Nongeostrophic Theory of Zonally Averaged Circulation. Part I: Formulation

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ABSTRACT

A nongeostrophic theory of zonally averaged circulation is formulated using the nonlinear primitive equations on a sphere, taking advantage of the more direct relationship between the mean meridional circulation and diabatic heating rate which is available in isentropic coordinates. Possible differences between results of nongeostrophic theory and the commonly used geostrophic formulation are discussed concerning (i) the role of eddy forcing of the diabatic circulation, and (ii) the "nonlinear nearly inviscid" limit versus the geostrophic limit.

A set of general diagnostic tools comparable in scope to their geostrophic counterparts is given in Part I, including (i) a generalized definition of Eliassen-Palm flux divergence (without restriction to small amplitudes, to steady state or to adiabatic flows), the vanishing of which is a necessary condition for nonacceleration, (ii) a generalized nonlinear Taylor formula that relates the flux of Ertel's potential vorticity to the Eliassen-Palm flux divergence and (iii) a relationship between the Eliassen-Palm flux divergence and isentropic mixing coefficient, K_{yy}, used in chemical tracer transport equations in isentropic coordinates. From the mean momentum budget, we give in Part II an estimate of the Eliassen-Palm flux divergence using fitted "observed" field of net radiative heating rate. From this an estimate of the magnitude and latitudinal/seasonal variation of K_{yy} is also provided.

1. Introduction

Great progress has been made in our understanding of the zonally averaged circulation in the atmosphere since the introduction of the concept of residual circulation by Andrews and McIntyre (1976) and Boyd (1976), which helps put in a proper perspective the role of "eddies" (i.e., deviations from zonal symmetry) in driving the mean (zonally symmetric) circulation. However, our present understanding is based largely on a geostrophic version of the general theory (see Edmon et al., 1980; Dunkerton et al., 1981; Palmer, 1981a,b; and Andrews et al., 1983). In its geostrophic form, the set of transformed zonally averaged equations of motion shows clearly that, in the absence of an Eliassen-Palm flux divergence, eddies do not accelerate the zonal mean flow, a result first recognized by Charney and Drazin (1961) and Dickinson (1969). A more controversial consequence of the geostrophic theory is that at equilibrium, an "almost frictionless" (molecular diffusion only say) stratosphere in the absence of large-scale eddies would be "extremely close" to *radiative equilibrium,* in which the absorption of solar insolation is simply balanced by an increase in local temperature, without inducing a meridional circulation (see Mahlman et al., 1984). It is therefore often concluded (see WMO/NASA, 1985, Chapter 6) that global pattern of rising and descending motions in the stratosphere "owes its existence to the presence of asymmetric motions".

While the role of the eddy driving of mean circulation is undeniably important, it is interesting to note that there exist a number of *zonally symmetric* calculations based on the nonlinear primitive (i.e., nongeostrophic) equations that produce realistic looking Hadley circulations in the absence of large-scale eddies but in the presence of small-scale mixing (i.e., viscosity) (Schneider and Lindzen, 1977; Schneider, 1977; Nakamura, 1978; Held and Hou, 1980).

While these "thought experiments" of a hypothetical axisymmetric atmosphere are useful in highlighting the qualitative difference between the geostrophic and nongeostrophic formulations, they do not address the more practical question: Is our atmosphere in the geostrophic regime? Or is the "nonlinear nearly inviscid" regime (Held and Hou, 1980) more applicable? These issues cannot be addressed using a formulation that adopts the a priori assumption of geostrophy.

It is an objective of the present work to point out (i) the different regimes an atmosphere can be in depending on the magnitudes of eddy forcing in a primitive equation formulation, and (ii) to find out which of these regimes our atmosphere is in during different seasons. Task (i) is discussed in the present Part I, while task (ii) is attempted in Part II.

Another objective of the present work is to develop a complete set of general diagnostic tools that are comparable in scope to the corresponding geostrophic diagnostics.[1] In order to make our diagnostics more easily

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[1] Such a set of diagnostics for nongeostrophic flow is hitherto not available, although a general (nongeostrophic) definition of the Eliassen-Palm flux divergence in pressure coordinates has been given by Andrews and McIntyre (1976, 1978a) and Boyd (1976) and used by Andrews et al. (1983).

adaptable to current procedures for data analysis, we have emphasized the derivation of relationships of various eddy diagnostic quantities to Ertel's potential vorticity (Ertel, 1942). McIntyre and Palmer (1983, 1984) have effectively argued for the usefulness of Ertel's potential vorticity on isentropic surfaces as a diagnostic tool for visualizing large-scale nonlinear dynamical processes in the stratosphere. Hoskins et al. (1985) have reviewed the use and significance of isentropic potential vorticity maps in a variety of atmospheric situations, not necessarily restricted to the stratosphere. In particular, they emphasized the Lagrangian conservation principle for Ertel's potential vorticity, which presumably is better conserved than that of the quasi-geostrophic potential vorticity along isobaric surfaces (Charney and Stern, 1962). In addition to the two main "principles" associated with the use of isentropic potential vorticity, namely, the conservation principle just mentioned, and the "invertibility" principle (see Hoskins et al., 1985), we add further that (i) the use of isentropic maps of potential vorticity allows us to diagnostically calculate the Eliassen–Palm flux pseudo-divergence, which represents the net eddy forcing of the zonal mean flow in isentropic coordinates, and (ii) the Lagrangian conservative properties of the isentropic potential vorticity endows it with the property of a passive (but nonconservative) tracer; the dual (dynamical and passive) character of the isentropic potential vorticity allows us, in principle, to deduce the transport characteristics of the atmosphere from its momentum budget.

Although the relationship between the flux of *geostrophic* potential vorticity and *geostrophic* Eliassen–Palm flux divergence is well-known in pressure coordinates (the so-called Taylor relationship, see Edmon et al., 1980, and references therein), the corresponding relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen–Palm flux divergence does not appear to have been derived. Partly as a consequence, Clough et al. (1985) have to resort to a comparison of the *isentropic* maps of Ertel's potential vorticity with the Eliassen–Palm flux divergence calculated based on the quasi-geostrophic definition of Edmon et al. (1980) for *pressure coordinates.* Although this somewhat inconsistent procedure is at present largely dictated by the retrieval process for the observational data, it would have been conceptually clearer if the relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen–Palm flux divergence in the same coordinates had been known and a direct (albeit approximate) comparison made of these two quantities. This is especially true for diagnosing model generated data: Although the numerical data are generated using a primitive equation model, the diagnostics by Dunkerton et al. (1981) are again based on the quasi-geostrophic formulation of Edmon et al. (1980).

A relationship between the flux of Ertel's potential vorticity along isentropic surfaces and the Eliassen–Palm flux divergence is here obtained[2] without the quasi-geostrophic approximation. This crucial relationship links the flux of a quasi-conservative wave property to the net wave forcing of the mean flow without restriction to small wave amplitudes, thus making it a useful diagnostic tool applicable even to the "surf zones" (McIntyre and Palmer, 1983) in the stratosphere.

The existence of a link between the net eddy forcing term in the mean zonal momentum budget and the Ertel's potential vorticity, which is quasi-conservative, allows us to deduce from the mean momentum budget and mean isentropic gradient of potential vorticity a quantity, K_{yy}, that appears ubiquitously in chemical tracer transport equations in isentropic coordinates as the isentropic diffusion coefficient (see Tung, 1982, 1984; Ko et al., 1985). Thus it appears that an assessment of the magnitude of the Eliassen–Palm flux divergence could also possibly afford us a preliminary look at the magnitude and latitudinal distribution of K_{yy}, that has been difficult to obtain by direct evaluation from Eulerian data on transient waves. It also turns out that the use of Ertel's potential vorticity in isentropic coordinates to deduce K_{yy} is less problem-prone than a similar procedure in isobaric coordinates using quasi-geostrophic potential vorticity (Newman et al., 1985, personal communication). This, diagnostic, part of the work is discussed in Part II.

The use of isentropic coordinates turns out to be important in our present formulation of a nongeostrophic theory of zonally averaged circulation. Although many of our formulae may have their counterparts in pressure coordinates, the role played by the eddy Eliassen–Palm flux divergence in the forcing of the residual zonal mean circulation becomes more difficult to understand in pressure coordinates due to the presence of mean ageostrophic meridional circulations. The simple, direct relations between the Eliassen–Palm flux divergence and the geostrophic residual mean circulation in pressure coordinates, as discussed succinctly by Edmon et al. (1980), are no longer available for nongeostrophic flows. Such a problem does not appear in isentropic coordinates. For our purposes, the advantage of using potential temperature

$$\theta \equiv T\left(\frac{p_{00}}{p}\right)^{R/c_p}$$

as the vertical coordinate (instead of pressure p, say) lies in the direct relationship between the vertical "velocity", $\dot{\theta} \equiv d\theta/dt$, and the diabatic heating rate, Q, as given by the thermodynamics equation

$$\dot{\theta} = \frac{\theta}{T} Q.$$

[2] Andrews (personal communication, 1986) has independently derived a formula similar to the present one.

Thus, there should be no mean circulation in the latitude–height plane for an atmosphere in radiative equilibrium ($Q \equiv 0$). Conversely the presence of a mean meridional circulation, driven by, say, eddy forcings in the form of Eliassen–Palm flux divergence, necessarily induces a diabatic heating, leading to an out-of-equilibrium situation for the atmosphere. Such a direct relationship between radiation and dynamics is one of the many reasons why it is *convenient* to adopt the isentropic coordinates for our discussions to follow. The relevant equations will also be recast into log-pressure coordinates later in appendix B.

2. The two-dimensional equations

A listing of the set of primitive equations in isentropic coordinates can be found, for example, in appendix A of Tung (1982). [There is however a typographical error in Eq. (A4) there, where $1/\cos\varphi$ should read $\cos\varphi$]. The so called two-dimensional (2-D) equations are obtained by taking the zonal average of the 3-D equations by the operation

$$\bar{h}(\varphi,\theta) \equiv \frac{1}{2\pi}\int_0^{2\pi} h(\lambda,\varphi,\theta)d\lambda, \qquad (2.1)$$

where λ is the longitude, φ the latitude and θ is the potential temperature, used here as the vertical coordinates. We shall use primes to denote deviations from zonal average, i.e.,

$$h' \equiv h - \bar{h}. \qquad (2.2)$$

The primed quantities arise from asymmetric perturbations, and are referred to here as "eddies" or "waves". In isentropic coordinates it is sometimes more useful to use density-weighted zonal averages, defined as (see Gallimore and Johnson, 1981a,b),

$$\hat{h} \equiv \overline{\rho_\theta h}/\bar{\rho}_\theta, \qquad (2.3)$$

where

$$\rho_\theta \equiv \rho\frac{\partial z}{\partial \theta}$$

is the density in isentropic coordinates, i.e., it is mass per unit pseudovolume (with dz replaced by $d\theta$). The deviation from the density-weighted average is denoted by an asterisk, i.e.,

$$h^* \equiv h - \hat{h}. \qquad (2.4)$$

(Note that $\overline{\rho_\theta h^*} \equiv 0$, but $\overline{h^*} \neq 0$ in general.) The difference between h^* and h' is $(\overline{\rho_\theta' h'})/\bar{\rho}_\theta$, which is quadratic in wave amplitudes. The distinction between the two is significant only for finite amplitude waves. But since we are interested in deriving relations that hold in the presence of finite amplitude perturbations, the distinction will be made in the present work.

As in Tung (1982), we use capital letters to denote mass flow rates, thus

$$U \equiv \rho_\theta u, \quad V \equiv \rho_\theta v\cos\varphi, \quad W \equiv \rho_\theta\dot{\theta}. \qquad (2.5)$$

With these definitions, the zonally averaged equations take the following form.

a. Equation of mass conservation

$$\frac{\partial}{\partial t}\bar{\rho}_\theta + \frac{\partial}{\partial y}\bar{V} + \frac{\partial}{\partial \theta}\bar{W} = 0 \qquad (2.6)$$

where $y \equiv a\sin\varphi$ (see Tung, 1982).

b. Thermodynamics equation

The thermodynamics equation

$$\frac{d}{dt}\ln\theta = Q/T$$

can be written as

$$\bar{W} = (\overline{q/\Gamma}) \approx \bar{q}/\bar{\Gamma} \qquad (2.7)$$

where $q \equiv \rho Q$ is the diabatic heating rate per unit physical volume divided by c_p, and Γ is the static stability parameter defined as

$$\Gamma \equiv \frac{T}{\theta}\frac{\partial \theta}{\partial z}.$$

The derivation of (2.7) was previously given by Tung (1982):

> To circumvent any reference to the height coordinate z in our present formulation in θ-coordinates, we note that
>
> $$\Gamma^{-1} = \frac{\theta}{T}\frac{\partial z}{\partial \theta} = \frac{\theta}{\rho T}\rho_\theta = -\frac{\theta}{g\rho T}\frac{\partial}{\partial \theta}p,$$
>
> where the definition $\rho_\theta \equiv \rho\partial z/\partial\theta$ and the hydrostatic relationship
>
> $$g\rho_\theta = -\frac{\partial p}{\partial \theta}$$
>
> have been used. A further use of the ideal gas law; $p = \rho RT$, then yields
>
> $$\Gamma^{-1} = -\left(\frac{R}{g}\right)\frac{\partial \ln p}{\partial \ln\theta} = \frac{c_p}{g}\left(1 - \frac{\partial \ln T}{\partial \ln\theta}\right).$$
>
> This last expression is the one we will use, assuming $T = T(\lambda, \varphi, \theta)$. However, in dealing with observational data, we usually have $\theta = \theta(\lambda, \varphi, p)$, in which case Γ can be obtained from
>
> $$\Gamma = -\frac{g}{R}\frac{\partial \ln\theta}{\partial \ln p} = \frac{g}{c_p} + \frac{\partial T}{H\partial \ln(p_{00}/p)},$$
>
> where
>
> $$H = RT/g.$$

c. Zonal momentum equation

$$\frac{\partial}{\partial t}(\bar{\rho}_\theta\hat{u}\cos\varphi) + \frac{\partial}{\partial y}(\bar{V}\hat{u}\cos\varphi) + \frac{\partial}{\partial \theta}(\bar{W}\hat{u}\cos\varphi) - f\bar{V} = \nabla\cdot\mathbf{F} + \bar{\rho}_\theta\hat{\mathcal{D}}\cos\varphi, \qquad (2.8a)$$

where $f \equiv 2\Omega \sin\varphi$ is the Coriolis parameter and $\mathcal{D}$ is the small-scale frictional force per unit mass for the zonal momentum. The derivation of Eq. (2.8) is given in appendix A.

1) Eliassen–Palm flux divergence

Of importance in our present discussion is the first term on the right-hand side of Eq. (2.8a), which represents a generalized version of the so-called Eliassen–Palm flux divergence. Written in component form (recalling $y \equiv a \sin\varphi$), it is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial \theta} F_\theta$$

with

$$F_y \equiv -\overline{\rho_\theta u^* v^*} \cos^2\varphi$$

and

$$F_\theta \equiv -\cos\varphi \left[\overline{\rho_\theta u^* \dot{\theta}^*} - \frac{1}{g} \overline{p' \frac{\partial}{\partial x} \Phi'} \right] \tag{2.9a}$$

($\partial/\partial x$ is defined as $\partial/a \cos\varphi \partial\lambda$, and Φ denotes the Montgomery streamfunction).

A physical interpretation of $\nabla \cdot \mathbf{F}$ can be given, following Andrews (1983) but generalizing to include diabatic eddies: $\nabla \cdot \mathbf{F}$ can be viewed as the x-component of the force exerted by eddies on a thin tube with its axis oriented along the x-direction bounded by the fixed lateral sides which are located at y and $y + dy$, and undulating bottom and top isentropes θ and $\theta + d\theta$. The horizontal component of the Eliassen–Palm flux, F_y, represents lateral flux of zonal momentum into the tube. The first term in the vertical component F_θ represents the vertical, i.e., cross isentropic, flux of zonal momentum into the tube, while the second term in F_θ represents the x-projection of the net pressure force pushing against the (slanted) isentropes (the so-called form drag). In addition to eddy forcing, the thin tube in general also experiences fluxes of mean absolute zonal momentum by the mean flow ($\bar{V}$, $\bar{W}$) and any momentum dissipative forces that may present.

The use of $(\ )^*$ instead of $(\ )'$ is necessary in (2.9a) in order for (2.8a) to hold at finite amplitudes. For small amplitude disturbances, the difference between $(\ )^*$ and $(\ )'$ becomes asymptotically small so that $(1/\rho_\theta)F_y \approx -\overline{u'v'} \cos^2\varphi$ is the usual horizontal momentum flux by the eddies.

The Eliassen–Palm flux divergence defined in (2.9a) appears to be the most general form possible. (It is essentially the same as in Tung, 1982, section 5, but with several typographical errors corrected.) Like in Andrews (1983), it is valid without restriction to small amplitude perturbations and quasi-geostrophic scaling. However, unlike in Andrews (1983), where *adiabatic* mean and eddy flows are assumed a priori, the present definition is valid for a general diabatic atmosphere. Furthermore, due to his use of the zonally averaged zonal momentum instead of the *density-weighted* zonal average, the eddy flux terms in his zonal momentum equation cannot be expressed in a pure divergence form as is done in our Eq. (2.8a), except under nonacceleration conditions. The present form permits the interpretation that eddy fluxes act to *redistribute* mean angular momentum (without net creation), and that when integrated over a volume bounded by a surface with no net outward eddy fluxes, eddy forcing of mean flow vanishes.

Nevertheless, the not-in-divergence form of eddy forcing terms of the *zonally averaged* momentum (which cannot be called an Eliassen–Palm flux divergence) is useful in a different aspect. It turns out that an exact relationship exists between this "Eliassen–Palm flux pseudodivergence", as we shall call it, and the flux of Ertel's potential vorticity along isentropic surfaces. Since the flux of Ertel's potential vorticity is not in general in a divergence form, no exact relationship exists between it and the Eliassen–Palm flux divergence defined in (2.9a).

2) Eliassen–Palm pseudodivergence

We shall now generalize Andrews' (1983) expression for the eddy forcing term for the *zonally averaged* zonal momentum to include diabatic terms. Using the zonal average of u instead of the density-weighted average of u, we have, instead of (2.8a), the following

$$\frac{\partial}{\partial t}(\bar{\rho}_\theta \bar{u} \cos\varphi) + \frac{\partial}{\partial y}(\bar{V} \bar{u} \cos\varphi) + \frac{\partial}{\partial \theta}(\bar{W} \bar{u} \cos\varphi) - f \bar{V} = \square \cdot \mathcal{F} + \bar{\rho}_\theta \bar{\mathcal{D}} \cos\varphi, \tag{2.8b}$$

where $(1/\bar{\rho}_\theta)\square \cdot \mathcal{F}$ represents the eddy forcing of the zonally averaged absolute angular momentum, and is given by

$$\frac{1}{\bar{\rho}_\theta} \square \cdot \mathcal{F} \equiv -\overline{v' \cos\varphi \frac{\partial}{\partial y} u' \cos\varphi} - \left(f - \frac{\partial}{\partial y} \bar{u} \cos\varphi \right) \overline{v' \cos\varphi \frac{\rho_\theta'}{\bar{\rho}_\theta}} - \overline{\dot{\theta}' \frac{\partial}{\partial \theta} u' \cos\varphi} + \frac{1}{\bar{\rho}_\theta} \overline{\rho_\theta' \dot{\theta}'} \frac{\partial}{\partial \theta} \bar{u} \cos\varphi. \tag{2.9b}$$

By comparing (2.8b) with (2.8a), and noting that

$$\hat{u} - \bar{u} = \overline{\rho_\theta' u'}/\bar{\rho}_\theta,$$

we see that

$$\square \cdot \mathcal{F} = \nabla \cdot \mathbf{F} - \frac{\partial}{\partial y}(\bar{V} \overline{\rho_\theta' u'} \cos\varphi/\bar{\rho}_\theta) - \frac{\partial}{\partial \theta}(\bar{W} \overline{\rho_\theta' u'} \cos\varphi/\bar{\rho}_\theta) - \frac{\partial}{\partial t} \overline{\rho_\theta' u'} \cos\varphi, \tag{2.10}$$

so that in component form, the "pseudodivergence" is given by

$$\Box\cdot\mathcal{F}=\frac{\partial}{\partial y}\mathcal{F}_y+\frac{\partial}{\partial\theta}\mathcal{F}_\theta+\frac{\partial}{\partial t}\mathcal{F}_t,$$

where

$$\left.\begin{aligned}\mathcal{F}_y&=F_y-\frac{1}{\bar\rho_\theta}\bar V\overline{\rho_\theta' u'}\cos\varphi=-\overline{V'u'}\cos\varphi\\ \mathcal{F}_\theta&=F_\theta-\frac{1}{\bar\rho_\theta}\bar W\overline{\rho_\theta' u'}\cos\varphi\\ &=-\overline{W'u'}\cos\varphi+\frac{1}{g}\overline{p'\frac{\partial}{a\partial\lambda}\Phi'}\\ \mathcal{F}_t&=-\overline{\rho_\theta' u'}\cos\varphi.\end{aligned}\right\}\quad(2.11)$$

The above expressions reduce to those of Andrews (1983) for adiabatic waves. The presence of the last time-derivative term in (2.10) is what makes $\Box\cdot\mathcal{F}$ not a pure divergence. (The same situation appears also in Andrews', 1983, expression for adiabatic atmospheres.) In pressure coordinates, there is no distinction between (2.9a) and (2.9b), because the "density", ρ_p, in pressure coordinates is $-1/g$ and so there is no density perturbation, i.e., $\rho_p'=0$. In the present case, $\Box\cdot\mathcal{F}$ can be put into a divergence form only if it is integrated with respect to time over a lifecycle of a conservative temporary disturbance.

Conservation of absolute angular momentum. A statement of the conservation of angular momentum appears to be more obvious if Eq. (2.8a) is combined with Eq. (2.6) to yield

$$\bar\rho_\theta\frac{\partial}{\partial t}\hat L+\bar V\frac{\partial}{\partial y}\hat L+\bar W\frac{\partial}{\partial\theta}\hat L=\nabla\cdot\mathbf{F}+\bar\rho_\theta\hat D\cos\varphi,\quad(2.12a)$$

where La is the absolute angular momentum per unit mass, with L defined as

$$L\equiv[u+\Omega a\cos\varphi]\cos\varphi.$$

When $\bar L$ is used instead of $\hat L$, we have

$$\bar\rho_\theta\frac{\partial}{\partial t}\bar L+\bar V\frac{\partial}{\partial y}\bar L+\bar W\frac{\partial}{\partial\theta}\bar L=\Box\cdot\mathcal{F}+\bar\rho_\theta\bar{\mathcal{D}}\cos\varphi.\quad(2.12b)$$

In the following we shall complete the listing of the zonally averaged equations for use in our later computations. Some of the additional approximations that will be introduced do not affect our above discussion on Eliassen–Palm fluxes, which are based only on the zonal momentum equation without approximation (beyond that made in deriving the primitive equations).

d. Meridional momentum equation

It appears, as shown in appendix B, that for a fast rotating planet like the earth, the dominant terms in the mean meridional momentum equation is expressed in the following balance

$$\tilde f\bar u=-\cos\varphi\frac{\partial}{\partial y}\bar\Phi\quad(2.13)$$

where

$$\tilde f\equiv\left(2\Omega+\frac{\bar u}{a\cos\varphi}\right)\sin\varphi$$

and Φ is the Montgomery streamfunction. Equation (2.13) becomes the geostrophic relationship if $\tilde f$ is approximated by f.

Within the same degree of approximation as (2.13), $\bar u$ can be replaced by $\hat u$ if (2.8a) is used instead of (2.8b).

e. Hydrostatic relationship

$$c_p\bar T=\theta\frac{\partial}{\partial\theta}\bar\Phi.\quad(2.14)$$

f. Balanced wind relationship

Combining (2.13) and (2.14), we find a diagnostic relationship between $\bar u$ and $\bar T$:

$$2\left(\Omega+\frac{\bar u}{a\cos\varphi}\right)\sin\varphi\theta\frac{\partial}{\partial\theta}\bar u=-\cos\varphi\frac{\partial}{\partial y}c_p\bar T.\quad(2.15)$$

Again, $\hat u$ can replace $\bar u$ in Eq. (2.15).

When the mean relative angular frequency $\bar u/(a\times\cos\varphi)$ is neglected when compared with the planetary rotational frequency Ω, (2.15) becomes the thermal wind relationship

$$f\theta\frac{\partial}{\partial\theta}\bar u=-\cos\varphi\frac{\partial}{\partial y}c_p\bar T,\quad(2.16)$$

which is a good approximation to (2.15) for Earth's atmosphere.

g. Equation of state

Using the hydrostatic equation

$$g\rho_\theta=-\frac{\partial}{\partial\theta}p\quad(2.17)$$

and the relationship between temperature and pressure from the definition of potential temperation

$$\theta\equiv T\left(\frac{p_{00}}{p}\right)^{R/c_p},$$

we can obtain a relationship between density and temperature as

$$\rho_\theta=-\frac{p_{00}}{g}\frac{\partial}{\partial\theta}\left(\frac{T}{\theta}\right)^{c_p/R}.\quad(2.18)$$

To facilitate the taking of zonal averages, temperature is decomposed into a radiative equilibrium state, $T_e(y,\theta)$, and deviations from it, ΔT, i.e.,

$$T\equiv T_e(y,\theta)+\Delta T,$$

and it is assumed that ΔT is small compared to $T_e(y,\theta)$. Then (2.18) yields

$$\bar{\rho}_\theta = -\frac{p_{00}}{g}\frac{\partial}{\partial\theta}\left[\left(\frac{T_e}{\theta}\right)^{c_p/R}\left(1+\frac{c_p}{R}\frac{\Delta\bar{T}}{\bar{T}_e}\right)\right]. \quad (2.19)$$

The assumption that $\Delta T/T_e(y, \theta)$ is small appears to be valid in the stratosphere even for finite amplitude disturbances. The largest deviation from radiative equilibrium temperature occurs in the Northern Hemisphere during polar night, with $\Delta\bar{T} \sim 20°$ to 30°K in climatological averages. Even here, $\Delta\bar{T}/T_e$ can still be considered small, except of course, during episodes of sudden warming.

For future reference, we list the expression for the perturbation density

$$\rho_\theta' = -\frac{p_{00}}{g}\frac{\partial}{\partial\theta}\left[\frac{c_p}{R}\left(\frac{T_e}{\theta}\right)^{c_p/R}\frac{T'}{T_e}\right]. \quad (2.20)$$

3. Nonacceleration theorem

The mean angular momentum equation (2.12a) is in the *form* of the so-called "generalized Eliassen–Palm theorem" (see Andrews and McIntyre, 1976, 1978a,b; Killworth and McIntyre, 1985), although we have not shown that our $-\hat{L}$ is a quadratic wave property. Furthermore, our $-\hat{L}$ is a momentum, not a wave pseudomomentum (see McIntyre, 1981). So strictly speaking (2.12a) is not a "generalized Eliassen–Palm Theorem." This is however not important for our purpose of obtaining necessary condition for nonacceleration. If we let

$$\frac{\hat{D}}{Dt} \equiv \frac{\partial}{\partial t} + \hat{v}\frac{\partial}{a\partial\varphi} + \hat{\dot{\theta}}\frac{\partial}{\partial\theta} \quad (3.1)$$

be the substantial derivative following the density-weighted mean circulation

$$(\hat{v}, \hat{\dot{\theta}}) \equiv \left(\frac{\bar{V}}{\bar{\rho}_\theta \cos\varphi}, \frac{\bar{W}}{\bar{\rho}_\theta}\right),$$

then Eq. (2.12a) is in the form

$$\bar{\rho}_\theta\frac{\hat{D}}{Dt}\hat{L} - \nabla\cdot\mathbf{F} = \bar{\rho}_\theta\hat{\mathcal{D}}\cos\varphi. \quad (3.2)$$

The advantage of using isentropic coordinates becomes clear when we consider the situation under radiative equilibrium conditions. In the absence of diabatic heating, the thermodynamics equation $\bar{W} = \bar{q}/\bar{\Gamma}$ implies

$$\bar{W} \equiv 0. \quad (3.3)$$

The continuity equation

$$\frac{\partial}{\partial t}\bar{\rho}_\theta + \frac{\partial}{\partial y}\bar{V} + \frac{\partial}{\partial\theta}\bar{W} = 0$$

together with a polar boundary condition

$$\bar{V} = 0 \quad \text{at} \quad y = a \text{ or } -a$$

then suggests that

$$\bar{V} \equiv 0 \quad (3.4)$$

at equilibrium. That is, there is no mean meridional circulation at radiative equilibrium. The same conclusion can be drawn in pressure coordinates only if geostrophic approximation is made in the zonal momentum equation. In nongeostrophic form in pressure coordinates, the presence of the advection of mean temperature by ageostrophic mean circulation complicates the relationship between diabatic heating and the mean meridional circulation. Because of the direct relationship that exists in isentropic coordinates between the mean meridional circulation and the presence of nonconservative process (such as diabatic heating), Eq. (3.2) can be written as

$$\bar{\rho}_\theta\frac{\partial}{\partial t}\hat{L} - \nabla\cdot\mathbf{F} = \bar{\rho}_\theta\hat{\mathcal{D}}\cos\varphi - \left(\bar{V}\frac{\partial}{\partial y}\hat{L} + \bar{W}\frac{\partial}{\partial\theta}\hat{L}\right) \quad (3.5)$$

where the right-hand side of Eq. (3.5) should vanish in the absence of nonconservative processes (frictional and diabatic forces). Using (3.5) we find that a *necessary condition for nonacceleration in a conservative atmosphere is the vanishing of Eliassen–Palm flux divergence:*

$$\nabla\cdot\mathbf{F} = 0 \quad (3.6)$$

along isentropes that extend to either the north or south pole. Since

$$\Box\cdot\mathcal{F} = \nabla\cdot\mathbf{F}$$

under "nonacceleration conditions",

$$\Box\cdot\mathcal{F} = 0 \quad (3.7)$$

is also a necessary condition for nonacceleration.

Equation (3.7) is similar to the condition derived by Andrews (1983) and appears to be the most general version of nonacceleration theorems presently available in Eulerian coordinates. It is valid for finite amplitude waves satisfying primitive equations on a sphere.

It should be pointed out that (3.6) is only a necessary condition for nonacceleration. Vanishing of the Eliassen–Palm flux divergence is in general not sufficient for nonacceleration. Furthermore, although adiabaticity, $\bar{q} = 0$, implies $\nabla\cdot\mathbf{F} = 0$ at steady state, as mentioned before, the converse is not necessarily true. That is, the vanishing of the Eliassen–Palm flux divergence does not guarantee that at steady state the atmosphere should be in local radiative (or radiative–convective) equilibrium, with $\bar{q} = 0$. This observation has important implications concerning the presumed role of large-scale waves in driving the atmosphere out of a radiative equilibrium. We will come back to this point in a later section.

Incidentally, it can be shown that in an adiabatic, frictionless atmosphere (which, however, may not necessarily be in equilibrium), our generalized Eliassen–Palm relationship reduces to

$$\frac{\partial}{\partial t}\bar{\rho}_\theta\hat{L} + \frac{\partial}{\partial y}(\hat{v}\cos\varphi\bar{\rho}_\theta\hat{L} + \overline{\rho_\theta u^* v^*}\cos^2\varphi) = 0. \quad (3.8)$$

This is in the same form as the finite-amplitude Eliassen–Palm relationship derived for a barotropic and quasi-geostrophic atmosphere by Killworth and McIntyre (1985), provided that $\bar{\rho}_\theta \bar{L}$ here is identified with their $-\bar{A}$. It should, however, be mentioned that the two quantities are not the same physically. It is just that their *fluxes* are the same so that it is *as if* the waves are carrying momentum (McIntyre, 1981).

4. Nonlinear nongeostrophic Taylor relationship

The relationship between the quasi-geostrophic Eliassen–Palm flux divergence in pressure coordinates and the flux of quasi-geostrophic potential vorticity is well known (Charney and Stern, 1962; Dickinson, 1969; Green, 1970; Edmon et al., 1980). If the quasi-geostrophic potential vorticity can be assumed to be conserved along isobaric surfaces then a parameterization of the flux, and hence of the Eliassen–Palm flux divergence, can be made in terms of the mean gradient of potential vorticity along isobaric surfaces. Such a procedure appears to be problematic because (i) the geostrophic assumption is too restrictive as it cannot account for mixing by gravity and equatorial waves in the stratosphere, (ii) quasi-geostrophic potential vorticity is in general a less-well conserved quantity along isobaric surfaces in the stratosphere than Ertel's potential vorticity along isentropes, and (iii) the quasi-geostrophic scaling requires that the cross-isobaric diffusion terms (such as K_{yz} and K_{zz}) be neglected; this is perhaps not justifiable (see Tung, 1984) in pressure coordinates, except in limited regions in the midstratosphere (see Newman et al., personal communication, 1985).

For these reasons it appears to be desirable to obtain a relationship between the nongeostrophic expression we have for the Eliassen–Palm flux divergence and the flux of Ertel's potential vorticity in isentropic coordinates. Then making use of the quasi-conservative nature of the isentropic potential vorticity, a parameterization of the eddy momentum forcing term (Eliassen–Palm flux divergence) can possibly be made.

a. Ertel's potential vorticity

Ertel's potential vorticity is defined as (e.g., see Pedlosky, 1979),

$$\Pi \equiv \frac{\zeta + f}{\rho_\theta} \tag{4.1}$$

where

$$\zeta \equiv \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u \cos\varphi$$

is the relative vorticity, and

$$\rho_\theta \equiv \rho \frac{\partial z}{\partial \theta} = -\frac{1}{g}\frac{\partial p}{\partial \theta}$$

is the "density" in isentropic coordinates. [Recall also, $y \equiv a \sin\varphi$, $\partial/\partial x \equiv (a\cos\varphi)^{-1}\partial/\partial\lambda$].

It can easily be shown, using (2.4), that the following identity holds for the perturbation (from its density-weighted average) potential vorticity

$$\rho_\theta \Pi^* = \zeta' - \left(f - \frac{\partial}{\partial y}\bar{u}\cos\varphi\right)\rho_\theta'/\bar{\rho}_\theta. \tag{4.2}$$

[Note that although (4.2) is of the same form as the linearized expression for $\bar{\rho}_\theta \Pi'$, no small amplitude approximation is made in deriving it]. Using the identity that

$$\overline{a^* b'} = \overline{a' b'},$$

we find that the northward flux of Ertel's potential vorticity is given by

$$\bar{\rho}_\theta \overline{(\rho_\theta \Pi^*) v^*}\cos\varphi = \bar{\rho}_\theta \overline{\rho_\theta \Pi^* v'}\cos\varphi$$

$$= -\bar{\rho}_\theta \overline{v' \cos\varphi \frac{\partial}{\partial y} u' \cos\varphi} - \left(f - \frac{\partial}{\partial y}\bar{u}\cos\varphi\right)\overline{\rho_\theta' v'}\cos\varphi. \tag{4.3}$$

The expression in (4.3) is to be compared with the Eliassen–Palm flux pseudodivergence, which from (2.9b) is

$$\Box \cdot \mathcal{F} = -\bar{\rho}_\theta \overline{v' \cos\varphi \frac{\partial}{\partial y} u' \cos\varphi} - \left(f - \frac{\partial}{\partial y}\bar{u}\cos\varphi\right)$$
$$\times \overline{\rho_\theta' v'}\cos\varphi - \bar{\rho}_\theta \overline{\dot{\theta}' \frac{\partial}{\partial \theta} u' \cos\varphi} + \overline{\rho_\theta' \dot{\theta}'}\frac{\partial}{\partial \theta}\bar{u}\cos\varphi. \tag{4.4}$$

b. Adiabatic waves

For *adiabatic waves* ($\dot{\theta}' = 0$), (4.3) and (4.4) are the same, so

$$\Box \cdot \mathcal{F} = \bar{\rho}_\theta \overline{(\rho_\theta \Pi^*) v^*}\cos\varphi. \tag{4.5}$$

Equation (4.5) provides an alternate diagnostic expression for the Eliassen–Palm flux pseudodivergence in (2.8b) (the net eddy forcing of the mean flow) which should be valid for time scales less than radiative damping times. Since no small amplitude assumption has been made, (4.5) should probably hold even for large-amplitude "breaking" planetary waves observed in the stratosphere (McIntyre and Palmer, 1983, 1984; Leovy et al., 1985), to the extent that such waves tend to mix predominantly along isentropes. However, the expression should not be expected to hold in the mesosphere in the presence of significant *breaking* gravity waves, which can mix momentum across isentropes. Note also that for adiabatic waves, (4.5) is an identity and its validity does not depend on the exact form of momentum dissipation adopted for the waves.

c. Diabatic waves

To give an estimate of the neglected terms in (4.5), we first write down the full expression without approximation,

$$\Box \cdot \mathcal{F} = \bar{\rho}_\theta \overline{(\rho_\theta \Pi^*) v^*}\cos\varphi - \bar{\rho}_\theta \overline{\dot{\theta}' \frac{\partial}{\partial \theta} u' \cos\varphi}$$
$$+ \overline{\rho_\theta' \dot{\theta}'}\frac{\partial}{\partial \theta}\bar{u}\cos\varphi. \tag{4.6}$$

The last two terms in (4.6), denoted by Δ, will be estimated as follows. Since

$$\bar{\rho}_\theta^2\left(\frac{1}{\rho_\theta}\frac{\partial}{\partial\theta}u\right)' \approx \bar{\rho}_\theta\frac{\partial}{\partial\theta}u' - \rho_\theta'\frac{\partial}{\partial\theta}\bar{u},$$

we have

$$\Delta \equiv -\overline{\bar{\rho}_\theta\dot{\theta}'\frac{\partial}{\partial\theta}u'}\cos\varphi + \overline{\rho_\theta'\dot{\theta}'}\frac{\partial}{\partial\theta}\bar{u}\cos\varphi \approx -\overline{\bar{\rho}_\theta^2\dot{\theta}'\left(\frac{1}{\rho_\theta}\frac{\partial}{\partial\theta}u\right)'}.$$

The eddy cross-isentropic velocity $\dot{\theta}'$ is estimated using the Newtonian cooling expression (see Tung, 1984):

$$\dot{\theta}' = -\frac{\alpha\theta T'}{\bar{T}}, \quad \text{where} \quad \alpha^{-1} \sim 5 \text{ days}$$

and the vertical derivative of u is estimated using the thermal wind relationship. Thus

$$\frac{\Delta}{\bar{\rho}_\theta} \approx -\frac{\alpha}{f_0\bar{T}}c_p\frac{\partial}{a\partial\varphi}\overline{\frac{1}{2}T'^2}.$$

Using a value of $T' \sim 10°\text{C}$ and a horizontal scale of a, we find

$$\frac{\Delta}{\bar{\rho}_\theta} \sim 10^{-6} \text{ m s}^{-2}.$$

This is about two orders of magnitude smaller than the Eliassen–Palm flux divergence, which appears to be of the order of

$$\Box\cdot\mathcal{F}/\bar{\rho}_\theta \sim 10^{-4} \text{ m s}^{-2}$$

during winter, in the extratropics, based on its value deduced by Clough et al. (1985) in pressure coordinates.

Thus, it appears that the relationship (4.5), which was found for adiabatic waves, should continue to hold for diabatic waves to a good degree of accuracy. Thus, we have

$$\Box\cdot\mathcal{F} \approx \bar{\rho}_\theta\overline{(\rho_\theta\Pi^*)v^*}\cos\varphi. \quad (4.7)$$

This is the diagnostic relationship that we have sought as a generalized version of the geostrophic Taylor (1915) relation (Edmon et al., 1980) commonly used.

5. A parameterization of Eliassen–Palm flux pseudo-divergence

To incorporate the effect of diabatic as well as adiabatic eddies, we start with the nonconservative form of Ertel's potential vorticity equation (see Pedlosky, 1979):

$$\rho_\theta\frac{d}{dt}\Pi = S, \quad (5.1)$$

where

$$S \equiv (f+\zeta)\frac{\partial}{\partial\theta}\dot{\theta}$$

is the diabatic source or sink of potential vorticity. Small-scale momentum mixing on the large-scale waves and a small term relating to the horizontal components of the absolute vorticity vector have been ignored. When written in the form of (5.1), the transport of Π obeys the same equation as that for (nonconservative) tracers. It turns out that because of the definition (4.1), the source term is proportional to the Ertel's potential vorticity itself, i.e.,

$$S = \rho_\theta\Pi\frac{\partial}{\partial\theta}\dot{\theta}. \quad (5.2)$$

Thus, in an inertially stable atmosphere (i.e., $\Pi > 0$), there is "production" of Π in the region where diabatic heating is increasing with altitude, and "loss" of Π in the region where diabatic heating is decreasing with altitude.

To obtain a parameterization of the flux of Ertel's potential vorticity, we take guidance from the small amplitude theory to derive the form of functional dependence on the mean quantity. The perturbation form of (5.1) is

$$\underset{\text{(a)}}{\rho_\theta\Big(\frac{\partial}{\partial t}} + \underset{\text{(b)}}{\frac{\hat{u}}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Big)\Pi^*} + \underset{\text{(c)}}{\rho_\theta'\frac{\partial}{\partial t}\hat{\Pi}} + \underset{\text{(d)}}{\bar{\rho}_\theta\frac{\hat{v}}{a}\frac{\partial}{\partial\varphi}\Pi^*} + \underset{\text{(e)}}{\bar{\rho}_\theta\hat{\dot{\theta}}\frac{\partial}{\partial\theta}\Pi^*}$$

$$+ \underset{\text{(f)}}{\rho_\theta\frac{v^*}{a}\frac{\partial}{\partial\varphi}\hat{\Pi}} + \underset{\text{(g)}}{\rho_\theta\dot{\theta}^*\frac{\partial}{\partial\theta}\hat{\Pi}} = \underset{\text{(h)}}{\rho_\theta\Pi^*\frac{\partial}{\partial\theta}\hat{\dot{\theta}}} + \underset{\text{(i)}}{\rho_\theta\hat{\Pi}\frac{\partial}{\partial\theta}\dot{\theta}^*}. \quad (5.3)$$

As usual, the advection of Π^* by the mean meridional circulation $(\hat{v}, \hat{\dot{\theta}})$ [terms (d) and (e)] is to be neglected when compared with the advection by the mean zonal wind [term (b)]. Under the same approximation, term (h) should also be neglected as it is of the same order as or smaller than term (e). An additional assumption is that the time scales of eddies are much shorter than that of the mean quantities, so term (c) should be neglected when compared to term (a), resulting in

$$\Pi^* = -\eta^*\frac{\partial}{a\partial\varphi}\hat{\Pi} - \phi^*\frac{\partial}{\partial\theta}\hat{\Pi} + \hat{\Pi}\frac{\partial}{\partial\theta}\phi^*, \quad (5.4)$$

where we have defined the displacements η^* and ϕ^* in the horizontal and vertical directions as

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+\hat{u}\frac{\partial}{\partial x}\right)\eta^* &= v^* \\ \left(\frac{\partial}{\partial t}+\hat{u}\frac{\partial}{\partial x}\right)\phi^* &= \dot{\theta}^*\end{aligned}\right\}. \quad (5.5)$$

Thus,

$$\overline{(\rho_\theta\Pi^*)v^*}\cos\varphi = -\overline{v^*(\rho_\theta\eta^*)}\cos^2\varphi\frac{\partial}{\partial y}\hat{\Pi}$$

$$-\overline{v^*\cos\varphi(\rho_\theta\phi^*)}\frac{\partial}{\partial\theta}\hat{\Pi} + \overline{v^*\cos\varphi\left(\rho_\theta\frac{\partial}{\partial\theta}\phi^*\right)}\hat{\Pi}. \quad (5.6)$$

Let

$$\bar{\rho}_\theta K_{yy} \equiv \overline{v^*(\rho_\theta \eta^*)} = \overline{\rho_\theta \frac{\partial}{\partial t}\frac{1}{2}\eta^{*2}} \qquad (5.7)$$

$$\bar{\rho}_\theta K_{y\theta} \equiv \overline{v^*(\rho_\theta \phi^*)} \qquad (5.8)$$

and

$$\bar{\rho}_\theta \hat{v}_E \equiv \overline{v^*\left(\rho_\theta \frac{\partial}{\partial \theta}\phi^*\right)}. \qquad (5.9)$$

Then (5.6) becomes

$$\overline{(\rho_\theta \Pi^*)v^*}\cos\varphi = -\bar{\rho}_\theta K_{yy}\cos^2\varphi \frac{\partial}{\partial y}\hat{\Pi} - \bar{\rho}_\theta K_{y\theta}\cos\varphi \frac{\partial}{\partial \theta}\hat{\Pi} + \bar{\rho}_\theta \hat{v}_E \cos\varphi \hat{\Pi}. \qquad (5.10)$$

Thus, (4.6) implies that

$$\frac{1}{\bar{\rho}_\theta}\Box\cdot\mathcal{F} = -\bar{\rho}_\theta K_{yy}\cos^2\varphi \frac{\partial}{\partial y}\hat{\Pi} - \bar{\rho}_\theta K_{y\theta}\cos\varphi \frac{\partial}{\partial \theta}\hat{\Pi} + \bar{\rho}_\theta \hat{v}_E \cos\varphi \hat{\Pi}. \qquad (5.11)$$

Since $\Box\cdot\mathcal{F}$ appears as the eddy forcing for the zonal mean absolute angular momentum $\bar{L}$ [see Eq. (2.12b)], (5.11) should also be rewritten in terms of $\bar{L}$. This can be accomplished by noting the following identity

$$\bar{\rho}_\theta \hat{\Pi} = f + \bar{\zeta} = -\frac{\partial}{\partial y}\bar{L}. \qquad (5.12)$$

In particular, the horizontal gradient of the Ertel's potential vorticity is given by

$$\bar{\rho}_\theta \frac{\partial}{\partial y}\hat{\Pi} = -\bar{\rho}_\theta \frac{\partial}{\partial y}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L} = -\frac{\partial^2}{\partial y^2}\bar{L} + \frac{\frac{\partial}{\partial y}\bar{\rho}_\theta}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L}$$

$$= \frac{2\Omega}{a} - \frac{\partial^2}{\partial y^2}(\bar{u}\cos\varphi) - f\frac{\left(f - \frac{\partial}{\partial y}\bar{u}\cos\varphi\right)\Gamma}{\cos\varphi g p_e} \times \frac{\partial}{\partial \ln\theta}\left[\frac{p_e}{\tilde{T}e}\frac{\partial}{\partial \ln\theta}\bar{u}\right], \qquad (5.13)$$

where

$$p_e(\theta) \equiv p_{00}\left(\frac{\tilde{T}e}{\theta}\right)^{c_p/R}.$$

Using the parameterization in (5.11), the mean zonal angular momentum equation becomes

$$\frac{\partial}{\partial t}\bar{L} + (\hat{v} + \hat{v}_E)\cos\varphi \frac{\partial}{\partial y}\bar{L} + \hat{\dot{\theta}}\frac{\partial}{\partial \theta}\bar{L} = K_{yy}\cos^2\varphi \bar{\rho}_\theta \frac{\partial}{\partial y}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L} + K_{y\theta}\cos\varphi \bar{\rho}_\theta \frac{\partial}{\partial \theta}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L}. \qquad (5.14)$$

The coefficients K_{yy} and $K_{y\theta}$ for the diffusion of mean angular momentum turn out to be the same as those for the diffusion of conservation tracers as described in, for example, Tung (1982; 1984) and Ko et al. (1985). Thus it appears that an assessment of the mean angular momentum budget or of the Eliassen–Palm flux divergence could also possibly afford us a preliminary look at the magnitude and latitudinal distribution of the diffusion coefficients used in zonally averaged model of tracer transports. These quantities are difficult to obtain by direct evaluation from data on transient waves in the atmosphere. Before we give a preliminary estimate of K_{yy} in Part II, a few remarks concerning Eq. (5.14) are given in the following.

(i) It is well-known that angular momentum L is not conservative and so cannot be treated in the same way as a conservative tracer. If L *were* a conservative quantity, i.e., if it were to satisfy

$$\frac{d}{dt}L = 0,$$

instead of the actual equation

$$\frac{d}{dt}L = -\frac{\partial}{a\partial\lambda}\Phi,$$

which we have used, (apart from the viscous term not included here) then the procedure outlined in Tung (1982) for conservative tracers would have given, for example, for the first term on the right-hand side of Eq. (5.14),

$$\frac{\partial}{\partial y}K_{yy}\cos^2\varphi \frac{\partial}{\partial y}\bar{L}$$

instead of

$$K_{yy}\cos^2\varphi \bar{\rho}_\theta \frac{\partial}{\partial y}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L}$$

actually appearing in Eq. (5.14). The difference should be attributed to the nonconservative nature of L, and not to the "quasi-adiabatic eddy approximation" used to obtain (4.6).

(ii) If we assume that the vertical (θ–) to horizontal (isentropic) gradient of the mean Ertel potential vorticity is (when converted into units of length assuming an isothermal atmosphere) of the order of the ratio of a scale height (~7 km) to the radius of the earth (6400 km), then estimates given in Tung (1984) would suggest that the magnitude of the $K_{y\theta}$ term in Eq. (5.14) [or (5.11)] is about 10% of the magnitude of the K_{yy} term in that same equation in the winter stratosphere in isentropic coordinates. (These two terms are comparable in pressure coordinates under the same circumstances.) Therefore the right-hand side of Eq. (5.14) is dominated by the K_{yy} term, during winter,

$$\frac{\partial}{\partial t}\bar{L} + (\hat{v} + \hat{v}_E)\cos\varphi \frac{\partial}{\partial y}\bar{L} + \hat{\dot{\theta}}\frac{\partial}{\partial \theta}\bar{L} = K_{yy}\cos^2\varphi \bar{\rho}_\theta \frac{\partial}{\partial y}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L}. \qquad (5.15)$$

(iii) The form of the right-hand side of Eq. (5.14) suggests that large-scale eddies act to diffuse mean absolute angular momentum. It is perhaps important to note here the difference between the diffusion of mean absolute angular momentum and the diffusion of mean zonal velocity. The latter mechanism has sometimes been used as a parameterization of the eddy forcing term in the mean zonal momentum equation. It can be shown that the right-hand side of (5.15) can be written as

$$K_{yy}\cos^2\varphi\bar{\rho}_\theta\frac{\partial}{\partial y}\frac{1}{\bar{\rho}_\theta}\frac{\partial}{\partial y}\bar{L}=K_{yy}\cos^2\varphi\left[\frac{\partial^2}{\partial y^2}(\bar{u}\cos\varphi)+\frac{f\{f-[\partial(\bar{u}\cos\varphi)/\partial y]\}\Gamma}{\cos\varphi g p_e}\frac{\partial}{\partial\ln\theta}\left(\frac{p_e}{\tilde{T}_e}\frac{\partial}{\partial\, ln\theta}\bar{u}\right)-\frac{2\Omega}{a}\right]. \quad (5.16)$$

If the mechanism of zonal velocity diffusion were adopted, the last term in (5.16)

$$-K_{yy}\cos^2\varphi\frac{2\Omega}{a} \quad (5.17)$$

would have been absent. The presence of this term, which is proportional to the planetary rotational frequency, has important physical implications. This term is negative independent of the sign of mean gradients provided that the flux of Ertel's potential vorticity is down the horizontal gradient (so that $K_{yy}>0$). Therefore, it provides a source of *easterly* relative momentum. The first two terms in the right-hand side of Eq. (5.16) act to diffuse relative angular momentum, thus serving to smooth horizontal and vertical gradients of the zonal wind $\bar{u}$, but providing no net source of mean relative momentum. The presence of (5.17) serves to *decelerate* the westerly jet, balancing the Coriolis acceleration associated with the poleward mean meridional flow that exists in the stratosphere in the winter hemisphere. Without the term (5.17), the stratospheric westerly jet would reach unrealistically large velocities. By thermal wind, the temperature near the winter pole would be too low. The simple parameterization of Rayleigh friction sometimes used (Leovy, 1964; Holton and Wehrbein, 1980):

$$\frac{1}{\bar{\rho}_\theta}\Box\cdot\mathcal{F}=-K_R\bar{u},$$

also provides the needed easterly momentum source in the westerly region. In this sense, Rayleigh friction is probably a better parameterization than the one based on diffusion of $\bar{u}$, i.e.,

$$\frac{1}{\bar{\rho}_\theta}\Box\cdot\mathcal{F}\sim\mu\nabla^2\bar{u}.$$

(iv) If the same balance of terms as in Eq. (5.15) were to prevail in the easterly region in the summer hemisphere, the presence of the easterly momentum source (5.17) would have led to an unrealistically strong easterly jet. However, the presence of the same easterlies prevents most of the stationary planetary waves from penetrating above 30 km so that in this region K_{yy} should be very small. Also the estimate of the ratio of $K_{y\theta}$ to K_{yy} terms mentioned earlier for winter stratosphere probably does not apply to the summer hemisphere. If the eddy forcing term is dominated by the effect of diabatic gravity waves with short vertical scales, the $K_{y\theta}$ term in Eq. (5.14) may even dominate over the K_{yy} term in the easterly region. Since $K_{y\theta}$ can have either sign, the problem of easterly acceleration in a easterly region does not necessarily arise.

(v) Strictly speaking, one should not treat (5.11) or (5.14) as a *parameterization* of the eddy forcing of the zonal mean flow unless the dependence of the coefficients K_{yy} and $K_{y\theta}$ on $\bar{u}$ or $\bar{L}$ is also known. This situation for the momentum equation should be contrasted with that for the tracer transport equation, where the K are independent of the concentration of the minor species being transported. Nevertheless, Eq. (5.11) or (5.14) is useful for diagnostic purposes as one can in principle deduce from the momentum budget, provided that $\hat{v}_E/\hat{v}$ is small, the same set of K that are used in tracer transport equations.

6. Relation between Eliassen–Palm flux divergence and diabatic heating

The linear relationship (see WMO/NASA, 1985; Chapter 6) between Eliassen–Palm flux term and the diabatic heating is obtained only when the quasi-geostrophic approximation is applied to the steady-state mean zonal momentum equation, which becomes[3]

$$-f\bar{V}=\Box\cdot\mathcal{F}. \quad (6.1)$$

The steady state continuity equation is, from (2.6),

$$\frac{\partial}{\partial y}\bar{V}+\frac{\partial}{\partial\theta}\bar{W}=0. \quad (6.2)$$

[Incidentally, it has been shown in Tung (1982) that (6.2) is a good approximation to (2.6) even without the steady-state assumption]. The thermodynamics equation is, in isentropic coordinates, from (2.7)

$$\bar{W}=\bar{q}/\bar{\Gamma}. \quad (6.3)$$

Given the eddy forcing, $\Box\cdot\mathcal{F}$, Eq. (6.1) determines the meridional velocity $\bar{V}$ locally at every point where the

[3] Note that under the geostrophic approximation, $\nabla\cdot\mathbf{F}$ and $\Box\cdot\mathcal{F}$ can be used interchangeably in Eq. (6.1).

eddy forcing is specified. The vertical velocity $\bar{W}$ is calculated from $\bar{V}$ through the continuity equation (6.2), given a boundary condition on $\bar{W}$ at some level. The thermodynamics equation then relates $\bar{W}$ to the diabatic heating field $\bar{q}$.

The physical implications of these relations have been taken to be that (i) it is the eddies that are responsible for driving the atmosphere away from local radiative equilibrium, and (ii) the degree of deviation from radiative equilibrium is directly proportional to the strength of the Eliassen–Palm flux divergence, the last point being inferred from the relationship

$$\frac{\partial}{\partial\theta}(\bar{q}/\bar{\Gamma}) = \frac{\partial}{\partial y}\left(\frac{1}{f}\Box\cdot\mathcal{F}\right), \tag{6.4}$$

obtained by combining (6.1), (6.2) and (6.3).

There are two points of caution that we wish to inject concerning the interpretation just mentioned. First, as mentioned above, $\bar{W}$ and hence $\bar{q}$, are influenced by boundary conditions as well as in situ eddy forcing, and so the above results apply only in the absence of significant boundary forcing in $\bar{W}$. Second, geostrophy is not necessarily a good approximation when applied to the zonally averaged momentum equation in the east–west direction. (It is a better approximation when applied to the meridional momentum equation, yielding the thermal wind relationship). Results on zonal mean circulations using the primitive equations are known in some cases to differ significantly from those obtained using geostrophic equations.

This model difference can be understood by examining the relative importance of various terms in the steady zonal momentum equation:

$$\bar{V}\frac{\partial}{\partial y}(\bar{u}\cos\varphi) + \bar{W}\frac{\partial}{\partial\theta}(\bar{u}\cos\varphi) - f\bar{V} = \Box\cdot\mathcal{F}. \tag{6.5}$$

In quasi-geostrophic approximation, the nonlinear advection terms, the first two terms in Eq. (6.5), are assumed to be much smaller than the Coriolis term, the third term in Eq. (6.5). Consequently the balance is between the Coriolis term and the eddy forcing term on the right-hand side [see Eq. (6.1)]. As $\Box\cdot\mathcal{F}$ is made smaller and smaller, the Coriolis term necessarily decreases in importance according to the quasi-geostrophic balance (6.1). This eventually leads to the breakdown of the geostrophic approximation as the neglected relative acceleration terms are no longer small when compared to the Coriolis force. The unanswered questions are: What is the threshold magnitude for eddy forcing below which the geostrophic balance (6.1) no longer holds, and the "nonlinear nearly inviscid regime" (Held and Hou, 1980) applies? Which regime is our atmosphere in? An assessment of the magnitude of $\Box\cdot\mathcal{F}$ and the validity of geostrophic approximation as applied to the mean zonal momentum equation are necessary for an understanding of these issues. The diagnostic assessments will be given in Part II. In this section, some conceptual issues concerning the state of the atmosphere in the absence of eddy forcing will first be addressed.

The steady state zonal momentum equation can be written as

$$J[\bar{\Psi}, \bar{L}] = \bar{X}, \tag{6.6}$$

where $\bar{\Psi}$ is the streamfunction of the meridional circulation defined by

$$\left.\begin{aligned} \bar{W} &= \frac{\partial}{\partial y}\bar{\Psi} \\ \bar{V} &= -\frac{\partial}{\partial\theta}\bar{\Psi} \end{aligned}\right\}. \tag{6.7}$$

The Jacobian operator appearing in (6.6) is defined by

$$J[A, B] \equiv A_y B_\theta - A_\theta B_y.$$

In (6.6) $\bar{X}$ represents the large-scale eddy forcing plus small-scale dissipative terms, i.e.,

$$\bar{X} = \Box\cdot\mathcal{F} + \nabla\cdot\mu\nabla\bar{L} \tag{6.8}$$

where μ is the coefficient of small-scale mixing or molecular diffusion. In the absence of zonal asymmetries, $\bar{X}$ is represented by only the (molecular) diffusion.

a. Inviscid symmetric states

If the hypothetical atmosphere is symmetric ($\Box\cdot\mathcal{F} = 0$) and inviscid ($\mu \equiv 0$), (6.6) becomes

$$J[\bar{\Psi}, \bar{L}] = 0, \tag{6.9}$$

which implies that lines of constant absolute momentum should coincide with the streamline. This is simply a statement that the inviscid symmetric circulation is angular momentum conserving. There is no constraint from the momentum equation that the solution should be in local radiative equilibrium defined by

$$\bar{\Psi} \equiv 0, \quad \bar{q} \equiv 0. \tag{6.10}$$

The exact solution to (6.9) is

$$\bar{\Psi} = G(\bar{L}); \tag{6.11}$$

the functional form of G is to be determined by boundary conditions. If the horizontal boundaries are in radiative equilibrium, i.e.,

$$\bar{W} = 0 \quad \text{at} \quad \theta_0 \text{ and } \theta_1, \tag{6.12}$$

we have, without loss of generality

$$\bar{\Psi} = 0 \quad \text{at} \quad \theta_0 \text{ and } \theta_1. \tag{6.13}$$

If every line of constant $\bar{L}$ intersects or touches either the upper or lower boundary, then (6.11) becomes, when evaluated using (6.13),

$$\bar{\Psi} = G \equiv 0. \tag{6.14}$$

This is the radiative equilibrium solution, with $\bar{q} \equiv 0$. The zonal momentum is related through the balanced wind (or thermal wind) relationship to the radiative equilibrium temperature T_e satisfying $\bar{q}(T_e) = 0$.

However, if in the interior of the domain there exists a region of closed contours of $\bar{L}$, then there may exist modonlike solutions for which G is not necessarily zero. There is in general a meridional circulation in such a region (an out-of-radiative equilibrium situation). Also, inside the closed circulation region the zonal momentum is no longer constrained by the radiative equilibrium temperature, and can thus in principle attain any distribution.

The inviscid symmetric circulation is *nonunique* because neither the shape of the closed region, the functional form of G, nor the distribution of $\bar{L}$ inside this region is determined without the imposition of additional constraints. A common misconception has been that the radiative equilibrium solution (6.14) is the only solution in the absence of eddies or friction. (Ironically, even Held and Hou, 1980, who have established that a realistic looking Hadley circulation can exist even in the absence of large-scale eddies, chose to interpret their deduced circulation as due to the presence of small but nonzero viscosity as a singular perturbation to the inviscid solution, which they assumed to be in radiative equilibrium.)

However, that the inviscid steady state recirculating solution can be nonunique is well known in other areas of fluid mechanics (e.g., see Batchelor, 1956, in the case of vorticity distribution in steady laminar flow, Stern, 1975, for modons, and Tung et al., 1982, for internal waves of permanent form). There are at least two ways of removing the arbitrariness. These depend on whether the steady inviscid solution is treated as the limit of $\mu \to 0$ and then $t \to \infty$, or as the limit of $t \to \infty$ and then $\mu \to 0$. In the first approach, one considers the quasi-steady limit of an inviscid flow (see Benney and Ko, 1978, and Tung et al., 1982) and G can take any arbitrary form depending presumably on initial conditions (if these are stable). In the second approach, one considers the nearly inviscid limit of a steady flow (see Batchelor, 1956). The presence of even a small amount of viscosity (which however acts for an infinitely long time) can place severe constraints on the distribution of G inside a recirculating region.

b. The nearly inviscid limit

The presence of a small amount of viscosity eliminates the radiative equilibrium solution (6.10). More precisely, (6.10), or an order μ modification of (6.10), cannot remain as a solution to the viscous equation even in the limit as $\mu \to 0$. The argument was given by Held and Hou (1980) using a version of Hide's (1969) theorem and assuming that the imposed radiative equilibrium temperature decreases poleward for all heights.

When a small amount of viscosity is present, the inviscid recirculating solution mentioned above remains valid to leading order except in certain boundary layers, in which it may be modified in different ways depending on the boundary condition. If the imposition of viscous boundary conditions, such as no-slip, is inconsistent with the inviscid solution, a (thin) boundary layer is needed. Inside the boundary layer, the effect of viscosity is important and (6.9) no longer holds. Here, the constraint from (6.11) that streamlines follow lines of constant absolute angular momentum is broken. Consequently, the streamlines can be closed, as demanded by mass conservation, while lines of constant angular momentum can be open and intersect the lower boundary, as required for example by the no-slip condition, $\bar{u} = 0$ (i.e., $\bar{L} = \Omega a \cos^2\varphi$).

Outside the boundary layer, the interior solution is almost inviscid, i.e.,

$$\bar{\Psi} \approx G(\bar{L}) \tag{6.15}$$

except now G is to be determined by matching to the boundary layer solution, and not by evaluating (6.15) at the boundary itself (where $\bar{\Psi} = 0$).

The detail matched asymptotic solution will be presented in a separate paper. Here we only wish to emphasize the fact that the interior nearly inviscid solution (6.15) can be very close to one of the inviscid recirculating solutions (6.11). And it is therefore more natural to interpret the nearly inviscid solution to be a small perturbation (at least in the interior) from a recirculating inviscid solution,[4] instead of it being a singular perturbation (with order one change) from the radiative equilibrium solution as suggested by Held and Hou (1980).

The conclusion that one can draw from the above discussion appears to be that eddies (large or small scales) are not needed to drive the atmosphere out of radiative equilibrium. Nevertheless, sufficient viscosity may be needed to maintain some form of stability for these nearly inviscid steady symmetric circulations. Nevertheless, one should not diminish the important role of large-scale eddies in driving the observed circulation simply based on the above discussion, which is intended for a hypothetical atmosphere.

7. Conclusions

A formulation is given for a nongeostrophic theory of zonally averaged circulation. Motivations for extending the commonly used geostrophic version of the general theory are many; some of them are

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[4] In this case the inviscid solution can be taken as a modonlike recirculating configuration with the lower interface brought arbitrarily close to the lower boundary. The lower interface will in general be a region of discontinuity in which the lines of constant $\bar{L}$ will complete the closed contour. The addition of a small amount of viscosity will introduce a thin boundary layer around the discontinuity and, depending on the lower boundary condition, lines of constant $\bar{L}$ do not necessarily have to close via this boundary layer.

(i) Quasi-geostrophy does not hold at and near the equator and so is not uniformly valid in a model of zonal mean global circulation.

(ii) Although there is some consensus that the mean zonal velocity is in geostrophic balance, there is no a priori justification for applying the quasi-geostrophic approximation for the zonally averaged north–south velocity. This is the case even if the nonaveraged north–south velocity can be considered as in geostrophic balance with the east–west gradient of pressure (or height) field.

(iii) In the eddy forcing terms for the mean circulation equatorial wave and gravity waves, which are important components of the forcing for, e.g., the semiannual and quasi-biennial oscillations, are filtered out in the quasi-geostrophic version of the theory.

(iv) The constraint of geostrophy is so strong that one is prevented from addressing some fundamental problems concerning zonally averaged circulations using the set of geostrophic equations. For example, it is probably not appropriate in using the set of geostrophic equations to deduce the state of the atmosphere in the absence of zonal asymmetry, because while these equations would lead one to conclude that the steady state symmetric atmosphere is extremely close to radiative equilibrium, the original primitive equations suggest otherwise.

A major hindrance in the past in adopting the more general (i.e., nongeostrophic) formulation of zonally averaged circulations lies not in technical difficulties associated with the formulation but in clear interpretation of the role of various eddy forcing terms. This problem is largely circumvented when the zonal averaging is performed on isentropic surfaces, as it is done in the present work. The role of eddies in the forcing of the mean flow is much better defined in isentropic coordinates without the presence in isobaric coordinates of mean temperature advection by the ageostrophic circulation.

The more general formulation suggests that a hypothetical atmosphere can be in different circulation regimes depending on the strength of the eddy Eliassen–Palm flux divergence. When the eddy strength is strong, the mean circulation may be in the geostrophic regime away from the equator. However, when the eddy strength is weak, the circulation may be close to the nearly inviscid nonlinear regime of Held and Hou (1980), which is not a continuation of the geostrophic regime as the eddy strength is reduced.

In Part II, we will address the question of which regime our atmosphere is in. Furthermore, we will use our present isentropic formulation and the derived relation between Eliassen–Palm flux pseudodivergence and the flux of Ertel's isentropic potential vorticity to deduce the isentropic diffusion coefficient, K_{yy} that is useful in tracer transport studies.

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APPENDIX A

The Zonally Averaged Momentum Equation in Isentropic Coordinates on a Sphere

Starting from the zonal momentum equation, where $\mathcal{D}$ is the viscous force per unit mass,

$$\frac{\partial}{\partial t}u+\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}u+\dot{\theta}\frac{\partial}{\partial\theta}u - \left(2\Omega+\frac{u}{a\cos\varphi}\right)\sin\varphi v=-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi+\mathcal{D} \quad \text{(A1)}$$

and the equation for the conservation of mass:

$$\frac{\partial}{\partial t}\rho_\theta+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\rho_\theta u +\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\rho_\theta v\cos\varphi)+\frac{\partial}{\partial\theta}(\rho_\theta\dot{\theta})=0, \quad \text{(A2)}$$

one has

$$\frac{\partial}{\partial t}\rho_\theta u+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\rho_\theta uu+\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\varphi}(\rho_\theta uv\cos^2\varphi) +\frac{\partial}{\partial\theta}(\rho_\theta u\dot{\theta})-2\Omega\sin\varphi\rho_\theta v=-\frac{\rho_\theta}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi+\rho_\theta\mathcal{D}. \quad \text{(A3)}$$

Taking the zonal average: $(\bar{\ })\equiv\int_0^{2\pi}(\ )d\lambda$, one obtains the zonally averaged momentum equation:

$$\frac{\partial}{\partial t}\overline{\rho_\theta u}+\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\varphi}(\overline{\rho_\theta uv}\cos^2\varphi)+\frac{\partial}{\partial\theta}(\overline{\rho_\theta u\dot{\theta}}) -2\Omega\sin\varphi\overline{\rho_\theta v}=-\overline{\frac{\rho_\theta}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi}+\overline{\rho_\theta\mathcal{D}}. \quad \text{(A4)}$$

We let, as in Tung (1982):

$$\bar{U}\equiv\overline{\rho_\theta u},\quad \bar{V}\equiv\overline{\rho_\theta v}\cos\varphi,\quad \text{and}\quad \bar{W}\equiv\overline{\rho_\theta\dot{\theta}} \quad \text{(A5)}$$

be the zonally averaged mass circulation, and define, as in Gallimore and Johnson (1981a) the density-weighted zonal average

$$\hat{u}\equiv\overline{\rho_\theta u}/\bar{\rho}_\theta=\bar{U}/\bar{\rho}_\theta,$$

and deviation from zonal average

$$u^*\equiv u-\hat{u}. \quad \text{(A6)}$$

Then

$$\rho_\theta\overline{uv}\cos^2\varphi = \overline{\rho_\theta[\hat u + u^*]\left[\frac{\bar V}{\bar\rho_\theta} + v^*\cos\varphi\right]\cos\varphi}$$

$$= \hat u\bar V\cos\varphi + \overline{\rho_\theta u^* v^*}\cos^2\varphi + \frac{1}{\bar\rho_\theta}\overline{\rho_\theta u^*}\cos\varphi\bar V + \hat u\cos^2\varphi\overline{\rho_\theta v^*}.$$

The last two terms are zero from definition (A6). Equation (A4) now becomes

$$\frac{\partial}{\partial t}\bar\rho_\theta\hat u + \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\varphi}(\hat u\cos\varphi\bar V) + \frac{\partial}{\partial\theta}(\hat u\bar W) - 2\Omega\sin\varphi\bar V/\cos\varphi = -\overline{\left(\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\Phi\right)\rho_\theta} - \frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\varphi}(\overline{\rho_\theta u^* v^*}\cos^2\varphi) - \frac{\partial}{\partial\theta}(\overline{\rho_\theta u^*\dot\theta^*}) + \overline{\rho_\theta\mathcal{D}}. \quad \text{(A7)}$$

The right-hand side of (A7), apart from the $\bar{\mathcal{D}}$ term, can be written as

$$\frac{1}{\cos\varphi}\nabla\cdot\mathbf{F} = \frac{1}{\cos\varphi}\left[\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}F_y + \frac{\partial}{\partial\theta}F_\theta\right]$$

where

$$\left.\begin{aligned} F_y &\equiv -\overline{\rho_\theta u^* v^*}\cos^2\varphi \\ F_\theta &\equiv -\left[\overline{\rho_\theta u^*\dot\theta^*} - \overline{\frac{1}{g}p\frac{\partial}{a\cos\varphi\partial\lambda}\Psi}\right] \end{aligned}\right\} . \quad \text{(A8)}$$

In order to write the first term on the right-hand side of (A7) apart from $\bar{\mathcal{D}}$ term into a vertical derivative term, we have made use of the hydrostatic equation

$$g\rho_\theta = -\frac{\partial}{\partial\theta}p$$

and the relationship between the Montgomery streamfunction Φ and pressure

$$\frac{\partial}{\partial\theta}\Phi = c_p\left(\frac{p}{p_{00}}\right)^{R/c_p}.$$

Finally, by multiplying both sides of Eq. (A7) by $\cos\varphi$, we obtain

$$\frac{\partial}{\partial t}(\bar\rho_\theta\hat u\cos\varphi) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\hat u\cos\varphi\bar V) + \frac{\partial}{\partial\theta}(\hat u\cos\varphi\bar W) - 2\Omega\sin\varphi\bar V = \nabla\cdot\mathbf{F} + \overline{\rho_\theta\mathcal{D}}\cos\varphi. \quad \text{(A9)}$$

Next, the mean momentum equation expressed in terms $\bar u$ (instead of $\hat u$ as in Eq. (A9) is derived. We start with Eq. (A1) and perform the zonal average,

$$\frac{\partial}{\partial t}(\bar u\cos\varphi) + \overline{\frac{v}{a}\frac{\partial}{\partial\varphi}u\cos\varphi} + \overline{\dot\theta\frac{\partial}{\partial\theta}u\cos\varphi} - 2\Omega\sin\varphi\bar v\cos\varphi = \bar{\mathcal{D}}\cos\varphi. \quad \text{(A10)}$$

We let $v = \hat v + v^*$, but $u = \bar u + u'$, then

$$\overline{\frac{v}{a}\frac{\partial}{\partial\varphi}u}\cos\varphi = \frac{\hat v}{a}\frac{\partial}{\partial\varphi}\bar u\cos\varphi + \overline{\frac{v^*}{a}\frac{\partial}{\partial\varphi}u'}\cos\varphi + \frac{\bar v^*}{a}\frac{\partial}{\partial\varphi}\bar u\cos\varphi.$$

Since

$$v^* = v' - \overline{\rho_\theta' v'}/\bar\rho_\theta$$

from the definition of v^*, and

$$\overline{a^* b'} = \overline{a'b'},$$

we have

$$\overline{\frac{v}{a}\frac{\partial}{\partial\varphi}u}\cos\varphi = \frac{\hat v}{a}\frac{\partial}{\partial\varphi}\bar u\cos\varphi + \overline{\frac{v'}{a}\frac{\partial}{\partial\varphi}u'}\cos\varphi - \frac{1}{\bar\rho_\theta}\overline{\rho_\theta' v'}\frac{1}{a}\frac{\partial}{\partial\varphi}\bar u\cos\varphi.$$

Similarly,

$$\overline{\dot\theta\frac{\partial}{\partial\theta}u\cos\varphi} = \hat{\dot\theta}\frac{\partial}{\partial\theta}\bar u\cos\varphi + \overline{\dot\theta'\frac{\partial}{\partial\theta}u'\cos\varphi} - \frac{1}{\bar\rho_\theta}\overline{\rho_\theta'\dot\theta'}\frac{\partial}{\partial\theta}\bar u\cos\varphi.$$

Substituting

$$\bar v = \hat v - \overline{\rho_\theta' v'}/\bar\rho_\theta$$

into (A10), we find

$$\frac{\partial}{\partial t}(\bar u\cos\varphi) + \frac{\hat v}{a}\frac{\partial}{\partial\varphi}\cos\varphi + \hat{\dot\theta}\frac{\partial}{\partial\theta}\bar u\cos\varphi - 2\Omega\sin\varphi\hat v\cos\varphi = -\overline{\frac{v'}{a}\frac{\partial}{\partial\varphi}u'\cos\varphi} - \overline{\dot\theta'\frac{\partial}{\partial\theta}u'\cos\varphi} - \left(2\Omega\sin\varphi - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\bar u\cos\varphi\right)\overline{\rho_\theta' v'}\cos\varphi/\bar\rho_\theta + \frac{1}{\bar\rho_\theta}\overline{\rho_\theta'\dot\theta'}\frac{\partial}{\partial\theta}\bar u\cos\varphi + \bar{\mathcal{D}}\cos\varphi. \quad \text{(A11)}$$

Using the zonally averaged continuity equation [from (A2)]:

$$\frac{\partial}{\partial t}\bar\rho_\theta + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar\rho_\theta\hat v\cos\varphi) + \frac{\partial}{\partial\theta}(\bar\rho_\theta\hat{\dot\theta}) = 0. \quad \text{(A12)}$$

We rewrite (A11) finally in the form

$$\frac{\partial}{\partial t}(\bar{\rho}_\theta\bar{u}\cos\varphi)+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\bar{\rho}_\theta\cos\varphi\bar{u}\cos\varphi)$$
$$+\frac{\partial}{\partial\theta}(\bar{\rho}_\theta\hat{\dot{\theta}}\bar{u}\cos\varphi)-2\Omega\sin\varphi\hat{v}\cos\varphi=\Box\cdot\mathcal{F}+\bar{\rho}_\theta\bar{\mathcal{D}}\cos\varphi, \qquad (\mathrm{A13})$$

where

$$\frac{1}{\bar{\rho}_\theta}\Box\cdot\mathcal{F}\equiv-\overline{\frac{v'}{a}\frac{\partial}{\partial\varphi}u'\cos\varphi}-\overline{\dot{\theta}'\frac{\partial}{\partial\theta}u'\cos\varphi}$$
$$-\left(2\Omega\sin\varphi-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\bar{u}\cos\varphi\right)\overline{\rho_\theta'v'\cos\varphi}/\bar{\rho}_\theta$$
$$+\frac{1}{\bar{\rho}_\theta}\overline{\rho_\theta'\dot{\theta}'}\frac{\partial}{\partial\theta}\bar{u}\cos\varphi. \qquad (\mathrm{A14})$$

APPENDIX B

A Modified Scaling Appropriate for Zonal Mean Circulations

In section 6, two extreme circulation regimes are discussed. One, the inviscid or nearly inviscid regime, pertains to the mean circulation in the presence of no or small eddy momentum forcing. The other, the quasi-geostrophic regime, applies away from the equator when eddy momentum forcing is strong. It was clear from the discussion that the quasi-geostrophic approximation, obtained using Rossby number scaling, is of only limited validity when applied to different zonally averaged circulation regimes.

In this section, a somewhat different scaling based on the strength of eddy Eliassen–Palm flux divergence is proposed. It turns out that the result is more general and can cover both extremes of circulation regimes mentioned above.

Since the following discussions are not restricted to isentropic coordinates only, the more commonly used equations in pressure coordinates will be used first, with modifications for isentropic coordinates given later.

1. Implications of Rossby number scaling

Implicit in scalings based on the smallness of Rossby number

$$\mathrm{Ro}|_l\equiv\frac{u_{00}/l}{2\Omega\sin\varphi_0}, \qquad (\mathrm{B1})$$

where u_{00} is a typical value of the horizontal velocity,[5] l a typical horizontal length scale of the flow, and $2\Omega\sin\varphi_0$ a typical (midlatitude) value for the Coriolis parameter f, is the assumption that the gradient of relative momentum be much smaller than the gradient of planetary momentum, resulting in the approximation

$$\frac{\bar{D}}{Dt}\cdot\bar{L}\simeq\bar{v}\frac{\partial}{\partial y}(\Omega a\cos^2\varphi)=-f\bar{v} \qquad (\mathrm{B2})$$

where

$$\frac{\bar{D}}{Dt}\equiv\frac{\partial}{\partial t}+\bar{v}\frac{\partial}{\partial y}+\bar{w}\frac{\partial}{\partial z}, \quad z\equiv H_{00}\ln\left(\frac{p_{00}}{p}\right), \quad w\equiv\frac{d}{dt}z$$

and

$$\bar{L}\equiv\bar{u}\cos\varphi+\Omega a\cos^2\varphi.$$

Equation (B2) is the so-called geostrophic approximation, and by its nature necessarily excludes the kind of absolute-angular-momentum-conserving flow cited previously and by Held and Hou (1980), which, by approximately conserving absolute angular momentum, $\bar{L}$, generates in its northward branch westerly zonal flows as the planetary angular momentum, $\Omega a\cos^2\varphi$, decreases northward. Therefore, a characteristic of such a flow is that the *variation* (with latitude) of the relative zonal angular momentum is comparable to the *variation* in the planetary angular momentum (even though $\bar{u}/(\Omega a\times\cos\varphi)$ is small). In other words, the Rossby number for such a flow should be order one and so cannot be used as an asymptotic parameter for this case.

In our present problem of calculating the zonal mean circulations, the scale of the mean circulation should be obtained as part of the solution instead of being specified a priori.

2. An alternate scaling

We propose an alternate scaling based on the assumption that for a rapidly rotating planet, such as the Earth, the zonally averaged flow should predominantly be in the zonal direction. Therefore, the ratio

$$\epsilon\equiv\frac{\bar{v}_{00}}{\bar{u}_{00}} \qquad (\mathrm{B3})$$

should be small. In (B3), $\bar{v}_{00}$ and $\bar{u}_{00}$ are the typical magnitudes of the zonally averaged meridional and zonal velocities respectively. (Note that there is no assumption that $|v/u|$ be small.) We shall further take $\bar{v}_{00}$ to be typified by the quasi-geostrophic zonal mean meridional velocity, which is in turn given by the strength of the eddy Eliassen–Palm flux divergence divided by the Coriolis parameter. Viewed this way, ϵ is then a measure of the strength of eddy forcing of zonal mean momentum.

If, as is usually done, one nondimensionalizes both horizontal velocities, $\bar{u}$ and $\bar{v}$, by the same $\bar{u}_{00}$, then the following scalings should result:

$$\left.\begin{aligned}\bar{u}_*&\equiv\bar{u}/\bar{u}_{00}=\bar{u}_0+\epsilon\bar{u}_1+\epsilon^2\bar{u}_2+\cdots\\ \bar{v}_*&\equiv\bar{v}/\bar{u}_{00}=\epsilon\bar{v}_1+\epsilon^2\bar{v}_2+\cdots\end{aligned}\right\}. \qquad (\mathrm{B4})$$

[5] Note that Rossby number scaling does not distinguish between the magnitudes of meridional and zonal velocities.

Let l_y and l_z be the typical meridional and vertical length scales, then based on the continuity equation, the mean vertical velocity $\bar{w}$ should have the scale, $(l_z/l_y)\,\bar{v}_{00}$, and so the usual dimensionless quantity

$$\bar{w}_* \equiv \bar{w}\left(\frac{l_y}{l_z}\right)\Big/\bar{u}_{00}$$

should be of order ϵ, i.e.,

$$\bar{w}_* = \epsilon\bar{w}_1 + \epsilon^2\bar{w}_2 + \cdots. \qquad \text{(B5)}$$

For mean transports in the meridional plane, time t should be scaled by $l_y/\bar{v}_{00}$, and so we define

$$t_* \equiv t\left(\frac{\bar{v}_{00}}{l_y}\right).$$

Starting with the zonally averaged zonal momentum equation, which we write in the following form

$$\frac{\bar{D}}{Dt}\bar{L} = \bar{X}, \qquad \text{(B6)}$$

where $\bar{X}$ includes the Eliassen–Palm flux divergence (if in pressure coordinates, the residual quantities are used for $\bar{v}$ and $\bar{w}$), and dividing by $\bar{u}_{00}{}^2/l_y$, we have

$$\left[\epsilon\frac{\partial}{\partial t^*} + (\epsilon\bar{v}_1 + \cdots)\frac{\partial}{\partial y_*} + (\epsilon\bar{w}_1 + \cdots)\frac{\partial}{\partial z_*}\right] \times [(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots)\cos\varphi] - (\epsilon\bar{v}_1 + \cdots)f_* = \epsilon\bar{X}_1 + \epsilon^2\bar{X}_2 + \cdots$$

where

$$y_* \equiv y/l_y \quad \text{and} \quad f_* \equiv f(l_y/\bar{u}_{00}). \qquad \text{(B7)}$$

In (B7), we have also defined and scaled

$$\bar{X}_* \equiv \bar{X}/(\bar{u}_{00}{}^2/l_y) = \epsilon\bar{X}_1 + \epsilon^2\bar{X}_2 + \cdots. \qquad \text{(B8)}$$

(The scaling with $\bar{X}_*$, being order one would have violated our assumption of ϵ small. This exceptional case will not be discussed here.) Therefore, to leading order in ϵ, (B7) yields 1:

$$\left(\frac{\partial}{\partial t^*} + \bar{v}_1\frac{\partial}{\partial y_*} + \bar{w}_1\frac{\partial}{\partial z_*}\right)(\bar{u}_0\cos\varphi) - \bar{v}_1 f_* = \bar{X}_1. \qquad \text{(B9)}$$

Note that both the relative acceleration and Coriolis force are retained in this approximation. In the y-direction, we start with the following zonally averaged meridional momentum equation

$$\frac{\bar{D}}{Dt}\bar{v} + \tilde{f}\bar{u} = -\frac{1}{a}\frac{\partial}{\partial\varphi}\bar{\Phi} + \bar{Y}, \qquad \text{(B10)}$$

where $\bar{Y}$ is the eddy forcing for the y-momentum and $\tilde{f} \equiv [2\Omega + (\bar{u}/a\cos\varphi)]\sin\varphi$. The same procedure as before yields

$$\left[\epsilon\frac{\partial}{\partial t^*} + (\epsilon\bar{v}_1 + \cdots)\frac{\partial}{\partial y_*} + (\epsilon\bar{w}_1 + \cdots)\frac{\partial}{\partial z_*}\right] \times [\epsilon\bar{v}_1 + \cdots] + (\tilde{f}_0 + \epsilon\tilde{f}_1 + \cdots)(\bar{u}_0 + \epsilon\bar{u}_1 + \cdots) = -\cos\varphi\frac{\partial}{\partial y_*}(\bar{\Phi}_0 + \epsilon\Phi_1 + \cdots) + (\epsilon\bar{Y}_1 + \epsilon^2 Y_2 + \cdots), \qquad \text{(B11)}$$

where

$$\left.\begin{aligned} \tilde{f}_0 &= f_* + \bar{u}_0\left(\frac{l_y}{a}\right)\tan\varphi \\ \tilde{f}_1 &= \bar{u}_1\left(\frac{l_y}{a}\right)\tan\varphi \end{aligned}\right\}.$$

To leading order in ϵ, (B11) is

$$\tilde{f}_0\bar{u}_0 = -\cos\varphi\frac{\partial}{\partial y_*}\bar{\Phi}_0, \qquad \text{(B12a)}$$

which is the balanced wind relationship. Modification to (B12) from the mean meridional advection terms arises at order ϵ^2. Thus, the balanced wind relations (B12) holds to high accuracy away from the equator provided the eddy meridional momentum forcing is not too strong (i.e., if $\bar{Y}_1 = 0$). Our scaling requires that $\partial/\partial y_*\bar{\Phi}$ be small very near the equator, otherwise the assumption of small ϵ breaks down.

Equation (B12) reduces to the geostrophic relation if one further assumes that the Rossby number based on the radius of the earth a,

$$\mathrm{Ro}|_a \equiv \frac{\bar{u}/\cos\varphi}{2\Omega a} \qquad \text{(B13)}$$

is small. This is generally true for a rotating planet whose planetary rotational frequency, Ω, is larger than the relative angular frequency, $\bar{u}/(a\cos\varphi)$, of the flow on it. (Note that this ratio does not involve the unknown length scale l_y.) Thus a special, relevant case of (B12) is

$$f_*\bar{u}_0 = -\cos\varphi\frac{\partial}{\partial y_*}\bar{\Phi}_0. \qquad \text{(B14a)}$$

This geostrophic relationship for the zonal flow is obtained without the Rossby number scaling, i.e., the smallness of the Rossby number defined in (B1) is not assumed.

Assuming $\bar{Y}_1 \equiv 0$, the next order in (B11) is

$$\left[f_* + 2\bar{u}_0\left(\frac{l_y}{a}\right)\tan\varphi\right]\bar{u}_1 = -\cos\varphi\frac{\partial}{\partial y_*}\bar{\Phi}_1, \qquad \text{(B12b)}$$

and if furthermore (B13) is small, (B12b) becomes again the geostrophic relation:

$$f_*\bar{u}_1 = -\cos\varphi\frac{\partial}{\partial y_*}\bar{\Phi}_1. \qquad \text{(B14b)}$$

The energy equation,

$$\frac{d}{dt}\ln\theta = \frac{Q}{T},$$

when rewritten in log-pressure coordinates, becomes

$$\frac{d}{dt}T + \frac{g}{c_p}\frac{T}{T_{00}}w = Q,$$

where T_{00} is the reference temperature associated with the reference scale height, $H_{00} \equiv RT_{00}/g$, used in the definition of z. The zonal mean of the above equation is

$$\frac{\bar{D}}{Dt}\bar{T} + \bar{w}\frac{\bar{T}}{T_{00}}\frac{g}{c_p} = \bar{Q} + \bar{Z}, \qquad \text{(B15)}$$

where $\bar{Z}$ contains the eddy heat fluxes. We shall assume that $(\bar{v}, \bar{w})$ is the residual mean circulation, so that $\bar{Z}$ is not of dominant importance in the energy equation. This assumption will be reflected in our scaling for $\bar{Z}$ to follow.

We choose

$$\left.\begin{aligned} \bar{T}_* &\equiv \bar{T}/T_{00} = \bar{T}_0 + \epsilon\bar{T}_1 + \cdots \\ \bar{Z}_* &\equiv \bar{Z}l_y/(\bar{u}_{00}T_{00}) = \epsilon\bar{Z}_1 + \epsilon^2\bar{Z}_2 + \cdots \\ \bar{Q}_* &\equiv \bar{Q}l_y/(\bar{u}_{00}T_{00}) = \bar{Q}_0 + \epsilon\bar{Q}_1 + \cdots \end{aligned}\right\}. \qquad \text{(B16)}$$

For our present purpose, we shall assume that the mean radiative heating rate $\bar{Q}$ is a function of the mean temperature, viz,

$$\bar{Q} = \bar{Q}(\bar{T}).$$

Therefore,

$$\bar{Q}_* = \bar{Q}_*(\bar{T}_*) = \bar{Q}_*(\bar{T}_0) + \epsilon\frac{\partial}{\partial\bar{T}_0}\bar{Q}_*(\bar{T}_0)\bar{T}_1 + \cdots \qquad \text{(B17)}$$

and thus

$$\bar{Q}_0 = \bar{Q}_*(\bar{T}_0) \qquad \text{(B18a)}$$

$$\bar{Q}_1 = \frac{\partial}{\partial\bar{T}_0}\bar{Q}_*(\bar{T}_0)\cdot\bar{T}_1, \cdots \qquad \text{(B18b)}$$

Equation (B15), when divided by $(\bar{u}_{00}T_{00})/ly$, becomes

$$\left[\epsilon\frac{\partial}{\partial t_*} + (\epsilon\bar{v}_1 + \cdots)\frac{\partial}{\partial y_*} + (\epsilon\bar{w}_1 + \cdots)\frac{\partial}{\partial z_*}\right] \times[\bar{T}_0 + \epsilon\bar{T}_1 + \cdots] + (\epsilon\bar{w}_1 + \cdots)(\bar{T}_0 + \epsilon\bar{T}_1 + \cdots) \times\left(\frac{l_z}{T_{00}}\frac{g}{c_p}\right) = \bar{Q}_0 + \epsilon\bar{Q}_1 + \cdots + \epsilon\bar{Z}_1 + \epsilon^2\bar{Z}_2 + \cdots. \qquad \text{(B19)}$$

The leading term in (B18) is

$$\bar{Q}_0 \equiv 0, \qquad \text{(B20)}$$

which defines the leading order temperature field via (B18a) as

$$\bar{T}_0 = T_e, \qquad \text{(B21a)}$$

where

$$\bar{Q}_*(T_e) \equiv 0, \qquad \text{(B21b)}$$

In other words, $\bar{T}_0$ is the radiative equilibrium temperature T_e satisfying (B21b). At the next order in ϵ, (B19) yields

$$\left(\frac{\partial}{\partial t_*} + \bar{v}_1\frac{\partial}{\partial y_*} + \bar{w}_1\frac{\partial}{\partial z_*}\right)\bar{T}_0 + \bar{w}_1\bar{T}_0\left(\frac{l_z}{T_{00}}\frac{g}{c_p}\right) = \bar{Q}_1 + \bar{Z}_1, \qquad \text{(B22)}$$

which together with (B18b), should determine $\bar{T}_1$, the mean temperature change (from radiative equilibrium) induced by the mean meridional circulation $(\bar{v}_1, \bar{w}_1)$.

3. Summary of scaled mean equations in isobaric coordinates

In dimensional form, we write the zonal mean temperature and zonal flow as

$$\bar{T} = T_e(y, z) + \Delta\bar{T} \qquad \text{(B23)}$$

$$\bar{u} = u_e(y, z) + \Delta\bar{u} \qquad \text{(B24)}$$

where T_e is the local radiative equilibrium temperature determined from

$$\bar{Q}(T_e) \equiv 0 \qquad \text{(B25)}$$

and u_e is related to T_e through the balanced wind relationship[6]

$$2\left(\Omega + \frac{\bar{u}}{a\cos\varphi}\right)\sin\varphi\frac{\partial}{\partial z}\bar{u} = -\cos\varphi\frac{R}{H_{00}}\frac{\partial}{\partial y}\bar{T}. \qquad \text{(B26)}$$

[Substitute u_e for $\bar{u}$ and T_e for $\bar{T}$ in (B26).] Equation (B26) is obtained by combining the hydrostatic relation

$$H_{00}\frac{\partial}{\partial z}\bar{\Phi} = R\bar{T}, \qquad \text{(B27)}$$

with the balanced wind equation

$$\left(2\Omega + \frac{\bar{u}}{a\cos\varphi}\right)\sin\varphi\cdot\bar{u} = -\cos\varphi\frac{\partial}{\partial y}\bar{\Phi}, \qquad \text{(B28)}$$

which is correct up to and including order ϵ.

The zonal momentum equation is

$$\left(\frac{\partial}{\partial t} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z}\right)u_e\cos\varphi - f\bar{v} = \bar{X}. \qquad \text{(B29)}$$

[6] Near the equator, (B.26) fails unless $\frac{\partial}{\partial y}\bar{T} = 0$ at $y = 0$.

Knowing u_e from (B25) and (B26), this is a linear equation for the meridional circulation ($\bar{v}$, $\bar{w}$) assuming that the eddy momentum forcing, $\bar{X}$, is known. The $\bar{v}$ and $\bar{w}$ are related to each other by the continuity equation, which is

$$\frac{\partial}{\partial y}(\bar{v}\cos\varphi) + \frac{1}{\rho_0 z}\frac{\partial}{\partial z}(\rho_0(z)\bar{w}) = 0, \qquad \text{(B30)}$$

where $\rho_0(z) \equiv \rho_0(0)e^{-z/H_{00}}$.

Knowing $\bar{v}$ and $\bar{w}$, the energy equation

$$\left(\frac{\partial}{\partial t} + \bar{v}\frac{\partial}{\partial y} + \bar{w}\frac{\partial}{\partial z}\right)T_e + \bar{w}(T_e/T_{00})\frac{g}{c_p} = \bar{Z} + \frac{\partial}{\partial T}\bar{Q}(T_e)\cdot\Delta\bar{T} \qquad \text{(B31)}$$

then yields the dynamically induced part of the mean temperature $\Delta\bar{T}$. And from $\Delta\bar{T}$, the balanced wind relation gives diagnostically the dynamical correction to u_e. Note that the solution procedure outlined above for the scaled equations is linear in every step, while the original set of mean equations is nonlinear.

Note also that the above set of equations include the geostrophic equations as a subset; one can make the additional assumption of small Rossby numbers, $\text{Ro}|_l$, and the geostrophic equations will result from the above set. It also includes the nearly inviscid regime mentioned in section 6; even for small or zero eddy forcing $\bar{X}$, a mean meridional circulation ($\bar{v}$, $\bar{w}$) and an out of equilibrium temperature distribution $T_e + \Delta\bar{T}$ can still be the solution of our scaled set of equations, given appropriate boundary conditions.

4. The scaled mean equations in isentropic coordinates

The set of scaled mean equations in isentropic coordinates is the same as stated in section 2, except

(i) with the radiative equilibrium temperature $T_e(y, \theta)$ replacing $\bar{T}$ in the static stability parameter appearing in the thermodynamics equation (2.7).

(ii) with u_e replacing $\hat{u}$ in zonal the momentum equation (2.8a) and $\bar{u}$ in (2.8b), and similarly in (2.12a) and (2.12b).

(iii) with $\bar{q}(\bar{T})$ approximated by $\bar{q} = (\partial/\partial T)\,\bar{q}(T_e)\cdot(\bar{T} - T_e)$ and

(iv) under the present scaling, there is no difference between $\nabla\cdot\mathbf{F}$ and $\square\cdot\mathcal{F}$.

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