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STUDIES IN APPLIED MATHEMATICS
Published by Elsevier North Holland, Inc.

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In this paper the weakly nonlinear theory of long internal gravity waves propagating in stratified media is extended to the fully nonlinear case by treating Long's nonlinear partial differential equation for steady inviscid flows without restriction to small amplitudes and long wavelengths. The existence of finite amplitude solutions of "permanent form" is established analytically for a large class of stratification profiles, and properties are calculated numerically for the case of a hyperbolic tangent density profile in a large range of fluid depths. The numerical results agree well with the experimental data of Davis and Acrivos over the full range of wave amplitudes measured; such agreement is not obtainable with existing weakly nonlinear theories.

Introduction

Large amplitude internal waves that effectively maintain their shape over long propagation distances are often observed in nature [10, 15, 16, 18, 23], and it has been suggested that they occur in the Jovian atmosphere as the Great Red Spot [29]. The surprising ease with which large amplitude internal waves on a pycnocline can be repeatedly produced in the laboratory and the durability of the waveshape both over long distances and in collisions with walls and other such waves have been noted by many researchers. (For example, Maxworthy [28] stated, "Quite general and uncontrolled mixing events create solitary wave trains and lead us to suspect that they should be excited under many circumstances in natural systems.") With a few exceptions [6] (including calculations for interfacial waves), most of the existing theories [3–5, 22, 32] of internal solitary waves, as they are called, are limited to the case of small amplitudes and long wavelengths. (The first assumption allows the nonlinear equation to be obtained in an asymptotic expansion in terms of amplitude, with the lowest order equation being linear, while the long wavelength assumption reduces the partial differential

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STUDIES IN APPLIED MATHEMATICS 66:1–44 (1982)

Published by Elsevier North Holland, Inc.
0022-2526/82/010001+44$02.75

equations of hydrodynamics to an ordinary differential equation.) In applications the results from the weakly nonlinear theories are often extrapolated into amplitude regimes in which the assumptions under which they are derived are violated. While it is a remarkable fact that the predictions of the weakly nonlinear theories for some flow quantities agree quite favorably with experimental data even for moderately large wave amplitudes, notable discrepancies, both quantitative and qualitative, exist for other important flow quantities. For example, the weakly nonlinear theories predict that the phase speed of a solitary wave becomes larger for larger wave amplitudes in direct proportion. However, the experimental data of Davis and Acrivos [13] show a definite and substantial slower rate of increase with amplitude as the amplitude passes the weakly nonlinear regime. Also, the weakly nonlinear theories predict that the wavelength of the solitary wave contracts for larger wave amplitudes, while observations show the opposite effect past a certain wave amplitude, especially when recirculation regions are present inside the wave.

In this paper we shall show, both analytically and numerically, that waves of permanent form can indeed exist under the hydrodynamic balance between nonlinearity and dispersion. As dispersion becomes strong for large amplitudes, the long wavelength assumption has to be discarded and a partial differential equation studied. Long's equation applicable to steady internal waves in a stratified fluid is used. The amplitudes of the solutions are not restricted to be small and the wavelengths are not required to be long as in the weakly nonlinear theories. We shall, however, make the Boussinesq approximation since density changes in the lower atmosphere and in the ocean are usually small (in the sense that $\sigma = \Delta\rho_0/\rho_0$ seldom exceeds 10^{-2} across a thermocline). The validity of such an assumption is also supported by the experimental evidence of Davis and Acrivos, as we shall also discuss.

Previous work on the existence of finite amplitude waves of permanent form has been principally concerned with surface waves, for which the governing equation is the much simpler Laplace equation (for an irrotational flow), although the boundary condition at the free surface is rather complicated. Nebrassov [30, 31] and Levi-Civita [24] established the existence of periodic progressive surface waves of finite amplitude in water of infinite depth. Struik [33] extended the proof to water of finite depth. Lichtenstein [25] and Gerber [17] also give alternative proofs. By transforming into a streamline coordinate system, Dubreil-Jacotin [14] extended the convergence proof of Levi-Civita to the case where the fluid vorticity is not zero but is a function of streamline only. Benjamin [2] later used the same technique in his study of weakly nonlinear solitary surface waves in the presence of mean flow vorticity. Using the integral equation derived by Byatt-Smith [7] for finite amplitude surface waves, Byatt-Smith and Longuet-Higgins [8] computed both the shape and the phase speed of large amplitude solitary waves. The governing equation for the internal wave problem is more complicated than for surface waves, since the density variations in combination with the passage of the wave can generate vorticity in the flow field which is not conserved along streamlines. To isolate the internal wave modes we have imposed a "rigid lid" upper boundary condition for the present study.

The problem to be considered is defined in Sec. 1, the existence of solitary waves of modes 1 and 2 is established in Secs. 2 and 3, and periodic waves are treated in the Appendix. Section 4 studies the dependence of the solitary wave's amplitude on its phase speed. It is found that slight changes in the ambient density stratification can produce quite different solutions at large amplitudes, though the *existence* of such solitary waves is not sensitive to such changes. We also succeeded in establishing numerically the existence of finite amplitude waves by computing solutions to Long's equation by a combination of Newton's method and numerical continuation techniques due to Keller [19]. The details of the numerical procedure will appear elsewhere [9]. In Sec. 5, we present and discuss these numerically computed waves for a hyperbolic tanh density profile in fluid depths ranging from shallow to deep. Calculated results for mode-2 waves compare favorably with the experimental measurements of Davis and Acrivos [13]. The improvement over the weakly nonlinear theory is substantial when the wave amplitude is of order 1 and regions of recirculation appear within the wave.

1. Problem definition

For two-dimensional incompressible flows, a stream function ψ can be defined:

$$\frac{\partial}{\partial y}\psi = u, \qquad \frac{\partial}{\partial x}\psi = -v, \tag{1.1}$$

where u, v are the fluid velocities in the horizontal ($x-$) and vertical ($y-$) directions, respectively. The governing equations of motion and continuity are

$$\rho\left\{\frac{\partial}{\partial t}\psi_y - J(\psi,\psi_y)\right\} = -p_x + \nu\nabla^2\psi_y, \tag{1.2}$$

$$\rho\left\{\frac{\partial}{\partial t}\psi_x - J(\psi,\psi_x)\right\} = p_y + \rho g + \nu\nabla^2\psi_x, \tag{1.3}$$

$$\frac{\partial}{\partial t}\rho - J(\psi,\rho) = \kappa\nabla^2\rho, \tag{1.4}$$

where ρ is the fluid density, p the pressure, and ν and κ the coefficients of viscosity and (temperature or salinity) diffusion, respectively. $J(\psi, a)$ represents $\psi_x a_y - \psi_y a_x$.

Though we are primarily concerned in this paper with steady inviscid flows, it is important to point out here the order in which the limits $t \to \infty$, $(\nu, \kappa) \to (0,0)$ are to be taken. We shall seek steady solutions of *inviscid* flows; that is, we first take the limit $(\nu, \kappa) \to (0,0)$ and then let $t \to \infty$. Physically we are to examine the waves at a time long enough so that the transient behavior is negligible but not sufficiently long for the small viscosity and diffusivity to develop significant influence on the waves. Such a quasi-steady interpretation has been previously adopted by Benney and Ko [6]. The opposite limiting process is first $t \to \infty$ and then $\nu, \kappa \to 0^+$. One is then seeking the steady state solution of a slightly viscous fluid. This approach, adopted previously by Batchelor [1], would yield the true steady state solution (as opposed to quasi-steady) if such a solution exists. It is known that the two approaches give different solutions, especially for large amplitude waves possessing recirculation regions.

We shall first assume that a progressive waveform exists in an inviscid fluid and that this waveform moves with a steady phase speed c. We then examine the resulting steady state equations and inquire about the existence and character of the solutions. We fix the coordinate system with respect to the steady waveform, so that there is a steady upstream ($x \to -\infty$) current of speed c flowing towards the positive x-direction. The fluid flow caused by the steady state finite amplitude wave is described by Long's equation [26], which can be easily derived from the steady inviscid form of (1.2)–(1.4) [6] and is given by

$$\nabla^2\psi + \frac{1}{\rho}\frac{d\rho}{d\psi}\frac{1}{2}\left(\psi_x^2 + \psi_y^2\right) + gy = H(\psi) \tag{1.5}$$

where

$$\rho = \rho(\psi) \qquad \text{and} \qquad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2.$$

$H(\psi)$ is a function to be determined from the upstream conditions. Using

$$\psi \to cy$$

and

$$\rho \to \rho_0(y) \qquad \text{as} \quad x \to -\infty, \tag{1.6}$$

one finds

$$H(\psi) = \frac{1}{\rho_0}\frac{d\rho_0}{d\psi}\frac{1}{2}c^2 + g\frac{\psi}{c}. \tag{1.7}$$

Thus (1.5) becomes

$$\nabla^2\psi + \frac{1}{\rho_0}\frac{d\rho_0}{d\psi}\frac{1}{2}\left(\psi_x^2 + \psi_y^2 - c^2\right) + g\left(y - \frac{\psi}{c}\right) = 0. \tag{1.8}$$

Introducing the dimensionless variables

$$y^* = y/h, \qquad x^* = x/h, \qquad \psi^* = \psi/ch,$$

where $2h$ is a characteristic length scale for the pycnocline thickness, and writing

$$\rho = \rho_0(0)(1 - \sigma F(\psi^*)),$$

where σ can be interpreted as the relative density change across the entire pycnocline, one has from Eq. (1.8)

$$\left(\frac{\partial^2}{\partial x^{*2}}+\frac{\partial^2}{\partial y^{*2}}\right)\psi^* - \lambda\frac{F'(\psi^*)}{1-\sigma F}(y^*-\psi^*) = \frac{1}{2}\frac{\sigma F'(\psi^*)}{1-\sigma F}\left\{\psi_{x^*}^{*2}+\psi_{y^*}^{*2}-1\right\}. \tag{1.9}$$

In (1.9), we have defined

$$\lambda \equiv \sigma g h/c^2, \tag{1.10}$$

and since $\sqrt{\sigma g h}$ is the correct scaling for linear interfacial wave speeds, we expect λ to be of order 1. It is convenient to express (1.9) in terms of the perturbation streamfunctions ϕ^*, defined by

$$\psi^* = y^* + \phi^*. \tag{1.11}$$

Thus one has, after dropping the asterisk notation,

$$\nabla^2\phi + \lambda\frac{F'(y+\phi)}{1-\sigma F}\phi = \frac{1}{2}\sigma\frac{F'(y+\phi)}{1-\sigma F}\left\{\phi_x^2+\phi_y^2+2\phi_y\right\}. \tag{1.12}$$

The boundary conditions are

$$\phi \to 0 \quad \text{as} \quad x \to \pm\infty \tag{1.13}$$

and

$$\phi_x = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2. \tag{1.14}$$

Applying (1.13) to (1.14) then gives

$$\phi = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2. \tag{1.15}$$

For a Boussinesq fluid, $\sigma \ll 1$, (1.12) reduces to the following simpler system:

$$\begin{gathered}\nabla^2\phi + \lambda F'(y+\phi)\phi = 0,\\ \phi \to 0 \quad \text{as} \quad x \to \pm\infty,\\ \phi = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2.\end{gathered} \tag{1.16}$$

For weak stratifications the system (1.16) is a valid approximation to Eq. (1.12) provided

(i) the wave amplitude is not too large, i.e., $|\phi|_{max}$ not much larger than unity;
(ii) the wavelength is not too short and the vertical mode number is not too high, i.e.,

$$|\phi_x^2 + \phi_y^2|_{max} \not> 0(|\phi|_{max});$$

(iii) the Brunt-Väisälä profile $F'(y)$ is not a constant.

For the special cases of constant $F'(y)$ i.e., linear and exponential density profiles, the governing equation (1.16) to the lowest order in σ is linear even for ϕ not small. The Boussinesq approximation for these two cases is problematic since the linear equation does not possess a solution in the infinite domain [3, 27].

For constant $F'(y)$ terms of higher order in σ have to be included, and solitary waves have been shown to exist in these two cases [6] for long waves. The case of nonconstant $F'(y)$ is treated in this study.

For obvious reasons infinite domains, such as that in (1.16), are not well suited to numerical calculations. Therefore, we shall first seek solutions in a finite "box" and then take the width of the box to be so large that the solutions are no longer sensitive to further width increases. It turns out also that a theoretical analysis of the equation in a strictly infinite strip is also rather delicate. Consequently we shall first analyze Eq. (1.16) in a finite domain and then take the limit as the width of the box tends to be very large.

Define the finite domain $\mathfrak{D}_L$ by

$$\mathfrak{D}_L = \{-H_2 < y < H_1,\ -L < x < L\}. \tag{1.17}$$

The solution in $\mathfrak{D}_L$ will, of course, depend on L. We shall show that both the eigenvalue and the eigenfunction depend on L continuously and that their limits exist as $L \to \infty$. The most natural conditions to be applied on the boundary of the region $\mathfrak{D}_L$ (denoted $\partial\mathfrak{D}_L$) are the periodic boundary conditions

$$\phi(-L, y) = \phi(L, y), \qquad \phi(x, -H_2) = 0 = \phi(x, H_1). \tag{1.18}$$

The solution (if it exists) to the equation subject to conditions (1.18) will be a periodic nonlinear wave (known as a *cnoidal wave*) whose wavelength is $2L$. We shall show, in the Appendix, that such periodic solutions exist. The proof is brief, as our main interest lies in the solitary wave case.

In principle, one can obtain the solitary wave solutions from the periodic wave solutions by letting the wavelength of the latter approach infinity. However, it turns out to be difficult to extract additional information concerning the solitary wave from such a limiting process. For example, since periodic waves have both positive and negative displacements in a full cycle, it is difficult to determine whether the solitary wave obtained in the limit of the wavelength tending to infinity would have a positive or a negative displacement or both. Thus, for the purpose of determining the properties of a solitary wave, we shall consider in Sec. 2 wave solutions in a half domain

$$-L/2 < x < L/2$$

and seek solutions that are of one sign (positive or negative) in this domain (apart from variations in y). Making use of, and modifying, the work of Keller and Cohen [20] on positone solutions for the nonlinear heat equation, we show that nonlinear wave solutions of either elevation or depression exist.

2. The existence of a mode-1 solitary wave solution

In this section we consider the existence and properties of mode 1 solitary waves, which are defined as solutions to Eq. (1.16) that do not have any nodes in the infinite strip $\mathcal{D} = \{-H_2 < y < H_1,\ -\infty < x < \infty\}$ and vanish on its boundaries. Positive solutions ($\phi > 0$ in $\mathcal{D}$) correspond to waves of depression, whereas negative solutions represent waves of elevation.

As mentioned in Sec. 1, we shall approach the infinite domain problem as the limit of the problem in a finite box;

$$\mathcal{D} = \lim_{L\to\infty} \mathcal{D}_{L/2} \qquad \text{where} \quad \mathcal{D}_{L/2} = \{-H_2 < y < H_1,\ -L/2 < x < L/2\}.$$

Thus we shall first consider the system

$$\begin{aligned} \nabla^2\phi + \lambda f(\phi, y) &= 0 \qquad \text{in} \quad \mathcal{D}_{L/2}, \\ \phi &= 0 \qquad \text{on} \quad \partial\mathcal{D}_{L/2}, \end{aligned} \tag{2.1}$$

where $f(\phi, y) = F'(y+\phi)\phi$ and seek solutions that do not change sign in the domain. The methodology that will be used here is similar to that developed by Keller and Cohen for positone solutions of the nonlinear heat equation for an electrically conducting medium with temperature dependent resistance:

$$-\nabla^2 u = \lambda f(u, \mathbf{x}),$$

where u represents the perturbation temperature and $f(u,\mathbf{x})$ is the electrical resistance. An underlying assumption in their work is

$$f(0,\mathbf{x}) \neq 0,$$

which reflects the fact that the electrical resistance should not vanish when the temperature perturbation is zero. For our case, however,

$$f(\phi,\mathbf{x}) = F'(y+\phi)\phi,$$

and therefore the situation where $f(0,\mathbf{x})=0$ has to be faced. In addition to this assumption, two other important restrictions in Keller and Cohen's formulation, namely,

$$f(u,\mathbf{x}) > 0$$

and

$$f(u_2,\mathbf{x}) > f(u_1,\mathbf{x}) \qquad \text{if} \quad u_2 > u_1 > 0,$$

also must be removed in the present analysis. For the heat problem the restriction $f(u,\mathbf{x})>0$ is a statement of the fact that the electrical resistance is always positive. For our problem, this condition translates to $F'(y)>0$, meaning that the stratification has to be everywhere positive (i.e., stably stratified). In relaxing this restriction, we allow $F'(y)$ to have a countable number of zeros in the domain, but we still assume that $F'(y)$ cannot be negative. The last restriction is made so that the medium in which steady wave propagation is to take place is not convectively unstable. The second restriction, the so-called monotonicity requirement $f(u_2,\mathbf{x})> f(u_1,\mathbf{x})$, is not always met in our case and is therefore also removed in the following study.

2.1 The existence of waves of depression

We seek positive solutions ($\phi>0$, $\lambda>0$) to the system defined by (2.1). The linearized problem is

$$\begin{aligned} \nabla^2\phi^{(0)} + \lambda F'(y)\phi^{(0)} &= 0 \qquad \text{in} \quad \mathcal{D}_{L/2}, \\ \phi^{(0)} &= 0 \qquad \text{on} \quad \partial\mathcal{D}_{L/2}. \end{aligned} \tag{2.2}$$

The linear solution with one peak can be written

$$\phi^{(0)} = a\cos(kx)\psi_0(y,k), \qquad 0 < a \ll 1, \tag{2.3}$$

$$k = \pi/L, \tag{2.4}$$

where $\psi_0>0$ satisfies

$$\begin{aligned} \frac{d^2}{dy^2}\psi_0 + \left[\lambda F'(y) - k^2\right]\psi_0 &= 0, \\ \psi_0 &= 0 \qquad \text{at} \quad y = H_1 \text{ and } -H_2. \end{aligned} \tag{2.5}$$

Since it is assumed that

$$\int_{-H_2}^{H_1} F'(y)h^2(y)\,dy > 0 \tag{2.6}$$

for any nontrivial square integrable continuous real function $h(y)$ vanishing at the boundaries, $\lambda > 0$ is assured. The system (2.5) is known to possess an infinite number of discrete positive eigenvalues,

$$\lambda = \lambda_n^{(0)}(k), \qquad n = 1,2,3,\ldots,$$

which can be arranged in the following way:

$$0 < \lambda_1^{(0)} < \lambda_2^{(0)} < \lambda_3^{(0)} < \cdots, \tag{2.7}$$

with the only limit point at $+\infty$. In the present problem $\lambda_1^{(0)}$ is proportional to the reciprocal of the square of the linear phase-speed of mode-1 wave, $\lambda_2^{(0)}$ to that for the mode-2 wave, etc. In this section only $\lambda_1^{(0)}$ is considered, and the subscript will be omitted for convenience.

Let

$$\tau_1(\mathbf{x}) = \operatorname*{lub}_{\phi>0}\{-\partial f(\phi, y)/\partial\phi\}, \tag{2.8}$$

where again $f(\phi, y) = F'(y+\phi)\phi$. From this definition we have the following inequality

$$\tau_1(\phi_2 - \phi_1) + f(\phi_2, y) - f(\phi_1, y) \geqslant 0, \not\equiv 0 \tag{2.9}$$

if $\phi_2 > \phi_1$.

We let $\mathcal{L}$ be the elliptic operator defined by

$$\mathcal{L} \equiv -\nabla^2 + \lambda\tau_1, \qquad \lambda > 0. \tag{2.10}$$

$\mathcal{L}$ is a positive operator[1] for $\tau_1 \geqslant 0$. If $\tau_1 < 0$, $\mathcal{L}$ is positive only for $\lambda < \mu$, where $\mu > 0$ is the least eigenvalue to the following system for $\tau_1 < 0$:

$$\begin{aligned} -\nabla^2\psi + \mu\tau_1\psi = 0 \qquad & \text{in} \quad \mathcal{D}_{L/2}, \\ \psi = 0 \qquad & \text{on} \quad \partial\mathcal{D}_{L/2}. \end{aligned} \tag{2.11}$$

We propose to establish the existence of the nonlinear positive solution for $\lambda < \lambda^*$,

$$\lambda^* = \begin{cases} \mu & \text{if } \tau_1 < 0, \\ \infty & \text{if } \tau_1 \geqslant 0, \end{cases} \tag{2.12}$$

[1] That is, if ϕ is twice continuously differentiable and satisfies $\mathcal{L}[\phi] \geqslant 0, \not\equiv 0$ in $\mathcal{D}_{L/2}$ and $\phi = 0$ on $\partial\mathcal{D}_{L/2}$, then $\phi > 0$ in $\mathcal{D}_{L/2}$.

by iteration starting from the linear positive solution, $\phi^{(0)}>0$, $\lambda^{(0)}>0$, satisfying (2.5). A sequence of iterates $\phi^{(n)}$ is defined by a scheme similar to that used by Cohen [11]:

$$\mathcal{L}[\phi^{(n)}] = \lambda\left[f(\phi^{(n-1)}, y) + \tau_1\phi^{(n-1)}\right] \quad \text{in } \mathcal{D}_{L/2},$$
$$\phi^{(n)} = 0, \qquad \phi^{(0)} = 0 \quad \text{on } \partial\mathcal{D}_{L/2}, \tag{2.13}$$

for $n=1,2,3,\ldots$.

The iterates defined this way turn out to be monotonically positive (or *positone*):

THEOREM 1. *The iterates $\phi^{(n)}$ defined in (2.13) for $\lambda<\lambda^*$ are increasingly positive functions of n. That is,*

$$0 < \phi^{(0)} < \phi^{(1)} < \phi^{(2)} < \cdots < \phi^{(n-1)} < \phi^{(n)} \cdots. \tag{2.14}$$

Proof: We first establish that $\phi^{(1)}$ is positive and greater than $\phi^{(0)}$. We have, from (2.13) and (2.9),

$$\mathcal{L}[\phi^{(1)}] = \lambda\left[\tau_1(\phi^{(0)} - 0) + f(\phi^{(0)}, y) - f(0, y)\right] \geqslant 0, \not\equiv 0$$

since $\phi^{(0)}>0$. Therefore, by the maximum principle for elliptic operators, we have

$$\phi^{(1)} > 0 \qquad \text{in } \mathcal{D}_{L/2}.$$

Observing that the linear solution $\phi^{(0)}$ can be made arbitrarily small, we can also have

$$\phi^{(1)} - \phi^{(0)} > 0.$$

Next, we show by induction that if

$$\phi^{(n)} - \phi^{(n-1)} > 0$$

is true for some n, then

$$\mathcal{L}[\phi^{(n+1)} - \phi^{(n)}] = \lambda\left[f(\phi^{(n)}, y) - f(\phi^{(n-1)}, y) + \tau_1(\phi^{(n)} - \phi^{(n-1)})\right] \geqslant 0, \not\equiv 0$$

[from (2.9)]. So applying the maximum principle again yields

$$\phi^{(n+1)} > \phi^{(n)}. \qquad \square$$

Since the $\phi^{(n)}$ are monotonically increasing functions, an obvious question is whether they are bounded. For stratifications satisfying $F'(y)y<\infty$ the following result establishes the boundedness of $\phi^{(n)}$:

THEOREM 2. *The* $\phi^{(n)}$ *defined by* (2.13) *are uniformly bounded in* $\mathfrak{D}_{L/2}$ *for all* n *and for all positive finite* λ, $\lambda<\lambda^*$.

Proof: We define a function

$$\Phi(x, y, \lambda)$$

by the following inhomogeneous equation:

$$\begin{aligned} -\nabla^2\Phi &= \lambda\tau_2(\mathbf{x}) \quad &\text{in } \mathfrak{D}_{L/2}, \\ \Phi &= 0 \quad &\text{on } \partial\mathfrak{D}_{L/2}, \end{aligned} \tag{2.15}$$

where $\tau_2(\mathbf{x})$ is the least upper bound of $f(\phi, y)$. That is,

$$f(\phi, y) \leqslant \tau_2(x, y) \tag{2.16}$$

for all ϕ and all y in $\mathfrak{D}_{L/2}$. It is assumed that τ_2 is finite and positive in $\mathfrak{D}_{L/2}$. Let $G^{(0)}$ be the Green's function for the operator $-\nabla^2$, satisfying $G^{(0)}=0$ on $\partial\mathfrak{D}_{L/2}$. Then the solution to (2.15) can be written for any λ

$$\Phi(x, y) = \lambda\int_{\mathfrak{D}_{L/2}} G^{(0)}(\mathbf{x}, \xi)\tau_2(\xi)\, d\xi. \tag{2.17}$$

The solution (2.17) is known to be positive and finite for positive finite λ. Subtracting (2.13) from (2.15) and assuming $\Phi>\phi^{(n-1)}$, one has

$$\mathfrak{L}\left[\Phi-\phi^{(n)}\right] = \lambda\left[\tau_1\left(\Phi-\phi^{(n-1)}\right)+\tau_2-f\left(\phi^{(n-1)}, y\right)\right] \geqslant 0, \not\equiv 0,$$

and so $\Phi>\phi^{(n)}$ by the maximum principle. To complete the proof by induction, we show $\Phi>\phi^{(1)}$ by observing

$$\mathfrak{L}\left[\Phi-\phi^{(1)}\right] = \lambda\left[\tau_1\left(\Phi-\phi^{(0)}\right)+\tau_2-f\left(\phi^{(0)}, y\right)\right] \geqslant 0, \not\equiv 0.$$

Therefore $\phi^{(n)}$ is uniformly bounded from above by Φ in $\mathfrak{D}_{L/2}$ for all n. □

The existence of a positive nonlinear solution to (2.1) is established by the following theorem.

THEOREM 3. *The iterates* $\phi^{(n)}$ *defined by* (2.13) *converge uniformly to a minimum positive solution of* (2.1) *for* $\lambda<\lambda^*$.

Proof: Equation (2.13) can be converted into the following integral form:

$$\phi^{(n)}(\mathbf{x},\lambda)=\lambda\int_{\mathcal{D}_{L/2}} d\xi\, G^{(0)}(\mathbf{x},\xi)\left[f(\phi^{(n-1)},\xi)+\tau_1\left(\phi^{(n-1)}(\xi)-\phi^{(n)}(\xi)\right)\right], \tag{2.18}$$

where $G^{(0)}$ is the Green's function for the operator $-\nabla^2$, satisfying $G^{(0)}=0$ on $\partial\mathcal{D}_{L/2}$. Since the $\phi^{(n)}$ are uniformly bounded, they converge in the limit $n\to\infty$ to a positive function:

$$\hat{\phi}(\mathbf{x},\lambda)=\lim_{n\to\infty}\phi^{(n)}(\mathbf{x},\lambda)>0.$$

The function $\hat{\phi}$ is also uniformly bounded; i.e., for some finite positive number M,

$$\hat{\phi}(\mathbf{x},\lambda)\leqslant M \qquad \text{in} \quad \mathcal{D}_{L/2}.$$

Therefore, the integrand in (2.18) is bounded by

$$G^{(0)}(\mathbf{x},\xi)\left[f(\hat{\phi},\xi)+\tau_1 M\right]\leqslant G^{(0)}(\mathbf{x},\xi)\left[\tau_2+\tau_1 M\right],$$

and thus the limit $n\to\infty$ can be taken under the integral sign to yield

$$\hat{\phi}(\mathbf{x},\lambda)=\lambda\int_{\mathcal{D}_{L/2}} G^{(0)}(\mathbf{x},\xi)f\left(\hat{\phi}(\xi,\lambda),\xi\right)d\xi. \tag{2.19}$$

It is clear, since $\hat{\phi}$ defined by (2.19) satisfies (2.1), that it is *a* solution to that system. Further, it is a *positive* solution as it is obtained from the positone iterals.

It can be established that $\hat{\phi}$ is a minimum positive solution of (2.1) by noting that if $\phi(\mathbf{x},\lambda)$ is *any* positive solution to (2.1) and if $\phi(\mathbf{x},\lambda)>\phi^{(n-1)}(\mathbf{x},\lambda)$ is true, then it follows that $\phi(\mathbf{x},\lambda)>\phi^{(n)}(\mathbf{x},\lambda)$ because

$$\mathcal{L}\left[\phi-\phi^{(n)}\right]=\lambda\left[f(\phi,y)-f(\phi^{(n-1)},y)+\tau_1\left(\phi-\phi^{(n-1)}\right)\right]\geqslant 0,\not\equiv 0.$$

Having $\phi(\mathbf{x},\lambda)>\phi^{(n)}(\mathbf{x},\lambda)$ for all n, if follows that $\phi(\mathbf{x},\lambda)\geqslant\hat{\phi}(\mathbf{x},\lambda)$, and therefore $\hat{\phi}(\mathbf{x},\lambda)$ is the minimum positive solution. □

2.2. The existence of waves of elevation

Mode-1 waves of elevation are given by solutions to (2.1) with $\phi<0$ in $\mathcal{D}_{L/2}$ and $\lambda>0$. Defining $u=-\phi$, one is again seeking positive solutions ($u>0$ in $\mathcal{D}_{L/2}$). Using $f(u,y)$ now to denote $F'(y-u)u$, the governing equation becomes

$$\begin{aligned} -\nabla^2 u &= \lambda f(u,y) && \text{in} \quad \mathcal{D}_{L/2}, \\ u &= 0 && \text{on} \quad \partial\mathcal{D}_{L/2}. \end{aligned} \tag{2.20}$$

This is in the same form as the system considered in Sec. 2.1; the results presented there can easily be carried over to the present case. We again define a sequence of iterates $u^{(n)}$,

$$\mathcal{L}[u^{(n)}] = \lambda\left[f(u^{(n-1)}, y) + \tau_1 u^{(n-1)}\right] \quad \text{in} \quad \mathcal{D}_{L/2}$$
$$u^{(n)} = 0 \quad \text{on} \quad \partial\mathcal{D}_{L/2}, \tag{2.21}$$

$n = 1, 2, 3, \ldots$, where τ_1 is now defined as the least upper bound of $(-\partial f(u, y)/\partial u)$ for $u > 0$, and $u^{(0)} > 0$ is the linear solution satisfying (2.2) with $\phi^{(0)}$ replaced by $u^{(0)}$.

The proofs of the following theorems are similar to those in Sec. 2.1 and will not be repeated.

THEOREM 1. *The iterates $u^{(n)}$ defined in* (2.21) *for* $0 < \lambda < \lambda^*$ *are increasing positive functions of n, i.e.*,

$$0 < u^{(0)} < u^{(1)} < u^{(2)} < \cdots < u^{(n-1)} < u^{(n)} < \cdots. \tag{2.22}$$

THEOREM 2. *The $u^{(n)}$ as defined by* (2.21) *are uniformly bounded in $\mathcal{D}_{L/2}$ for all n and all λ in* $0 < \lambda < \lambda^*$.

THEOREM 3. *The iterates $u^{(n)}$ defined by* (2.21) *converge uniformly to a minimum positive solution of* (2.20).

2.3. Solitary waves in the infinite domain

Having established the existence of positive and negative solutions in the finite domain $\mathcal{D}_{L/2}$, it is desired to extend the domain to

$$\mathcal{D} = \{-H_2 < y < H_1, \ -\infty < x < \infty\},$$

as we take the limit $L \to \infty$. Let $k \equiv \pi/L$ as in (2.4). We observe that as $k \to 0$ the linear eigenvalue has a positive finite limit,

$$\lim_{k \to 0} \lambda^{(0)}(k) = \lambda^{(0)}(0), \qquad 0 < \lambda^{(0)}(0) < \infty. \tag{2.23}$$

For a hyperbolic tangent density profile (i.e., $F(y) = \tanh(y)$), $\lambda^{(0)}(0) = 2$ if H_1 and H_2 are both large.

The linear eigenfunction also exists in the limit $k \to 0^+$. Thus, take (2.3):

$$\phi^{(0)} = a \cos kx\, \psi_0(y, k). \tag{2.24}$$

It satisfies the zero boundary conditions at $x = \pm L/2$.

Now take the limit $k \to 0^+$. Then

$$\lim_{k \to 0^+} \phi^{(0)} = a\psi_0(y, 0^+) \neq 0 \qquad \text{for finite } x. \tag{2.25}$$

Thus, the limit exists (though nonuniform in x), and it is nontrivial; i.e.,

$$\phi^{(0)} \not\equiv 0.$$

Given that the linear eigenvalue and eigenfunction exist and are nontrivial, our iteration scheme would establish the existence of the nonlinear solution provided that the maximum principle used in the proofs can be extended to the infinite domain. There is a delicate mathematical problem concerning the maximum principle in the infinite domain. While there is little doubt that for our problem the limit $L \to \infty$ can be taken and the maximum principle remains valid, we shall avoid the mathematical details by treating our results in a "very large box."

3. The existence of mode-2 solitary wave solution

A mode-2 solitary wave is defined as a steady finite amplitude wave in $\mathcal{D}$ with only one node in the vertical direction. For a symmetric density stratification it can be shown that such a node is located at the point of symmetry. That is, if

$$F'(y) = F'(-y), \tag{3.1}$$

then

$$\phi(y) = -\phi(-y). \tag{3.2}$$

For the antisymmetric waves defined by (3.2), the analysis of Sec. 2 for waves of one sign can be directly applied to the upper and lower half domains:

$$\mathcal{D}_+ = \{0<y<H, -\infty<x<\infty\} \tag{3.3}$$

and

$$\mathcal{D}_- = \{-H<y<0, -\infty<x<\infty\}, \tag{3.4}$$

respectively. Specifically, it can be proved that a wave of elevation ($\phi<0$) exists in the upper domain $\mathcal{D}_+$ and a wave of depression ($\phi>0$) exists in the lower domain $\mathcal{D}_-$, thus implying the existence of a nontrivial mode-2 solitary wave for a symmetric stratification.

The more general case of a nonsymmetric density stratification has not been considered for mode-2 waves. [Note that the symmetry condition (3.1) is not used in Section 2 for mode-1 waves.]

4. Some properties of finite amplitude solitary waves

The dependence of the eigenfunction ϕ on the eigenvalue will now be considered. Differentiating (2.1) with respect to λ yields an inhomogeneous equation for the quantity $\phi_\lambda \equiv \partial\phi/\partial\lambda$:

$$\begin{aligned} \left[-\nabla^2 - \lambda f_\phi(\phi, y)\right]\phi_\lambda &= f(\phi, y) \quad \text{in } \mathcal{D}, \\ \phi_\lambda &= 0 \quad \text{on } \partial\mathcal{D}. \end{aligned} \tag{4.1}$$

The function

$$f_\phi \equiv \frac{\partial}{\partial\phi} f(\phi, y) = F'(y+\phi) + F''(y+\phi)\phi \tag{4.2}$$

turns out to be an important variable in determining many properties of ϕ_λ and hence of $\phi(\lambda)$. The specific shape of the undisturbed density profile $F(y)$ plays an important role in determining f_ϕ. Such a sensitivity to the shape of the density profile also appears in the two solutions obtained by Benney and Ko [6], though the equation they consider is different.

We also consider in this section the weakly nonlinear limit of our solutions and show how this limit is approached from the finite amplitude solutions. The dependence of the solutions on the sign of $F''(y)$ is also exhibited.

4.1. The weakly nonlinear limit

As $\phi \to \phi^{(0)}$, $\lambda \to \lambda^{(0)}$, (2.1) reduces to the following linear system to lowest order in $\phi^{(0)}$ ($|\phi^{(0)}| \ll 1$);

$$\begin{aligned} -\nabla^2\phi^{(0)} &= \lambda^{(0)} f_\phi(0, y)\phi^{(0)} \quad \text{in } \mathcal{D}_{L/2}, \\ \phi^{(0)} &= 0 \quad \text{on } \partial\mathcal{D}_{L/2}, \end{aligned} \tag{4.3}$$

$L \to \infty$, where $f_\phi(0, y) = F'(y)$.

To the next order in wave amplitude, one has, by subtracting (4.3) from (2.1),

$$\left[-\nabla^2 - \lambda^{(0)} f_\phi(0, y)\right]\left(\phi - \phi^{(0)}\right) = \lambda^{(0)}\tfrac{1}{2} f_{\phi\phi}(0, y)\phi^{(0)^2} + \left(\lambda - \lambda^{(0)}\right) f_\phi(0, y)\phi^{(0)}, \tag{4.4}$$

where $\frac{1}{2} f_{\phi\phi}(0, y) = F''(y)$. Multiplying (4.4) by $\phi^{(0)}$ and integrating over the domain then gives the following solvability condition:

$$\frac{(\lambda - \lambda^{(0)})}{\lambda^{(0)}} = -\frac{\displaystyle\iint_{\mathcal{D}_{L/2}} \tfrac{1}{2} f_{\phi\phi}(0, y)\phi^{(0)^3}\, d\mathbf{x}}{\displaystyle\iint_{\mathcal{D}_{L/2}} f_\phi(0, y)\phi^{(0)^2}\, d\mathbf{x}} + O(a^2) \tag{4.5}$$

$$= -\frac{\tfrac{8}{3} a \displaystyle\int_{-H_2}^{H_1} F''(y)\psi_0^3(y)\, dy}{\displaystyle\int_{-H_2}^{H_1} F'(y)\psi_0^2(y)\, dy} + O(a^2) \tag{4.6}$$

when the linear solution to (4.3),

$$\phi^{(0)} = a\cos kx\,\psi_0(y), \qquad 0 < a \ll 1, \quad k = \pi/L, \quad L \to \infty,$$

is substituted into (4.5). $\psi_0(y)$ satisfies the vertical structure equation (2.5).

The denominator in (4.6) is always positive because of condition (2.6). The sign of the numerator is now examined. The sign of ψ_0 is given in the following:

$\psi_0 < 0$ in $-H_2 < y < H_1$ for mode-1 waves of elevation,

$\psi_0 > 0$ in $-H_2 < y < H_1$ for mode-1 waves of depression, and

$\psi_0 < 0$ in $0 < y < H$ and $\psi_0 > 0$ in $-H < y < 0$ for mode-2 waves.

When the density stratification is symmetric about $y=0$, i.e.,

$$F'(y) = F'(-y),$$

$F''(y)$ is antisymmetric (Fig. 1):

$$F''(y) = -F''(-y).$$

Consider first the case where the stratification is strongest at the center of the pycnocline at $y=0$, and decreases away from it, i.e.,

$$F''(y)\begin{cases} <0 & \text{for } y>0, \\ >0 & \text{for } y<0, \end{cases} \tag{4.7}$$

an example of which is given by $F(y)=\tanh(y)$. We then have

$$\int_{-H}^{H} F''(y)\psi_0^3(y)\,dy \begin{cases} >0 & \text{for mode-2 waves,} & (4.8) \\ =0 & \text{for mode-1 waves.} & (4.9) \end{cases}$$

The dependence of λ on amplitude for the mode-2 wave is depicted in Fig. 2. It is seen that the phase speed of the weakly nonlinear wave is larger than its linear value since

$$c^2/c_0^2 = \lambda^{(0)}/\lambda > 1. \tag{4.10}$$

For the mode-1 wave, (4.9) implies that $\lambda - \lambda^{(0)}$ does not depend on the wave amplitude linearly. It turns out that the dependence is quadratic, similar to that for Stokes's surface wave and the periodic waves treated in the Appendix. Figure 3 depicts such dependence for the case of $F'''(y)>0$ (solid line) and the case of $F'''(y)<0$ (dashed line).

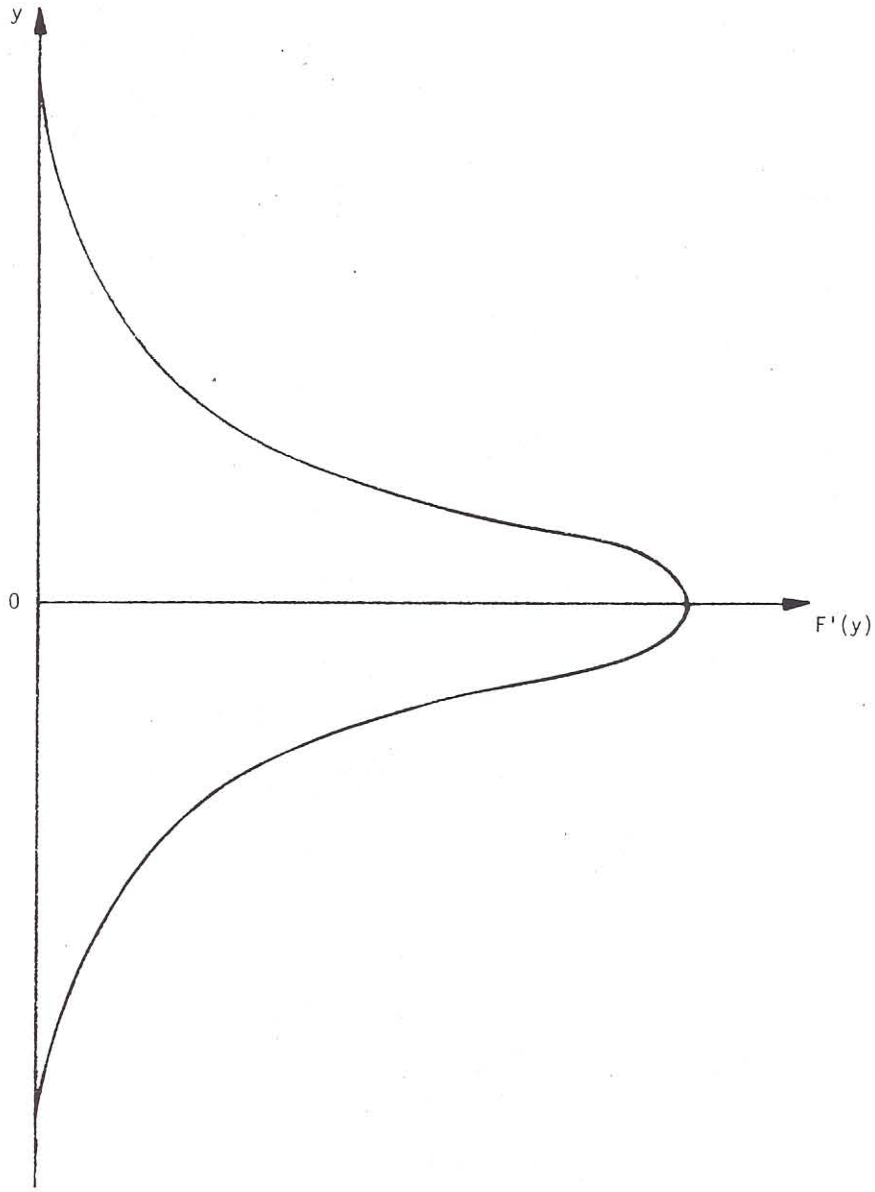


Figure 1. Density stratification profile.

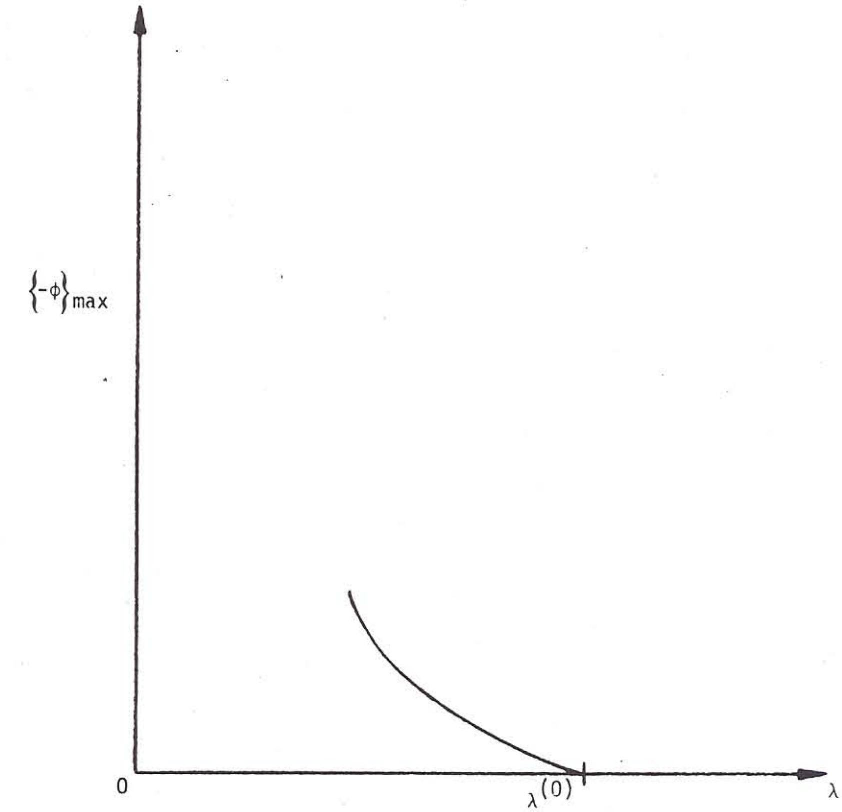


Figure 2. Solution branch in the amplitude versus λ plane for a mode-2 wave.

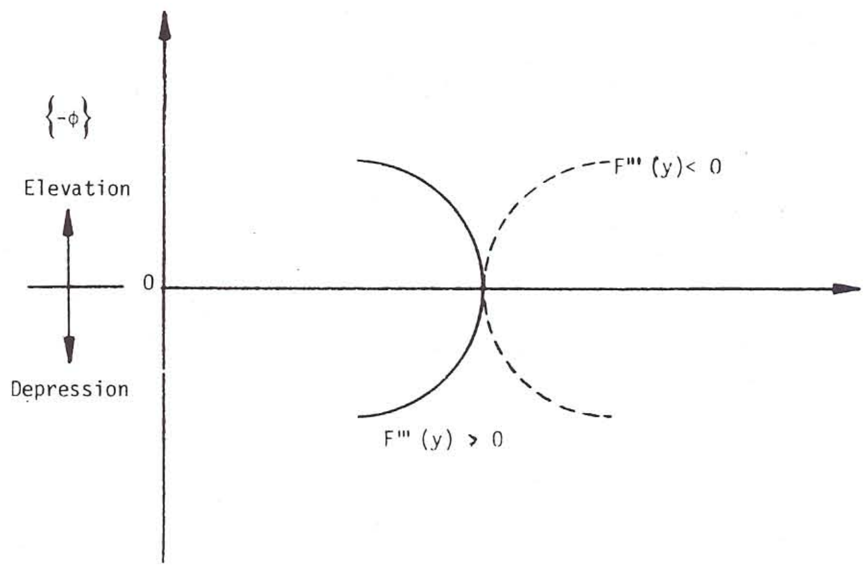


Figure 3. Solution branch in the amplitude versus λ plane for a mode-1 wave in a symmetric stratification.

The dependence of $\lambda - \lambda^{(0)}$ on wave amplitude can become linear even for mode-1 waves if the stratification is not symmetric. For the case depicted in Fig. 4, where the pycnocline is located near the water surface, the slope of $-\phi$ versus λ is positive (see Fig. 6). The slope becomes negative (see Fig. 7) when the pycnocline is located near the bottom surface, as in Fig. 5. Weakly nonlinear theory [22] for long waves predicts only waves that travel faster than the linear speed, and thus in Figs. 6 and 7 only the solid curves are known. The existence proofs given in the previous sections apply only for $\lambda < \lambda^*$. For weakly nonlinear waves, $\lambda^* = \lambda^{(0)}$, the linear eigenvalue. Therefore, only solutions with $\lambda - \lambda^{(0)} < 0$ (the solid lines in Figs. 6 and 7) have been established. This observation is consistent with the result from the weakly nonlinear theory that waves of elevation (depression) cannot exist if the pycnocline is situated near the surface (bottom). Solutions represented by the dashed curves in the figures therefore do not exist.

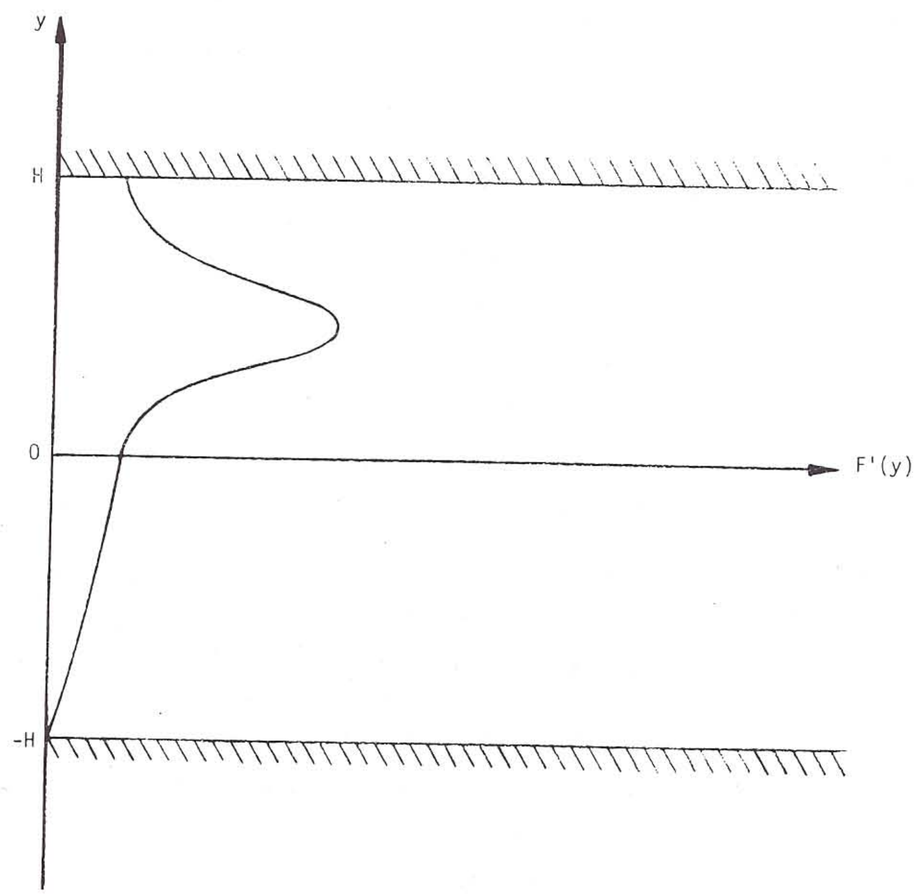


Figure 4. Stratification profile for a pycnocline located near the surface.

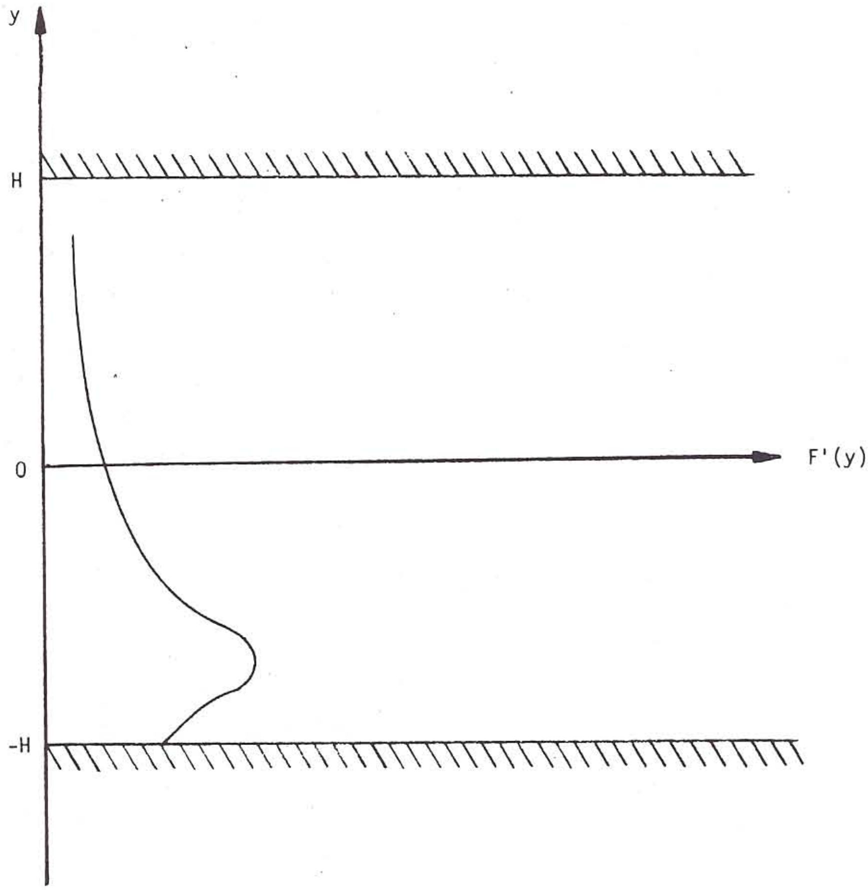


Figure 5. Stratification profile for a pycnocline located near the bottom.

4.2. Limit points and recirculation regions

We now return to a consideration of finite amplitude waves described by Eq. (2.1).

Consider as an example a mode-2 wave in the presence of symmetric stratification such as the one depicted in Fig. 1. As the wave amplitude is increased, λ decreases from its linear value (see Fig. 2). Therefore, larger amplitude mode-2 waves travel faster than small amplitude waves. The phase speed is not expected to increase indefinitely however, and there is the possibility that a *limit point* exists beyond which the phase speed ceases to increase as amplitude increases (see Fig. 8).

A limit point in the ϕ-λ plane is defined as that point at which the slope of $|-\phi_{\max}|$ versus λ is infinite; i.e.,

$$|\phi_\lambda| = \infty \qquad \text{at the limit point.} \tag{4.11}$$

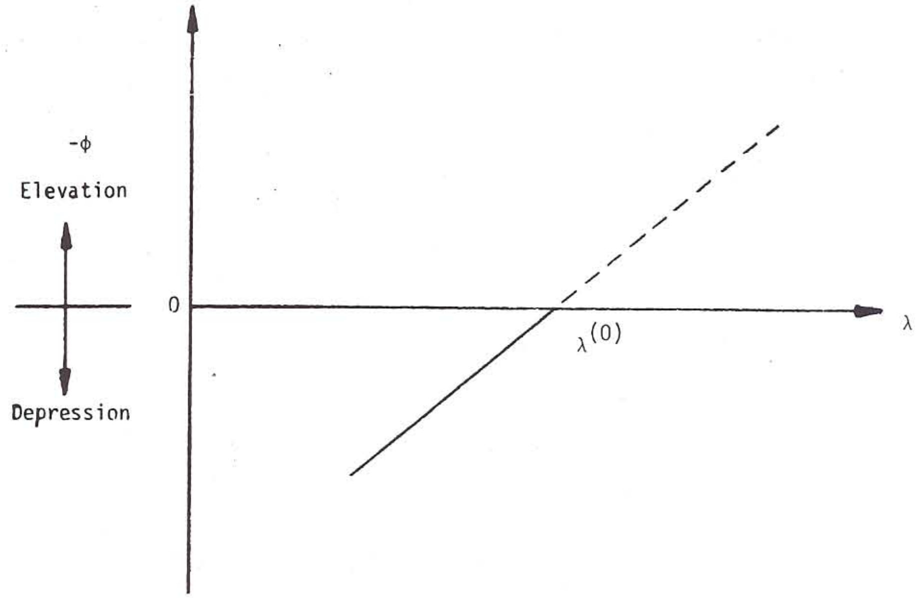


Figure 6. Small amplitude solution branch for a mode-1 wave in a pycnocline located near the surface.

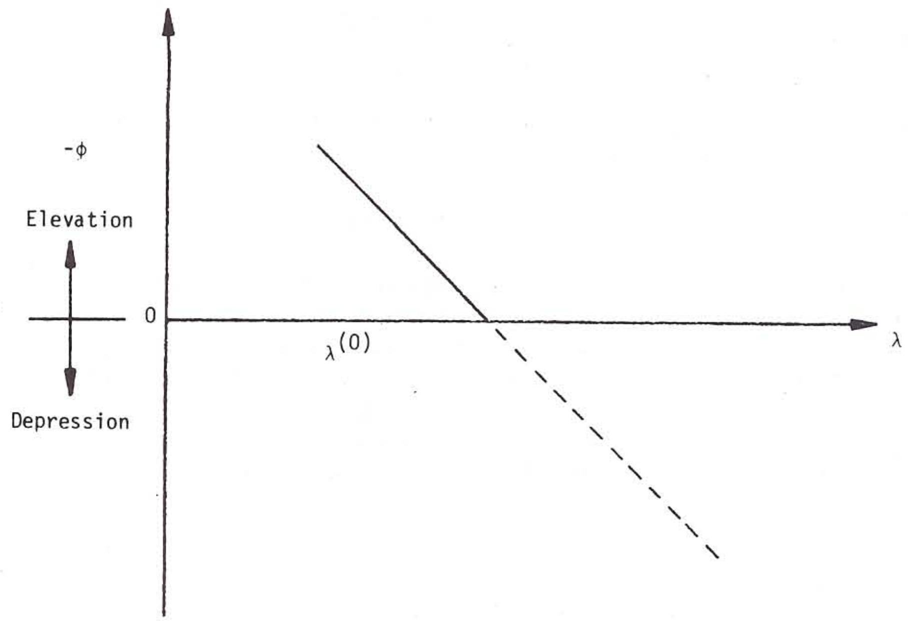


Figure 7. Small amplitude solution branch for a mode-1 wave in a pycnocline located near the bottom.

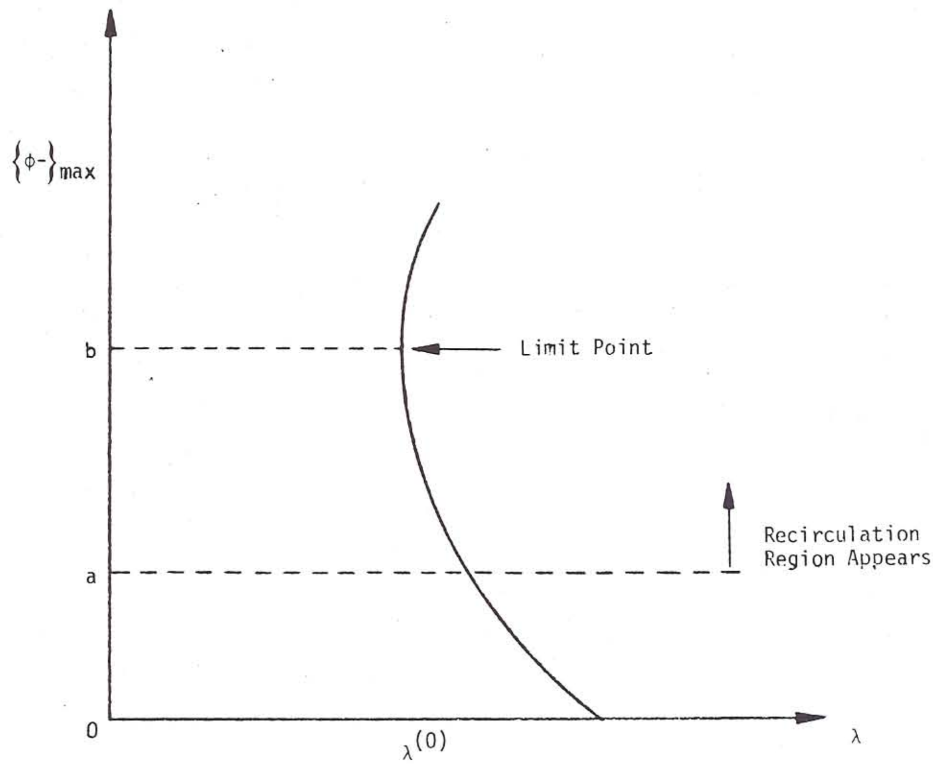


Figure 8. Amplitude versus λ solution branch for a mode-2 wave. Flow has closed streamlines for amplitudes larger than a. Point b is the limit point.

Examining (4.1), the equation for ϕ_λ, suggests that ϕ_λ is infinite only at the singular ("zero") points of the operator

$$G_\phi \equiv -\nabla^2 - \lambda f_\phi(\phi, y). \tag{4.12}$$

The operator $-\nabla^2$ in (4.12) is posiuive; in the domain D_+ which is of interest for the symmetric mode-2 wave, its least eigenvalue is $(\pi/H)^2$. For stratifications that are stable inside the domain (i.e., $F'(y)>0$), $f_\phi(\phi, y)$ as given by (4.2) is positive for small amplitudes. Let us examine the operator (4.12) first along the λ-axis in Fig. 2. The smallest eigenvalue of G_ϕ is equal to $(\pi/H)^2$. As one moves to the right, the operator G_ϕ decreases (in the sense that its eigenvalue is becoming smaller than $(\pi/H)^2$), until, at $\lambda = \lambda^{(0)}$, G_ϕ becomes zero since $\lambda^{(0)}$ solves

$$G_\phi \phi^{(0)} = 0.$$

This is one of the singular points of G_ϕ. It is, however, a bifurcation point because the right-hand side of (4.2) is zero also. There is more than one solution branch meeting at the bifurcation point, though for our problem only one branch has any physical significance (a second branch gives a mode-2 wave of depression, and the third is the trivial solution $\phi=0$). Continuing along the branch of interest, indicated on Fig. 2, we have already shown in Sec. 4.1 that the slope is negative (i.e., $-\phi_\lambda<0$ in $\mathcal{D}_+$), so that as one moves away from $\lambda^{(0)}$ along this branch λ decreases. The operator G_ϕ becomes positive, at least initially, for the weakly nonlinear waves, for which f_ϕ is positive. Limit points cannot exist in the weakly nonlinear regime; they have to be a finite amplitude phenomenon. At large amplitude $\lambda f_\phi(\phi, y)$ may become an increasing function of amplitude, and there is a possibility that at $(\phi, \lambda)=(\phi_l, \lambda_l)$, G_ϕ becomes a singular operator; i.e., the operator

$$G_\phi = -\nabla^2 - \lambda_l f_\phi(\phi_l, y)$$

has a zero eigenvalue.

Usually, the amplitude for which a limit point occurs is so large that a *recirculation* region has already developed within the flow domain. A recirculation region for mode-2 waves is defined as region where

$$\text{sign}\{\Psi\} \neq \text{sign}\{y\} \qquad \text{somewhere in } \mathcal{D}. \tag{4.13}$$

In (4.13) $\Psi \equiv y+\phi$ is the total stream function. The situation is represented by Figs. 8 and 9.

The steady state problem is complicated considerably with the appearance of closed streamlines in the flow domain. Because the streamlines in the recirculation region are not connected to the upstream flow, the assumption made in deriving Eq. (1.8) from (1.5) is violated. Furthermore, as pointed out by Batchelor [1], viscosity, which is neglected in obtaining the present solutions, should play an important role in the dynamics of a steady state recirculating flow. As mentioned in Sec. 1, we shall consider solutions in this study at a time long enough for a "quasi-steady" waveform to be established, but not long enough for viscous effects to be felt. The solution to (2.1) is such a solution, as it satisfies the Navier-Stokes equations (1.2)–(1.4) for $\kappa\equiv0\equiv\nu$ and the boundary conditions.

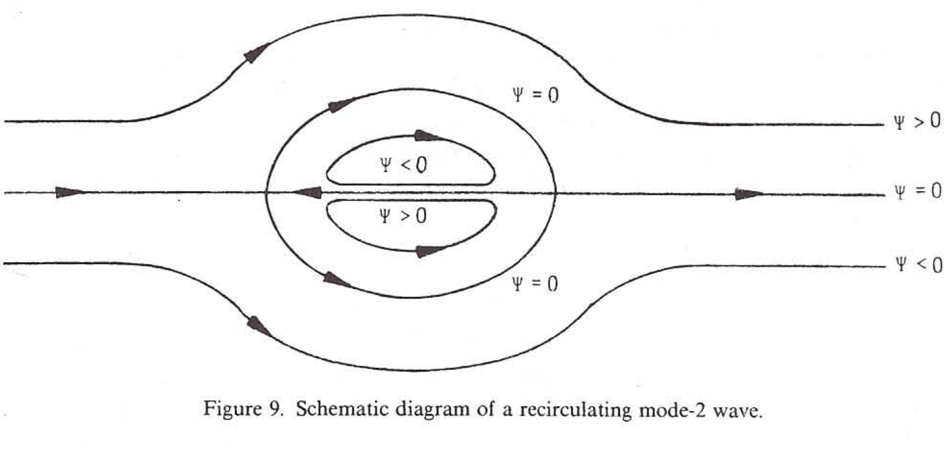


Figure 9. Schematic diagram of a recirculating mode-2 wave.

One can easily infer one more feature of the solutions from Eq. (4.2). Since $\phi_\lambda = 0$ cannot be a solution of (4.1) for nonzero $f(\phi, y)$, one can state the following theorem:

THEOREM. *A necessary condition for the existence of more than one solution with different phase speeds but with the same amplitude is that points of zero stratification exist in the density profile, i.e.,* $F'(y) = 0$ *at some* y.

It also follows that closed-loop solution branches in the ϕ-λ plane cannot exist unless $F'(y)$ has zero points.

For the case where the stratification is strongest at the center of the pycnocline at $y = 0$ and decreases away from it (see Fig. 1), one has for mode-2 waves

$$F'(y+\phi_1) > F'(y+\phi_2) \qquad \text{if} \quad |\phi_1| > |\phi_2| \tag{4.14}$$

provided that

$$\operatorname{sign}\{\Psi\} = \operatorname{sign}\{y\} \qquad \text{in} \quad \mathcal{D}$$

(i.e., no regions of recirculations exist [cf. (4.13)]).

Using a proof by Cohen and Laetsch [12] we now show that the mode-2 waves satisfying (4.14) are unique. Let ϕ_1 and ϕ_2 be two solutions of (2.1) for the same λ and $|\phi_1| > |\phi_2|$; i.e., $-\phi_1 > -\phi_2$ for $y > 0$ and $\phi_1 > \phi_2$ for $y < 0$. Then

$$-\phi_1 \nabla^2\phi_2 + \phi_2 \nabla^2\phi_1 = \lambda[\phi_1 f(\phi_2, y) - \phi_2 f(\phi_1, y)]$$

yields

$$0 = \lambda \int\!\!\int_{\mathcal{D}} \phi_1\phi_2[F'(y+\phi_2) - F'(y+\phi_1)]\, d\mathbf{x}. \tag{4.15}$$

However, by (4.14) the right-hand side of (4.15) is less than zero, a contradiction; therefore, ϕ_1 cannot be different from ϕ_2, and so *nonrecirculating mode-2 solutions on a pycnocline are unique.*

Since the governing equation and boundary conditions are symmetric with respect to x (i.e., invariant under the transformation $x \to -x$), a wave solution with fore-aft symmetry exists. The above result shows that such a symmetric waveform is the only solution for mode-2 wave on a pycnocline if its amplitude is not so large as to allow the appearance of recirculating regions. It is possible that for larger wave amplitudes an asymmetric waveform can appear in addition to the symmetric one.

5. Numerical results

In the previous sections we establish existence and discussed some properties of finite amplitude waves as solutions to Long's equation (1.16). In order to demonstrate the existence of these large amplitude waves and to determine their properties and behavior, we have also computed solutions to Eq. (1.16) numerically. As is typical of nonlinear eigenvalue problems, and in contrast to most linear eigenvalue problems, there will in general exist a continuous range of eigenvalues λ for which nonlinear eigenfunctions can be found, with λ being a (single or multiple valued) function of the solution amplitude. We have adopted a numerical scheme that is based on Newton's method with the capability to "continue" a solution along a solution branch once a neighboring solution is known. Thus, starting with, say, the known linear solution, or an approximate guess, scores of nonlinear solutions can be generated having increasingly larger amplitude, and a whole solution branch can be traced out. A finite difference discretization is used and the resulting system of nonlinear algebraic equations is linearized by Newton's method. A pseudo-arc-length continuation technique, due to Keller [19], is also employed to continue a solution branch past a limit point which was found for a shallow water case. The method is accurate and efficient provided the initial guess is close to the exact solution for a given value of λ. It should be mentioned also that when there is more than one solution for a given value of λ, the solution that the scheme converges to is often the one that is closest to the initial guess both in shape and in amplitude. All the initial guesses used in the present calculation possess fore-aft symmetry. The details of the numerical procedure will appear elsewhere [9]. In this section we shall present and discuss the properties of these computed finite amplitude waves. We have chosen the mode-2 wave for detailed computation here, mainly because the experimental measurements available are for that mode.

In all the calculations presented in this section, the density stratification profile used is given by

$$F'(y) = \operatorname{sech}^2 y \qquad \text{for} \quad -H < y < H,$$

and H is taken to be 4, 10, 20, 30, and 40 in units of half pycnocline thickness. In the present numerical calculation, the horizontal domain is truncated to a large but finite extent. If the solution obtained for such a "box" is to represent a solitary wave, then it should not be altered if the domain is enlarged. This we shall check numerically for each solution calculated. For most of our computations, a domain of width $2L=40$ was found to be large enough to contain the solitary wave. In Figure 10 we show a series of contour plots of the total streamfunction Ψ for a mode-2 wave with increasing amplitude (or decreasing eigenvalue λ) for $H=4$, the shallow case. Note the appearance of closed streamlines in the wave core when the wave amplitude exceeds a certain value.

Horizontal cross-sections cut along a line passing through the y-location of the wave maximum are displayed in Figure 11. The uppermost line is a portion of the extremely large amplitude wave found at $\lambda = 1.1615379$, past the limit point. This wave probably is not "solitary," as it fills most of even the largest computational domain we have used. The next curve from the top is the wave shape right at the limit point ($\lambda = 0.979114117$). The flat top and sharp ends of this wave are rather intriguing. For a given wave amplitude, the flat top disappears when the top boundary is moved further away (i.e., for increasing H). The third profile from

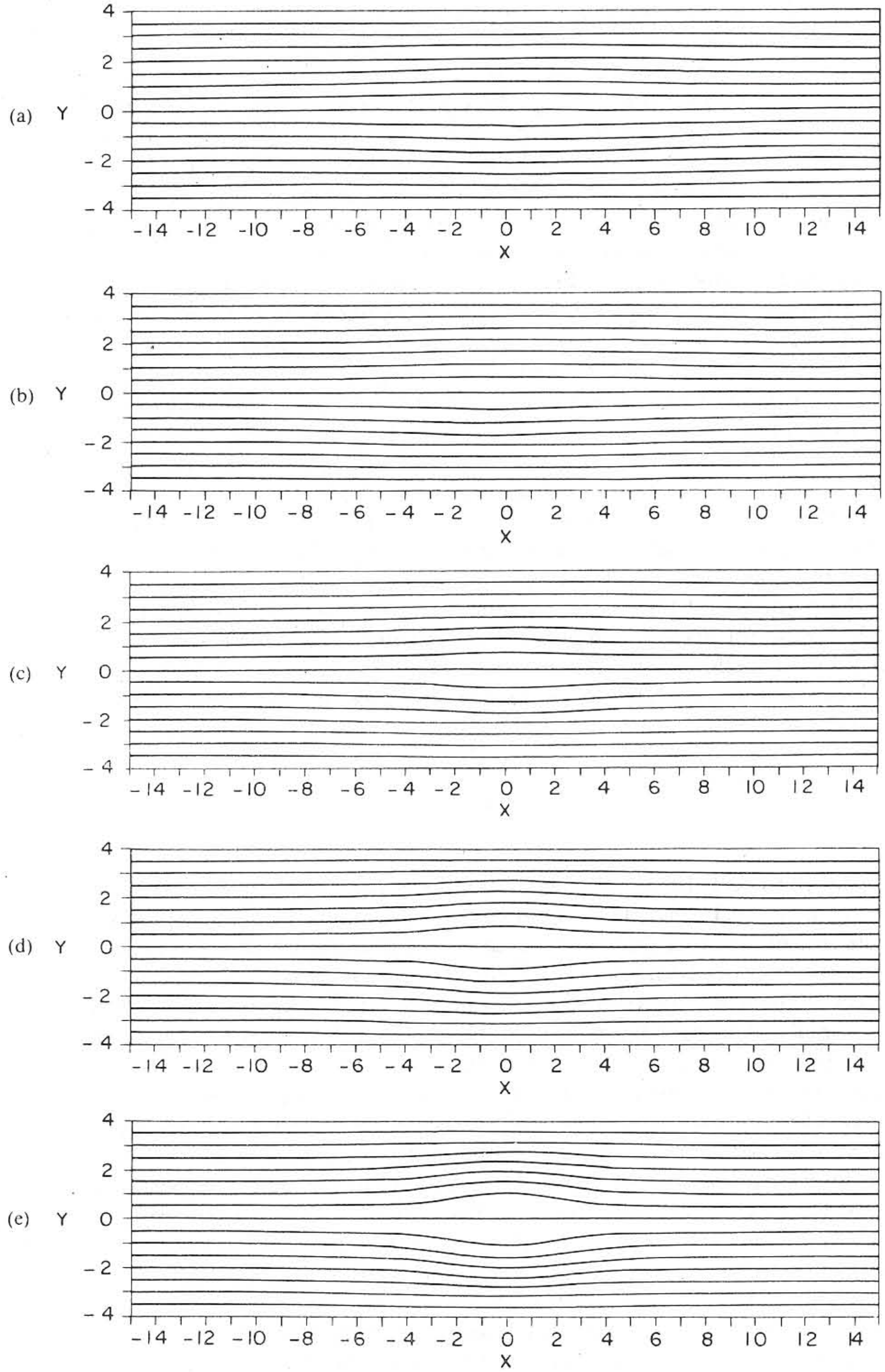


Figure 10. Contour plots of the total stream function Ψ: (a) $\lambda=2.70$, (b) $\lambda=2.60$, (c) $\lambda=2.50$, (d) $\lambda=2.25$, (e) $\lambda=2.00$, (f) $\lambda=1.50$, (g) $\lambda=1.20$, (h) $\lambda=1.10$, and (i) $\lambda=0.98$.

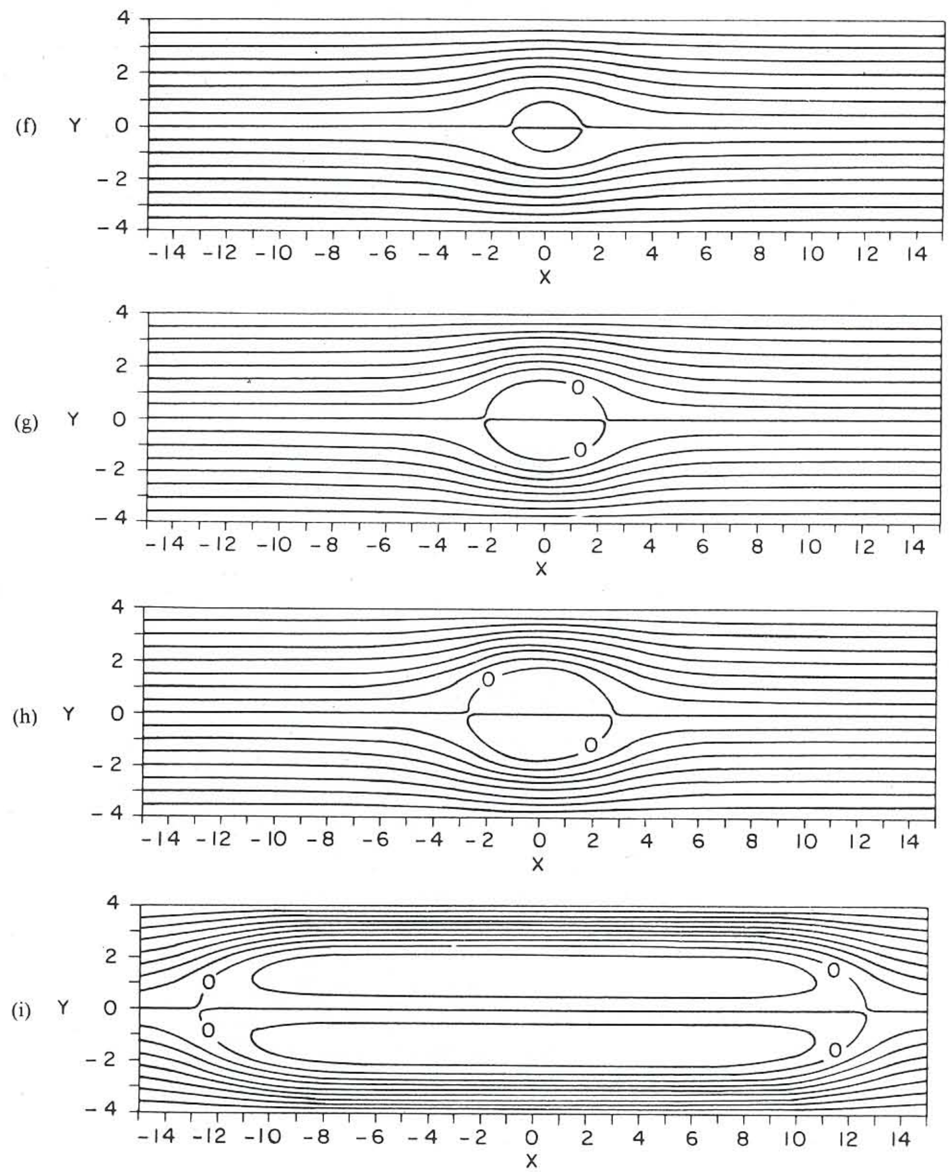


Figure 10. *(Continued)*

the top is for $\lambda = 1.1$ (below the limit point) and the other profiles, in order of decreasing amplitude, are for $\lambda = 1.2$, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0, 2.1, 2.25, 2.4, 2.6, and 2.7.

Figure 12 is the same as Fig. 11 except that the wave amplitudes are all normalized by their respective maxima. In Fig. 12a are plotted the lower 14 curves on Fig. 11. With their amplitudes normalized, it is clearly seen that as the λ decreases the relative width of the wave also decreases. The rate of decrease slows as the limit point is approached, and the last (the inner) few curves are almost

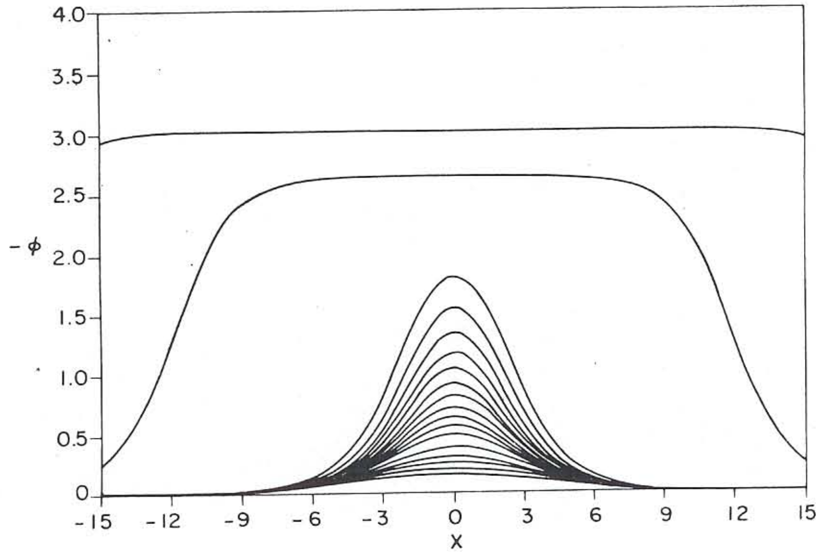


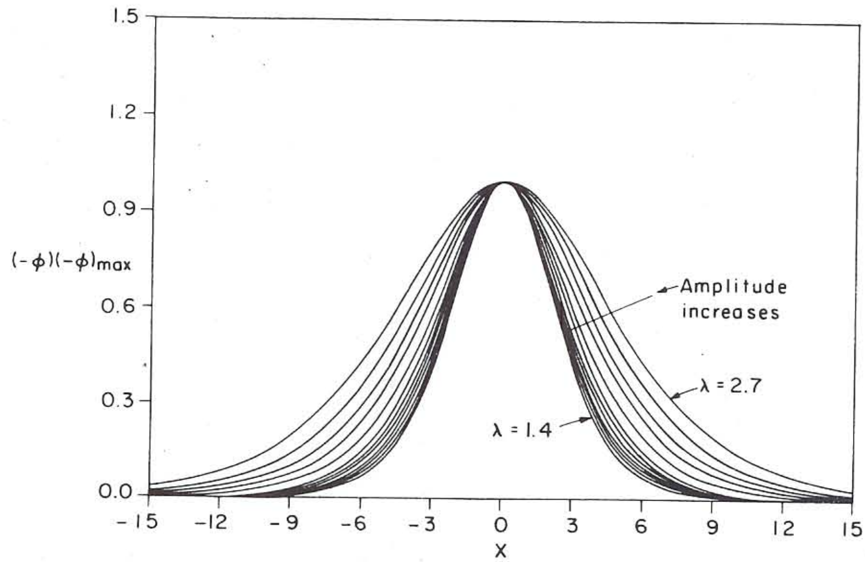
Figure 11. Horizontal cross-sections for solutions with different amplitudes.

indistinguishable from one another. Increasing the amplitude further (i.e., going to the upper branch of the solution curve) gives waves whose widths again increase. This is shown in Fig. 12b, which is for the top six curves in Fig. 11.

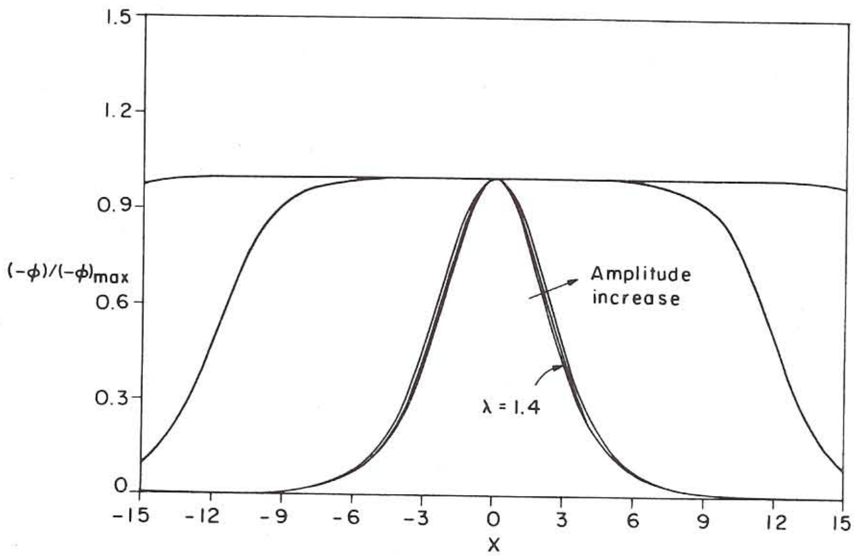
Figure 13 depicts vertical cross-sections along $x=0$ for the waves depicted in Fig. 11. It is seen that the wave maximum moves closer to the boundaries as the amplitude is increased. This fact may account for the flat top shape of the largest waves shown in Fig. 11.

Figure 14 is a plot of the maximum amplitude $(-\phi)_{\max}$ versus λ for various H. The curves for $H=20$ and 30, though also computed, are not presented here as they crowd near the curve for $H=40$. Each value along a curve is a nonlinear solution such as those depicted in Fig. 10. To construct the curve, say for $H=4$, several dozen of the solutions in Fig. 10 were used. The limit point for $H=4$ is located at $\lambda=0.979114117$, with $(-\phi)_{\max}=2.6470$. Near this point a slight variation in λ would produce a large variation in the amplitude. We have obtained several solutions beyond the limit point. They are not plotted on this curve for the reasons mentioned previously. It should also be mentioned that recirculating flows develop for waves whose amplitudes are larger than $(-\phi)_{\max} \simeq 0.9$.

Figure 15 is a plot of the wave amplitude $(-\phi)_{\max}$ versus the wavelength represented by $l_{1/2}$, the wave's *half-width*. The half-width is the horizontal width measured between the two locations where the wave amplitude is one-half its maximum value. Other width definitions (e.g., $l_{1/4}, l_{1/e}, l_{1/e^2}$) have also been examined but look very similar (Fig. 15). It is clearly seen that as the waves amplitude increases, its width first decreases as $\sim 1/(\text{amplitude})^{1/2}$, as predicted by weakly nonlinear theories, but the width contraction stops as a recirculation region is developed in the wave. It is perhaps important to note that the limit



(a)



(b)

Figure 12. Normalized horizontal cross-sections.

point in the amplitude versus $l_{1/2}$ plot occurs at a lower amplitude ($(-\phi)_{max} \simeq 1.2$ for $H=4$) than the limit point in the amplitude versus λ plane, which occurs at a much larger amplitude ($(-\phi)_{max} = 1.647$ for $H=4$). Thus, it probably can be inferred that the two limit points are caused by different mechanisms. It is probably safe to suggest that the limit point in the $(-\phi)_{max}$ versus $l_{1/2}$ plane is a result of the recirculating flow, while the limit point in the $(-\phi)_{max}$ versus λ

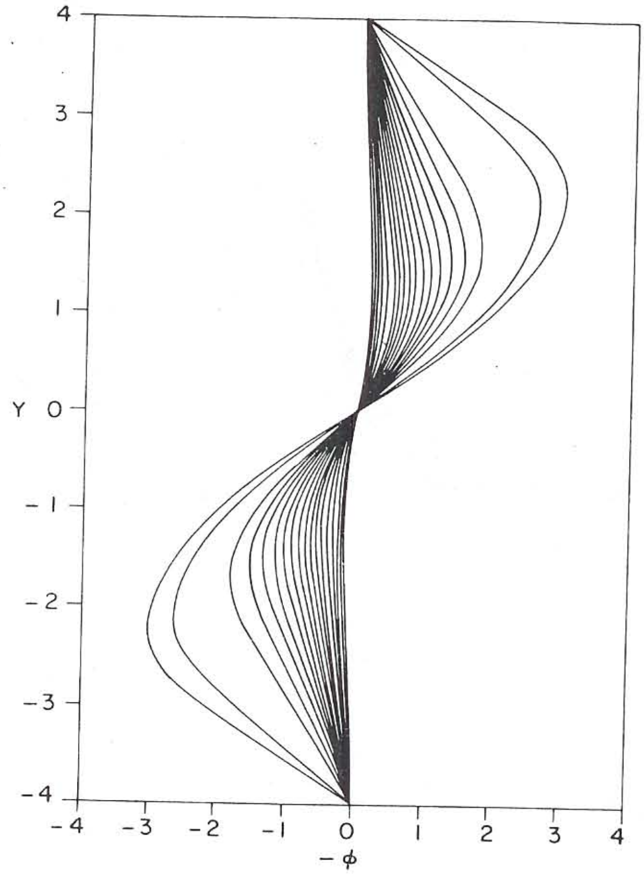


Figure 13. Vertical cross-sections for different amplitudes.

plane is probably due to the presence of the rigid boundary near the wave maximum.

In Fig. 14 results for other values of fluid depth are also presented. As can be expected, the curves for $H>10$ are not too different from each other, suggesting that the deep-water limit is almost approached for practical purposes. Contour plots for $H=40$, the deep water case, are shown in Fig. 16. Davis and Acrivos [13] presented experimental measurements of $(-\phi)_{max}$ versus λ for deep water, $H=40$. The density profile in their experiment can be well represented by $F(y)=\tanh y$, a profile used in our calculations. In Fig. 17 their data are compared with our calculated result for $H=40$. Good agreement is obtained.[2] Note also that the experimental results for various values of $\sigma\equiv\Delta\rho/\rho_R$ differ little from each other and also from our Boussinesq results, even for σ as large as

[2] The straight line, labeled "weakly nonlinear analysis," is taken from Davis and Acrivos, who obtained it for a fluid of infinite depth for which the limiting eigenvalue $\lambda^{(0)}$ for small amplitude is 2.0. For $H=40$ $\lambda^{(0)}$ is slightly larger than 2.0. This may be the reason why the straight line is not tangent to our calculated curve, as the weakly nonlinear curve should be.

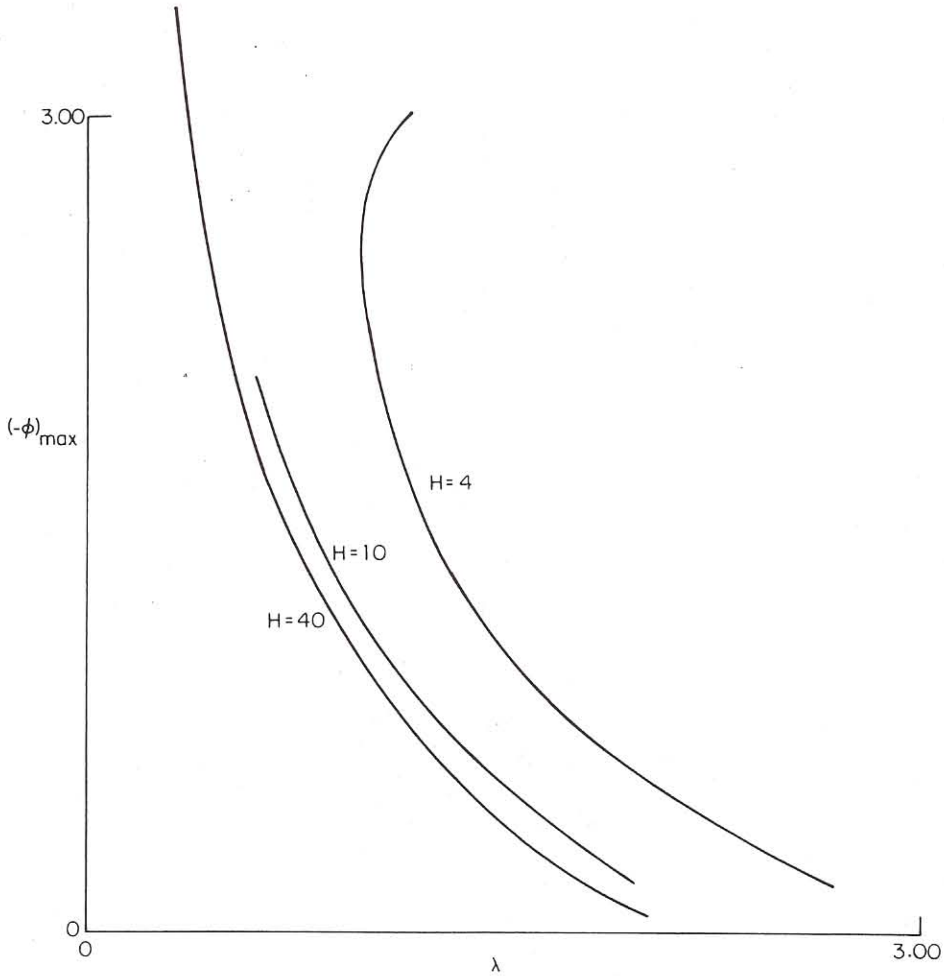


Figure 14. Amplitude versus λ solution branches.

16%. This may suggest that the Boussinesq approximation used in the present study is a good assumption.

6. Discussion

We have shown in this paper, both theoretically and numerically, that finite amplitude internal waves of permanent form can exist. Some of the results can be anticipated from weakly nonlinear theories, while the results for large amplitude waves are rather unexpected but have been qualitatively verified by experiments. Weakly nonlinear theories predict that long waves, whose dispersion is weak, can maintain their permanent form with weak nonlinearities. Shorter waves of small amplitude disperse while long waves of larger amplitude break. The possibility that shorter waves of larger amplitude can form solitary waves is thus indicated

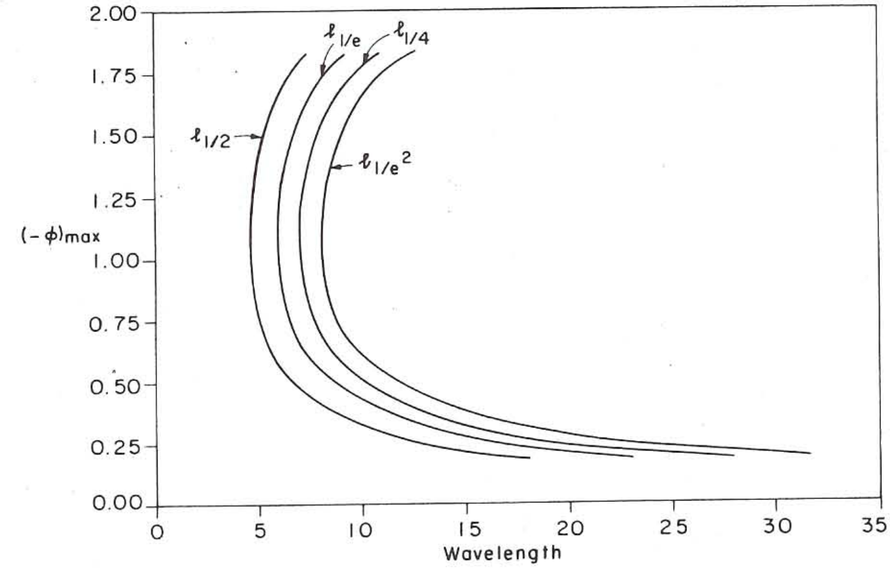
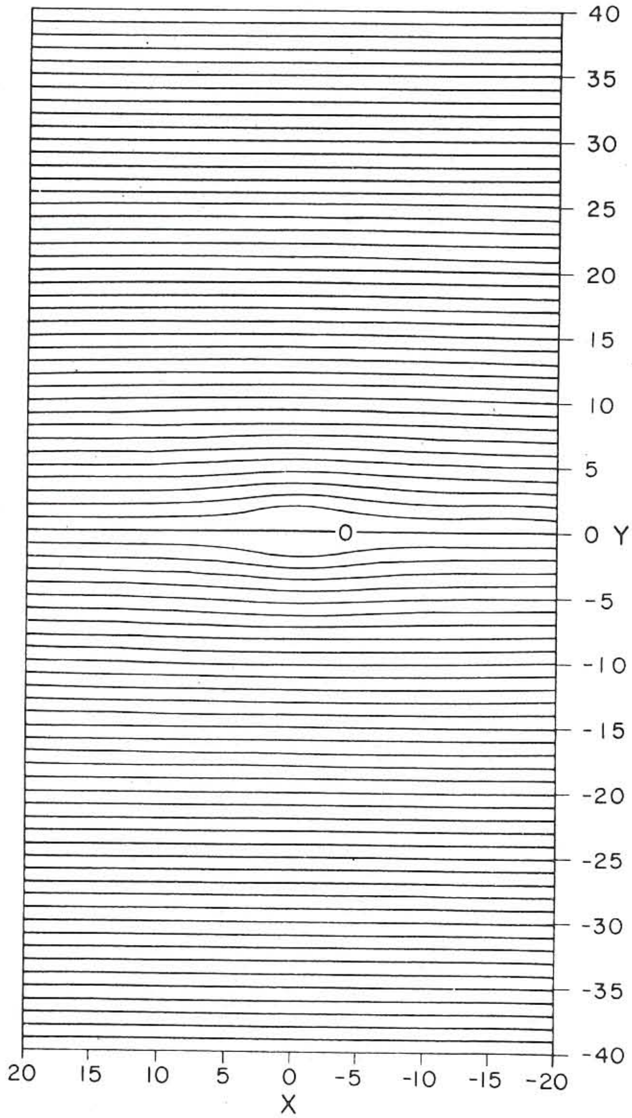


Figure 15. Amplitude versus half-width.

by these theories. We show here that this is indeed the case. A larger amplitude internal wave must have a smaller wavelength in order to maintain a permanent form. Larger amplitude waves also travel faster than smaller amplitude waves. These general trends are expected from weakly nonlinear theories, though we also found that the *rate* of contraction in wavelength and the rate of increase in phase speed gradually decrease with increasing amplitude. When the wave's amplitude is of the order of the pycnocline thickness, the wave ceases to contract and, for still larger amplitudes, starts to increase its wavelength. The shortest wavelength achieved is about 2–3 times the pycnocline thickness. Examining the solution reveals that a recirculation region has already developed in the wave. For the symmetric mode-2 wave we have considered, the flow inside the closed streamline is similar in appearance to the flow field associated with a vortex pair. There is a pair of fluid regions rotating with equal and opposite vorticity enclosed by the closed streamline. The size of the recirculation region increases with increasing amplitude, and the wavelength increase appears to be a natural consequence of such enlargement in size. The recirculating flows may also be expected to act to stabilize the propagation of large amplitude waves. Although we have not completed a stability analysis for these waves, large solitonlike internal waves with recirculation regions are easily produced in the laboratory and appear to be stable, while surface waves with comparable magnitudes are known to break.

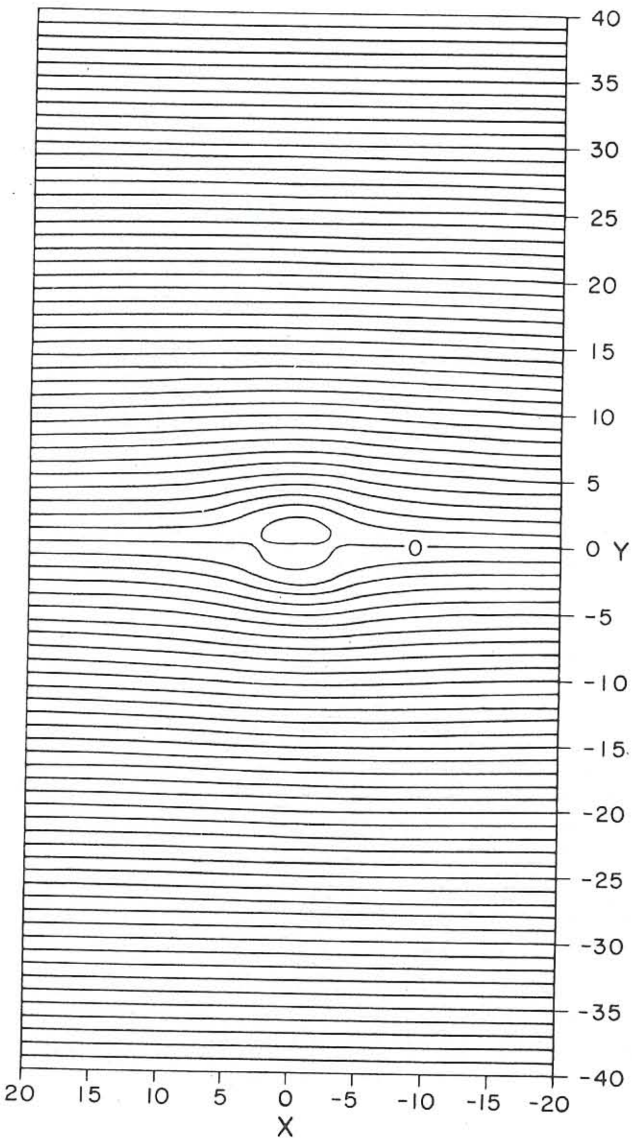
Existing weakly nonlinear descriptions of internal waves are traditionally divided into shallow water and deep water theories, with the Korteweg-deVries equation governing the former case and the Benjamin-Ono equation for the latter. Recently, Kubota et al. [22] considered the more general case of fluid of finite



(a)

Figure 16. Contour plots of the total stream function for $H=40$: (a) $\lambda \approx 1.0$, (b) $\lambda \approx 0.5$, and (c) $\lambda \approx 0.16$.

depth, which includes shallow and deep fluids as subcases. For finite amplitude steady waves we have treated fluid depths H ranging from 4 to 40 pycnocline thicknesses, with the case $H=4$ typifying a shallow depth medium and $H=40$ a deep fluid. That the deep fluid limit ($H \to \infty$) is indeed reached for practical purposes at $H=40$ can be discerned from Fig. 14, where it is seen that results become asymptotically close to the $H=40$ case when $H>10$.

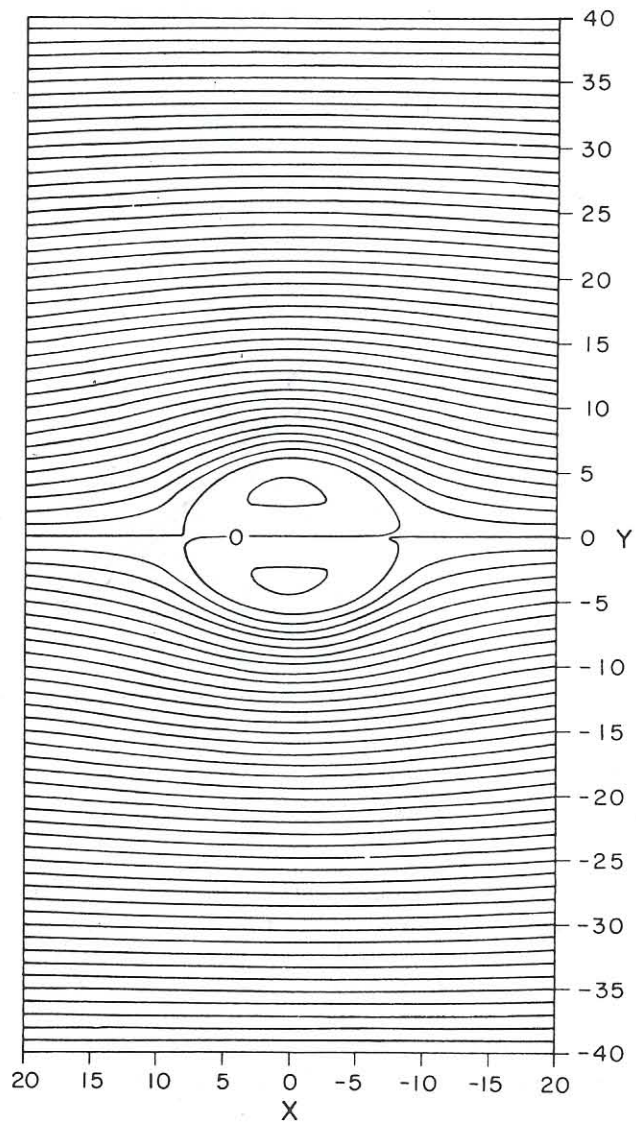


(b)

Figure 16. *(Continued)*

7. Concluding remarks

Internal gravity waves of "permanent form" in a stratified medium are studied. The weakly nonlinear theory for long waves is extended into the finite amplitude regime. Fluid depths ranging from shallow to deep have been incorporated, and in the case of deep water where experimental data are available the calculated and observed results compare very favorably.



(c)

Figure 16. *(Continued)*

Our results seem to indicate that the existing weakly nonlinear theories, derived based on an asymptotic expansion in wave amplitude assuming that the wave displacements are small compared with the scale of variation in the undisturbed density profile, (i.e., $a/h \ll 1$), remain surprisingly good up to $a/h \sim 0.5$. For larger amplitudes the agreements deteriorate. At $a/h \sim 1$ recirculation regions appear in the flow domain and some of the predictions of the weakly nonlinear theories become not even qualitatively correct. For example, the

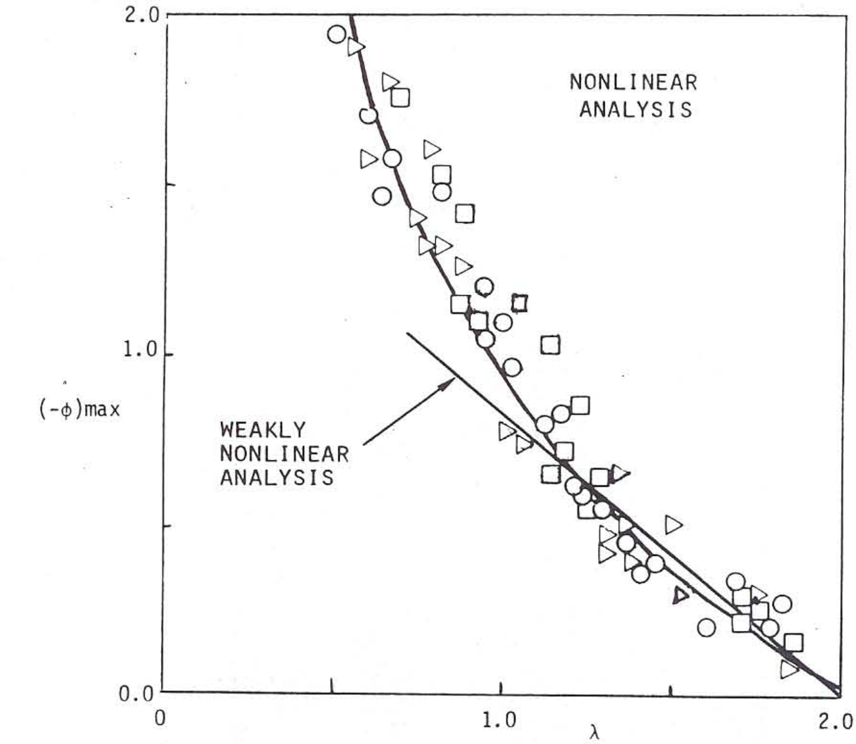


Figure 17. Comparison with the data of Davis and Acrivos [13]: $\rho_1 = 1.000$ g/cm^3; triangles, $\rho_2 = 1.052$; squares, $\rho_2 = 1.095$; circles, $\rho_2 = 1.168$.

contraction in the half-widths with increasing amplitudes predicted by the weakly nonlinear theories ceases to be correct for $a/h \gtrsim 1$ as recirculation regions seem to bring about an expansion in wave widths for increasing amplitudes. The fully nonlinear description presented here is expected to be important in sharp pycnocline regions.

Calculations for various stratification and current profiles and for mode-1 waves are in progress and will be reported in the near future.

Appendix. The existence of "periodic" wave solutions

In this section, the following nonlinear eigenvalue problem with periodic boundary conditions is considered:

$$\begin{aligned} -\nabla^2\phi &= \lambda f(\phi, y) \qquad \text{in } \mathcal{D}_L = \{-H_2 < y < H_1, -L < x < L\}, \\ \phi(-L, y) &= \phi(L, y), \\ \phi(x, -H_2) &= \phi(x, H_1) = 0, \end{aligned} \tag{A.1}$$

where $f(\phi, y) = F'(y+\phi)\phi$. Let $\omega(x, y)$ be a harmonic function satisfying the boundary conditions.

$$\omega(-L, y) = \phi(-L, y) = \omega(L, y), \qquad \omega(x, -H_2) = 0 = \omega(x, H_1). \tag{A.2}$$

Define

$$v \equiv \phi - \omega \tag{A.3}$$

and

$$g(v, \mathbf{x}) \equiv f(\omega + v, y). \tag{A.4}$$

Then (A.1) reduces to the following standard nonlinear system with homogeneous boundary conditions:

$$\begin{aligned} -\nabla^2 v &= \lambda g(v, \mathbf{x}) \quad \text{in} \quad \mathfrak{D}_L, \\ v &= 0 \quad \text{on} \quad \partial\mathfrak{D}_L. \end{aligned} \tag{A.5}$$

The existence proof for such a system is complicated but standard[3] (see e.g., Courant and Hilbert, Vol. II). We shall only briefly sketch the essential steps.

We assume that the function $g(v, \mathbf{x})$ has continuous first derivatives in y in $\mathfrak{D}_L + \partial\mathfrak{D}_L$ and for all v, and further assume that $g(v, \mathbf{x})$ is bounded, i.e.,

$$|g(v, \mathbf{x})| \leqslant M(x, y) < \infty \qquad \text{in} \quad \mathfrak{D}_L. \tag{A.6}$$

For example, in the numerical calculations presented in the text $F'(h) = \operatorname{sech}^2 y$ is used, so

$$f(\phi, y) = \operatorname{sech}^2(y+\phi)\phi,$$

which is bounded for all (real) y and all (real) ϕ. Thus (A.6) is satisfied.

Let

$$\kappa = \operatorname{lub}\left(-\frac{\partial}{\partial v} g(v, \mathbf{x})\right), \tag{A.7}$$

so that by the mean value theorem

$$\kappa \geqslant -[g(v_2, \mathbf{x}) - g(v_1, \mathbf{x})]/(v_2 - v_1),$$

[3]With important modifications to deal with an *eigenvalue* problem.

or

$$g(v_2, \mathrm{x}) - g(v_1, \mathrm{x}) + \kappa(v_2 - v_1) \geq 0 \tag{A.8}$$

if $v_1 \geq v_2$.

Let $\mathcal{L}$ be the operator defined by

$$\mathcal{L} \equiv -\nabla^2 + \lambda\kappa, \qquad \lambda > 0. \tag{A.9a}$$

It is a positive operator[4] if $\kappa \geq 0$. For $\kappa < 0$, $\mathcal{L}$ is a positive operator only for

$$\lambda < \mu \tag{A.9b}$$

where μ is the largest eigenvalue of

$$\begin{aligned} [-\nabla^2 + \mu\kappa]\Psi &= 0 \quad \text{in} \quad \mathcal{D}_L, \\ \Psi &= 0 \quad \text{on} \quad \partial\mathcal{D}_L. \end{aligned} \tag{A.9c}$$

Equations (A.9c) can be solved easily, to yield for $\kappa < 0$

$$\mu = \frac{(\pi/L)^2 + (n\pi/2H)^2}{-\kappa}, \qquad n = 1, 2, 3, \ldots, \tag{A.9d}$$

where n denotes vertical mode number. In the following we seek solutions with eigenvalues in the range

$$0 < \lambda < \lambda^* \tag{A.9e}$$

where

$$\lambda^* = \begin{cases} \mu & \text{if} \quad \kappa < 0, \\ \infty & \text{if} \quad \kappa \geq 0. \end{cases}$$

Define a sequence of functions $v^{(n)}$ by

$$\mathcal{L}[v^{(n+1)}] = \lambda[g(v^{(n)}, \mathrm{x}) + \kappa v^{(n)}] \quad \text{in} \quad \mathcal{D}_L, \tag{A.10}$$

$$v^{(n)} = 0 \quad \text{on} \quad \partial\mathcal{D}_L \tag{A.11}$$

with

$$v^{(0)} = V(x, y) \tag{A.12}$$

[4]That is, if ϕ is twice continuously differentiable and satisfies $\mathcal{L}[\phi] \geq 0, \not\equiv 0$ in $\mathcal{D}_L$ and $\phi = 0$ on $\partial\mathcal{D}_L$, then $\phi > 0$ in $\mathcal{D}_L$.

In (A.12), $V(x, y)$ is the solution of

$$\begin{aligned} -\nabla^2 V &= \lambda M(x,y) \quad \text{in } \mathcal{D}_L, \\ V &= 0 \quad \text{on } \partial\mathcal{D}_L. \end{aligned} \tag{A.13}$$

THEOREM 1. *For* $0<\lambda<\lambda^*$, *the iterates* $v^{(n)}$ *are uniformly bounded by* V *in the following manner*:

$$-V \leqslant v^{(n-1)} \leqslant v^{(n)} \leqslant V.$$

Proof: First we have

$$\mathcal{L}[v^{(1)}] = \lambda[g(V,\mathbf{x}) + \kappa V] \leqslant \lambda[M + \kappa V],$$

so $\mathcal{L}[v^{(1)}] \leqslant \mathcal{L}[V]$, $\mathcal{L}[v^{(1)}] \not\equiv \mathcal{L}[V]$. Therefore $v^{(1)} \leqslant V$ by the maximum principle for positive elliptic operators. Also, since

$$\begin{aligned} \nabla^2 v^{(1)} &= \lambda[-g(V,\mathbf{x}) + \kappa(v^{(1)} - V)] \\ &\leqslant -\lambda g(v^{(1)},\mathbf{x}) \leqslant \lambda M = -\nabla^2 V, \end{aligned}$$

then by the minimum principle for elliptic operators, $v^{(1)} \geqslant -V$. Having demonstrated that

$$-V \leqslant v^{(1)} \leqslant V, \tag{A.14}$$

we next show by induction that

$$-V \leqslant v^{(n-1)} \leqslant v^{(n)} \leqslant V, \qquad n = 1,2,3,\ldots. \tag{A.15}$$

If (A.15) holds for some n, then

$$\begin{aligned} \mathcal{L}[v^{(n+1)} - v^{(n)}] &= \lambda[-g(v^{(n-1)},\mathbf{x}) + g(v^{(n)},\mathbf{x}) - \kappa(v^{(n-1)} - v^{(n)})] \\ &\geqslant 0, \end{aligned}$$

so $v^{(n+1)} \geqslant v^{(n)} \geqslant -V$.

To show that $v^{(n+1)} \leqslant V$, consider

$$\begin{aligned} -\nabla^2 v^{(n+1)} &= \lambda[g(v^{(n)},\mathbf{x}) + \kappa(v^{(n)} - v^{(n+1)})] \\ &\leqslant \lambda g(v^{(n+1)},\mathbf{x}) \leqslant \lambda M = -\nabla^2 V. \end{aligned} \tag{A.16}$$

Thus $v^{(n+1)} \leqslant V$, and (A.15) is proved. □

Equation (A.15) implies that the functions $|v^{(n)}|$ are uniformly bounded in $\mathfrak{D}_L$ for all n. Omitting the technical details, which can be found in Courant and Hilbert, Vol. II, it can be shown that this fact is sufficient to guarantee the convergence of the sequence $\{v^{(n)}\}$. We state without proof the following result:

THEOREM 2. *The sequence* $\{v^{(n)}\}$ *defined by* (A.11) *and* (A.12) *is uniformly convergent in* $\mathfrak{D}_L$.

With this result it is easy to show that the sequence converges to a solution of (A.5).

THEOREM 3. *Let* $\hat{v}(\mathbf{x},\lambda)=\lim_{n\to\infty} v^{(n)}(\mathbf{x},\lambda)$ *for* λ *in the range* $0<\lambda<\lambda^*$; *then* $\hat{v}$ *is a solution to* (A.5).

Proof: Equation (A.10) can be solved to yield

$$\hat{v}^{(n+1)} = \lambda \iint_{\mathfrak{D}_L} G(\mathbf{x},\xi)\left[g\left(v^{(n)}(\xi),\xi\right)+\kappa v^{(n)}(\xi)\right] d\xi, \tag{A.17}$$

where G is the Green's function for the operator $\mathfrak{L}$ satisfying $G=0$ on $\partial\mathfrak{D}_L$. Since the limit $n\to\infty$ exists and the integrand in (A.17) is bounded, effecting the limit on both sides of (A.17) gives

$$\hat{v} = \lambda \iint_{\mathfrak{D}_L} G(\mathbf{x},\xi)\left[g(\hat{v}(\xi),\xi)+\kappa\hat{v}(\xi)\right] d\xi. \tag{A.18}$$

It is easily verified that $\hat{v}$ given by (A.18) is a solution of (A.5). Furthermore, it is bounded:

$$-V(\mathbf{x},\lambda) \leqslant \hat{v}(\mathbf{x},\lambda) \leqslant V(\mathbf{x},\lambda). \quad \square \tag{A.19}$$

Having shown that a solution to (A.5) exists, a question still remains: Is this solution different from the trivial one?

In Eq. (A.5) for v, $\phi(-L,y)$ enters as an *external* function determining the harmonic function ω through (A.2), and ω in turn enters into Eq. (A.5) through (A.4). Relation (A.3) of v to $\phi(x,y)$ is not needed and can be discarded. The system (A.5), (A.4), and (A.2) is self-determined as a function of the externally specified function $\phi(-L,y)$. From the known properties of the Laplace operator, the harmonic function ω exists and is different from zero if $\phi(-L,y)\not\equiv 0$. For a nontrivial ω (A.5) does *not* permit $v\equiv 0$ as a solution.[5] Therefore the solution $\hat{v}$

[5]Unless $\lambda\equiv 0$. However, we have from (A.1) that

$$\lambda\iint_{\mathfrak{D}_L} F'(y+\phi)\phi^2\,d\mathbf{x} = \iint_{\mathfrak{D}_L} |\nabla\phi|^2\,d\mathbf{x} > 0$$

if $\phi(-L,y)\not\equiv 0$. And, since the stratification $F'(y)$ is assumed to be "mostly" positive, we also have

$$\int_{\mathfrak{D}_L} F'(y+\phi)\phi^2\,d\mathbf{x} > 0.$$

It then follows that $\lambda>0$ and cannot be identically zero.

that we have proved to exist is a nontrivial solution. [If, on the other hand, $\phi(-L, y)$ is specified as zero, then the maximum principle for the Laplace operator implies that $\omega \equiv 0$. Consequently, $v \equiv 0$ *can* be a solution to (A.5). However, it is not known whether $\hat{v}$ is trivial, since the solutions to (A.5) may be nonunique.] Thus, specifying $\phi(-L, y)$ to be nonzero guarantees the existence of a solution $v = \hat{v} \neq 0$.

Because we do not have a priori knowledge of the sign of a periodic solution (v can, in fact, change sign in $\mathcal{D}_L$), we do not get as much information on the dependence of v on the eigenvalue λ as we can for the solitary wave case discussed in Sec. 4. Nevertheless, we do know that the dependence of $\lambda - \lambda^{(0)}$ (where $\lambda^{(0)}$ is the linear eigenvalue) on wave amplitude is quadratic for small amplitudes, similar to the case of nonlinear Stokes waves.

For small amplitudes, one can expand

$$\phi = a\phi_0 + a^2\phi_1 + a^3\phi_2 + O(a^4), \tag{A.20}$$

$$\lambda = \lambda^{(0)} + a\lambda^{(1)} + a^2\lambda^{(2)} + O(a^3), \tag{A.21}$$

where $|a| \ll 1$ is a measure of wave amplitude. The lowest order solution satisfies the linear equations

$$\begin{aligned} &\nabla^2\phi_0 + \lambda^{(0)}F'(y)\phi_0 = 0, \\ &\phi_0(-L, y) = \phi_0(L, y), \\ &\phi_0(x, -H_2) = 0 = \phi_0(x, H_1). \end{aligned} \tag{A.22}$$

The periodic solution can be written

$$\phi_0(x, y) = \cos kx\, \psi_0(y) \tag{A.23}$$

where

$$k = \pi/L > 0, \tag{A.24}$$

and $\psi_0(y)$ satisfies

$$\begin{aligned} &\frac{d^2}{dy^2}\psi_0 + \left[\lambda^{(0)}F'(y) - k^2\right]\psi_0 = 0, \\ &\psi_0 = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2. \end{aligned} \tag{A.25}$$

For the tanh y profile it can be shown that, for the mode-2 wave,

$$\psi_0(y) = \frac{\sinh y}{\cosh^{k+1} y}, \qquad \lambda^{(0)} = (k+1)(k+2),$$

if $H_1 \gg 1$ and $H_2 \gg 1$.

To the next order, we have

$$\nabla^2\phi_1 + \lambda^{(0)}F'(y)\phi_1 = -\lambda^{(0)}F''(y)\phi_0^2 - \lambda^{(1)}F'(y)\phi_0,$$
$$\phi_1(-L, y) = \phi_1(L, y), \tag{A.26}$$
$$\phi_1(x, -H_2) = \phi_1(x, H_1).$$

The solvability condition is

$$\lambda^{(1)}/\lambda^{(0)} = -\iint_{\mathcal{D}_L} F''(y)\phi_0^3\,d\mathbf{x} \Big/ \iint_{\mathcal{D}_L} F'(y)\phi_0^2\,d\mathbf{x} = 0 \tag{A.27}$$

since

$$\int_{-\pi}^{\pi} \cos^3 kx\,dx = 0,$$

so ϕ_1 is given by

$$\phi_1(x, y) = \psi_1(y) + \cos 2kx\,\psi_2(y), \tag{A.28}$$

where ψ_1 and ψ_2 satisfy

$$\frac{d^2}{dy^2}\psi_1 + \lambda^{(0)}F'(y)\psi_1 = -\tfrac{1}{2}\lambda^{(0)}F''(y)\psi_0^2,$$
$$\psi_1 = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2 \tag{A.29}$$

and

$$\frac{d^2}{dy^2}\psi_2 + \left[\lambda^{(0)}F'(y) - 4k^2\right]\psi_2 = -\tfrac{1}{2}\lambda^{(2)}F''(y)\psi_0^2,$$
$$\psi_2 = 0 \quad \text{at} \quad y = H_1 \text{ and } -H_2. \tag{A.30}$$

To third order in amplitude, one has

$$\nabla^2\phi_2 + \lambda^{(0)}F'(y)\phi_2 = -\lambda^{(0)}\left[\tfrac{1}{2}F'''(y)\phi_0^3 + 2F''(y)\phi_0\phi_1\right] - \lambda^{(2)}F'(y)\phi_0,$$
$$\phi_2(-L, y) = \phi_2(L, y), \qquad \phi_2(x, -H_2) = \phi_2(x, H_1). \tag{A.31}$$

The solvability condition for (A.31) is

$$\lambda^{(2)}/\lambda^{(0)} = -\frac{\iint_{\mathcal{D}_L}\left[\frac{1}{2}F'''(y)\phi_0^4+2F''(y)\phi_0^2\phi_1\right]d\mathbf{x}}{\iint_{\mathcal{D}_L}F'(y)\phi_0^2\,d\mathbf{x}}$$

$$= -\frac{\int_{-H}^{H}\left[\frac{3}{8}F'''(y)\psi_0^4+F''(y)(2\psi_1+\psi_2)\psi_0^2\right]dy}{\int_{-H}^{H}F'(y)\psi_0^2\,dy}$$

$$\equiv S_2. \tag{A.32}$$

Thus

$$\lambda = \lambda^{(0)}\left[1+a^2S_2+O(a^3)\right], \tag{A.33}$$

where S_2 can be explicitly calculated given $F'(y)$.

Acknowledgments

The authors would like to thank Professors David Benney of M.I.T. and Donald Cohen of Caltech for their many helpful critical comments, Dr. Denny Ko of dynamics Technology, Inc., for his direction, support, and encouragement, and Dr. Robert Gran who reviewed an earlier draft of the paper and made numerous suggestions for improvement of the text. The authors also acknowledge having useful conversations with Dr. Nino Pereira and Dr. Vincent Panico during the early stages of the work.

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(Received 17 March 1980)