, which are in fact present both downscale and upscale of injection. However, the inequalities (2.6) and (2.7) imply that the transition wavenumber k_t where the subleading cascade would be exposed is in the corresponding dissipation range, thereby hiding the subleading cascade in the energy spectrum.

This argument can be extended to the case of α-turbulence, which includes 2D and SQG turbulence. The spectrum of $A(k)$ and $B(k)$ are, in the downscale inertial range:

$$\begin{aligned} A(k) &= C_1(\Pi_A)^{2/3} k^{-\frac{7}{3}+\frac{1}{3}\alpha} + C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha} \\ B(k) &= |k|^\alpha A(k). \end{aligned}$$

If we assume that the transition wavenumber k_t is in the inertial range, then $k_t \sim (\Pi_B/\Pi_A)^{1/\alpha}$, and by applying the inequality (2.6), which yields $\Pi_A k^\alpha < \Pi_B$ for any k in the inertial range, we find that $k < (\Pi_B/\Pi_A)^{1/\alpha} \sim k_t$ for *all* k in the inertial range, in contradictton with our assumption. Therefore, in the downscale range, there is no observable transition and:

$$A(k) \cong C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}-\frac{1}{3}\alpha}, \quad B(k) \cong C_2(\Pi_B)^{2/3} k^{-\frac{7}{3}+\frac{2}{3}\alpha}. \tag{2.8}$$

In the upscale range, the inequality in (2.7) is the reverse of (2.6). The spectrum becomes then

$$A(k) \cong C_1(\Pi_A)^{2/3} k^{-\frac{7}{3}+\frac{1}{3}\alpha}, \quad B(k) = k^\alpha A(k). \tag{2.9}$$

This argument proves more rigorously a conjecture by Smith (2004) that in 2D turbulence (with $\alpha = 2$), on the downscale side of injection, we have no transition to shallower scaling $E(k) \sim k^{-5/3}$. His other conjecture, that the same result also holds for two-layer QG model, appears to be false. The energy spectrum for that model, where the subleading direct energy cascade can be observed, entails a violation of the inequality (2.6). For the SQG model, we show in the next section that the visible energy spectrum actually has the shallower -5/3 slope even though the Danilov inequality is also satisfied, because $B(k)$ is the 2D energy spectrum and $A(k)$ the 3D energy spectrum; there is no enstrophy cascade.

It should be noted that in the argument above it is *assumed* that inertial ranges exist either upscale or downscale of injection. Unlike the case of 3D turbulence, where the downscale energy cascade is very robust, it is well known that in 2D turbulence there are circumstances where the leading inverse energy cascade (Danilov 2005; Danilov & Gurarie 2001*a*,*b*; Gkioulekas & Tung 2005*b*) or the leading downscale enstrophy cascade (Tran & Bowman 2003, 2004; Tran & Shepherd 2002) may fail to appear as expected. At least some of these issues also apply to the case of α-turbulence (Tran 2004).

3. Physical Interpretation of $A(k)$ and $B(k)$

There has been considerable confusion over the reported spectral shape for the energy spectrum in SQG. This is due partly to the problem of identifying one of the two conserved quantities as the energy. As derived by Charney (1971), 3D QG flow conserves the 3D potential vorticity ξ, which is advected horizontally by the streamfunction ψ. Here, both ψ and ξ are 3D fields. For constant Coriolis parameter f, the conservational law for ξ takes the form:

$$\frac{\partial \xi}{\partial t} + J(\psi, \xi) = 0, \tag{3.1}$$

with ξ given by

$$\xi = \Delta\psi + \frac{f^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial \psi}{\partial z}\right) \equiv \mathcal{P}\psi,$$

where $\rho_0(z)$ is the ambient air density, and $N^2(z)$ the Brunt-Väisälä frequency. In SQG this potential vorticity ξ is assumed, a priori, to be identically zero for $z > 0$. The streamfunction ψ is solved from $\xi = \mathcal{P}\psi = 0$. With ρ_0 and N^2 taken to be constants, the horizontal Fourier transform of $\psi(x, y, z, t)$ is obtained as

$$\hat{\psi}(\mathbf{k}, z, t) = \hat{\psi}_0(\mathbf{k}, t)e^{-|\mathbf{k}|(N/f)z}, \tag{3.2}$$

using the boundedness boundary condition as $z \to \infty$.

Most of the dynamics in this model are occurring at the surface, where the boundary condition of vanishing vertical velocity, w, applied to the temperature (T) equation leads to:

$$\frac{\partial \Theta}{\partial t} + J(\psi, \Theta) = \mathcal{D} + \mathcal{F}, \tag{3.3}$$

where $\Theta \equiv (g/N)(T/T_0) = (f/N)(\partial\psi/\partial z)$ now plays the role of the conserved quantity ζ in (2.1), $\mathcal{D} = \nu\Delta\Theta$ is the thermal diffusion, and $\mathcal{F} = Q$ is the thermal heating (see Tung & Orlando (2003*b*)). This equation is to be solved on a 2D surface $z = 0$. It has the same form as the vorticity equation for 2D turbulence (e.g. (2.1)), except that the spectral relationship between the advected quantity Θ and the advecting field ψ is given instead by $\hat{\Theta}(\mathbf{k}) = -|\mathbf{k}|\hat{\psi}(k)$. That is, $L(k) = k$.

It can be shown (Charney 1971; Tung & Orlando 2003*b*) that the 3D QG energy density

$$\mathcal{E} \equiv \frac{1}{2}\rho_0\left[|\nabla\psi|^2 + \frac{f^2}{N^2}\left(\frac{\partial \psi}{\partial z}\right)^2\right], \tag{3.4}$$

is an invariant (i.e. independent of time),*when integrated over the 3D domain.* $\mathcal{E}$ is the sum of kinetic energy density: $\mathcal{E}_K = (1/2)\rho_0(u^2 + v^2) = (1/2)\rho_0(\nabla\psi)^2$ and available potential energy density: $\mathcal{E}_P = (1/2)(f^2/N^2)\rho_0(\partial\psi/\partial z)^2$. For SQG, by Parseval's equality, the energies integrated over the horizontal surface is

$$E_P \equiv \|\mathcal{E}_P\| = \rho_0 \iint \frac{1}{2}k^2|\hat{\psi}|^2 dk_x dk_y = \rho_0\|\frac{1}{2}\Theta^2\|,$$
$$E_K \equiv \|\mathcal{E}_K\| = \rho_0 \iint \frac{1}{2}k^2|\hat{\psi}|^2 dk_x dk_y = \rho_0\|\frac{1}{2}\Theta^2\|.$$

It is thus seen that the kinetic energy density and the available potential energy density, when integrated horizontally, are equipartitioned in flows such as SQG where potential vorticity is zero, and that $2B \equiv \|\Theta^2\| = (E_P + E_K)/\rho_0 = E/\rho_0$ is the total energy at the lower surface. The 3D energy is, instead.

$$\begin{aligned}
\int_0^\infty \|\mathcal{E}\|dz = \int_0^\infty \rho_0 dz\|\Theta^2\| &= \int_0^\infty \rho_0 dz \iint dk_x dk_y \left|\hat{\Theta}|_{z=0}\right|^2 e^{-2|\mathbf{k}|(N/f)z} \\
&= \frac{1}{2}\rho_0 \iint dk_x dk_y \frac{f}{N|\mathbf{k}|}\hat{\Theta}\hat{\Theta}^*\bigg|_{z=0} \\
&= \frac{1}{2}\rho_0 \iint \frac{f}{N}\left(-\hat{\psi}\hat{\Theta}^*|_{z=0}\right) dk_x dk_y = \frac{1}{2}\rho_0\frac{f}{N}\,\|(-\psi\Theta|_{z=0})\|.
\end{aligned}$$

This expression for 3D energy turns out to be $\rho_0(f/N)A$, with A defined earlier as $A \equiv (1/2)\|(-\psi\Theta)\|$. Previous authors have made use of the similarity between the form of vorticity equation (2.1) in 2D turbulence and the temperature equation (3.3) in SQG turbulence to identify, by analogy, A as the "energy" and B as the "enstrophy" (Held, Pierrehumbert, Garner & Swanson 1995; Pierrehumbert, Held & Swanson 1994). As pointed out in Tung & Orlando (2003*b*) and also here, $2B$ is the total energy integrated over the lower surface, and includes kinetic plus available potential energy. The physical interpretation for A was not given, but can now be seen to be the total energy integrated over the 3D domain. There is no potential enstrophy ($\xi^2/2$) per se in SQG turbulence, because potential vorticity ξ has been taken to be zero identically. There is also no flux of potential enstrophy in SQG.

In an inertial range where both fluxes of A and B are simultaneously present, the energy spectrum is the same as that for $2B(k)$ in α-turbulence with $\alpha = 1$:

$$E(k) = 2B(k) = C_1(\Pi_A)^{2/3}k^{-1} + C_2(\Pi_B)^{2/3}k^{-5/3}.$$

From the Danilov inequality we learn that the visible energy spectrum in the inertial range downscale from the injection scale is given by

$$E(k) \cong C_2\epsilon^{2/3}k^{-5/3}, \tag{3.5}$$

where $\Pi_B(k) = \epsilon$, a positive constant in the downscale range. This $k^{-5/3}$ energy spectrum is now predicted by our theory. The flux ϵ is not the "enstrophy" flux, nor is it the flux of 3D energy A, but is the 2D flux of 2D energy $2B = E$. This $k^{-5/3}$ shape is also seen in numerical simulations (Hoyer & Sadourny 1982).

For the spectral region upscale from that of injection the energy spectrum should have the form:

$$E(k) \cong C_1(\Pi_A)^{2/3}k^{-1},$$

where Π_A is the inverse flux of the 3D total energy in the horizontal spectral direction. Constantin (2002) had earlier provided a k^{-2} bound, which was later refined to k^{-1} by Tran (2003).

4. Two-layer QG models

The results in the previous sections demonstrate that QG turbulence can exhibit a variety of behaviors. Barotropic models, as typified by CHM model, possess an energy spectrum with -3 spectral slope, as in 2D turbulence, while SQG turbulence, which is baroclinic (with its exponential decay with height), has a spectrum with a $-5/3$ slope, in the downscale inertial range. Two-layer QG models have both a barotropic and a baroclinic component (see Salmon 1998), and we therefore expect a mixture of -3 and $-5/3$ slopes depending on the degree of baroclinicity. Its governing equation does not satisfy the form of the conservation law with $\zeta = -\mathcal{L}\psi$, unless we make $\mathcal{L}$ into a matrix and ψ into a column vector. The results obtained so far for scalar relations do not hold in this case. We will discuss the more general case of multi-layers in a future paper. Here we wish to demonstrate how the Danilov inequality fails.

A relatively realistic version of two-layer model applicable to studying atmospheric turbulence in the troposphere was adopted in Tung and Orlando (2004a). In this model forcing is due to thermal heating, which injects energy directly into the baroclinic part of the total energy. The two-layer fluid sits atop of an Ekman boundary layer near the ground, which introduces Ekman pumping in the lower layer (Holton 1979) but *not* in the upper layer. If one artificially removes these two baroclinic physical mechanisms, it can be shown that Danilov's inequality applies. The inequality is violated when they are retained. The physical implication of the latter case is important, as will be discussed later.

Two-layer QG models conserve potential vorticity in each layer in the absence of forcing and damping. In their presence, the governing equations are, from Tung & Orlando (2003*a*):

$$\text{Top layer:} \qquad \frac{\partial \zeta_1}{\partial t} + J(\psi_1, \zeta_1) = \mathcal{D}_1 + \mathcal{F}_1 \tag{4.1}$$

$$\text{Bottom layer:} \qquad \frac{\partial \zeta_2}{\partial t} + J(\psi_2, \zeta_2) = \mathcal{D}_2 + \mathcal{F}_2, \tag{4.2}$$

where

$$\zeta_1 = \Delta\psi_1 - \frac{k_R^2}{2}(\psi_1 - \psi_2), \;\; \zeta_2 = \Delta\psi_2 + \frac{k_R^2}{2}(\psi_1 - \psi_2),$$

are the potential vorticity in each layer. k_R is the Rossby radius of deformation wavenumber and is taken as a given constant. The dissipation terms, $\mathcal{D}_i$, includes momentum dissipation of relative vorticity, $\Delta\psi_i$, in each layer, and Ekman pumping from the lower boundary layer:

$$\mathcal{D}_1 = \nu(-\Delta)^{p+1}\psi_1, \quad \mathcal{D}_2 = \nu(-\Delta)^{p+1}\psi_2 - \nu_E \Delta\psi_2.$$

The two inviscid quadratic invariants are the total energy $A = E$ and potential enstrophy $B = G$, defined as

$$E \equiv \frac{1}{2}\| - (\psi_1\zeta_1 + \psi_2\zeta_2)\| = \frac{1}{2}\|\{|\nabla\psi_1|^2 + |\nabla\psi_2|^2 + \frac{k_R^2}{2}(\psi_1 - \psi_2)^2\}\|,$$

where the first two terms represent the kinetic energy in each layer, and the last term is the so-called available potential energy; and

$$G \equiv \frac{1}{2}\|(\zeta_1^2 + \zeta_2^2)\| = \frac{1}{2}\|\{(\Delta\psi_1)^2 + (\Delta\psi_2)^2 + \frac{k_R^2}{4}[k_R^2(\psi_1 - \psi_2)^2 + 2|\nabla\psi_1 - \nabla\psi_2|^2]\}\|.$$

In the above defined conserved quantities, the contributions from the two layers are summed, which allows us to discuss the spectral fluxes of E and G through the horizontal wavenumber domain $\mathbf{k}$. The governing spectral equations now have the same form as (2.2) and (2.3), but the appropriate spectral dissipation rates are different. They are given by $D_E(k) = 2\pi k d_E(k)$ and $D_G(k) = 2\pi k d_G(k)$, where $d_E(k)$ and $d_G(k)$ are the one-dimensional dissipation rates which read

$$d_E(k) = \nu k^{2p} k^2 |\hat{\psi}_1|^2 + (\nu k^{2p} + \nu_E) k^2 |\hat{\psi}_2|^2 > 0$$

$$\begin{aligned} d_G(k) &= (k^2 + k_R^2) d_E(k) - k^2 k_R^2 [\nu k^{2p} + \frac{1}{2}\nu_E] Re\{\hat{\psi}_1 \hat{\psi}_2^*\} \\ &= k^2 d_E(k) + (2\nu k^{2p} + \nu_E)\frac{k_R^2}{2} k^2 [|\hat{\psi}_1|^2 + |\hat{\psi}_2|^2 - Re(\hat{\psi}_1\hat{\psi}_2^*)] + \nu_E \frac{k_R^2}{2} k^2 [|\hat{\psi}_2|^2 - |\hat{\psi}_1|^2]. \end{aligned}$$

It is seen that the sign of $D_G(k)$ is indeterminant, depending on the difference in kinetic energy in each layer ($\frac{1}{2}k^2|\hat{\psi}_1|^2$ vs $\frac{1}{2}k^2|\hat{\psi}_2|^2$). So

$$\begin{aligned} k^2\Pi_E(k) - \Pi_G(k) &= \int_k^\infty 2\pi q [k^2 d_E(q) - d_G(q)] dq \\ &= \int_k^\infty 2\pi q \left\{ (k^2 - q^2) d_E(q) + \nu_E \frac{k_R^2}{2} q^2 [|\hat{\psi}_1(q)|^2 - |\hat{\psi}_2(q)|^2] \right. \\ &\qquad \left. -(2\nu q^{2p} + \nu_E)\frac{k_R^2}{2} q^2 [|\hat{\psi}_1(q)|^2 + |\hat{\psi}_2(q)|^2 - Re(\hat{\psi}_1(q)\hat{\psi}_2^*(q))] \right\} dq. \end{aligned} \tag{4.3}$$

The first and third term in the integrand are negative. It follows that Danilov's inequality holds, i.e.

$$k^2\Pi_E(k) - \Pi_G(k) < 0, \tag{4.4}$$

if either the top layer has no more kinetic energy than the lower layer or if $k_R = 0$. There is a wider sufficient condition in terms of ν_E, which can be derived by requiring that the last two terms in (4.3) should be negative. This leads to the following *sufficient* condition to *satisfy* (4.4):

$$\nu_E < 2\nu k_{\max}^{2p},$$

where $k_{\max}$ is either truncation wavenumber in the numerical model, or, in the theoretical case of infinite resolutions, is the hyperviscosity dissipation wavenumber, beyond which the spectral enstrophy dissipation rate becomes negligible.

Therefore a *necessary* condition to *violate* (4.4) is $\nu_E > 2\nu k_{\max}^{2p}$. It is interesting to note that the algorithm adopted by Tung & Orlando (2003*a*) for determining the magnitude of the hyperviscosity coefficient is $\nu_E \gg \nu k_{\max}^{2p}$, for all but the last twenty wavenumbers k in the dissipation range. They obtained the compound slope with a transition wavenumber k_t occuring in the inertial range downscale from injection.

5. Conclusion

The classical KLB theory of 2D turbulence relies for its mathematical simplicity and elegance on two unrealistic assumptions: that the domain is infinite, and that the Reynolds number approaches infinity. What we have shown here is that although the theory becomes much more complicated when these two assumptions are relaxed, the energy spectrum in the more general set-up of finite domain with finite dissipation still reduces to the KLB prediction visually in many cases. There are however important exceptions.

The presence of finite domains creates many important departures from the KLB theory: There may be no cascades at all if there is no infrared sink of energy, which needs to be remedied by the introduction of hypodiffusion (Tran & Bowman 2003, 2004; Tran

& Shepherd 2002). The boundary effect can introduce a particular solution which may overwhelm the universal behavior of the inertial range solution (Danilov 2005; Danilov & Gurarie 2001*a*,*b*; Gkioulekas & Tung 2005*b*). With suitably chosen hypodiffusion, these two effects can be minimized. The solution then yield a non self-similar compound energy spectrum, one part involving the energy flux and another part involving the enstrophy flux. Both fluxes coexist in the same inertial range (Gkioulekas & Tung 2005*a*,*b*).

Since both the energy and enstrophy are dissipated by the same dissipation, there must be a relationship between the two. Knowing this relationship will then tell us which part of the energy spectrum is dominant. The resulting "Danilov inequality" and its extensions are an important, and so far, missing, component of the general theory of double cascades in 2D and QG turbulence. Such an inequality is supplied here for several types of barotropic turbulence, and the difference for baroclinic turbulence, such as surface QG and 2-layer turbulences, is also noted.

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