On the Nonlinear Versus Linearized Lower Boundary Conditions for Topographically Forced Stationary Long Waves

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ABSTRACT

For quasi-geostrophic stationary long waves forced by topography, the nonlinear lower boundary condition is derived in terms of the geopotential height and compared with the linearized version. The common practice of replacing terms describing the flow over and around a mountain by upstream zonal flow over the mountain and evaluating the resulting condition at sea level is found to be a good approximation for the cases considered and does not need to be modified as sometimes suggested. Specifically, it is found that this approximation does not affect, for most cases, the lower boundary condition expressed in terms of the geopotential height provided that the stationary wave is not near resonance. At resonance, the eddy advection terms may become important for large-amplitude waves when dissipation and surface diabatic heating are taken into account.

1. Introduction

In calculations for the response of stationary long waves in the atmosphere to topographic forcing, a commonly used lower boundary condition is the specification of the vertical velocity w_f induced by flow over mountains. If $h(x, y)$ is the height distribution of the topography, then from kinematic considerations, the above-mentioned boundary condition takes the form[1]

$$w_f = \mathbf{u}(x, y, h) \cdot \nabla_H h \quad \text{at} \quad z = h, \qquad (1.1)$$

with $\mathbf{u}$ being the vector velocity field of the flow relative to the surface of the earth and ∇_H being the horizontal gradient operator. The linearized version of (1.1) is usually given in the form[2]

$$w_f = \bar{u}(y, 0) \frac{\partial}{\partial x} h \quad \text{at} \quad z = 0, \qquad (1.2)$$

where $\bar{u}(y, 0)$ is the mean zonal flow at the sea level, and is assumed to be given.[3]

Three approximations are made in going from (1.1) to (1.2):

a) The term $v(\partial/\partial y)h$, commonly referred to as due to flow *around* the mountain, is dropped in the linearized version. The total upstream zonal flow $\bar{u}$, is forced to flow over the mountain without veering, as far as w_f is concerned.

b) It is the flow at the sea-level that is used in (1.2) to force a vertical velocity, instead of that at the top of the elevation.

c) The boundary condition is specified at the sea-level, instead of at the actual lower surface $z = h$.

It has often been noted that these approximations may be inaccurate even when the linearization assumption can be justified for the interior flow (see Dickinson, 1980, for a brief review). In this article, the effects of these approximations on the forced stationary long waves will be assessed separately.

It has often been argued that the effect of approximation a) is to *overestimate* the vertical velocity forcing for the stationary waves, especially in the presence of high elevations, because it assumes that the zonal flow is not impeded by the rise of the mountain. Saltzman and Irsch (1972) compared the vertical velocity calculated using the *observed* total near-surface wind with that calculated using the observed zonal wind in (1.2), and found that (1.2) generally overestimated the magnitude of the vertical forcing. Their suggested remedy of including the eddy meridional wind v' in the expression for w_f, i.e.,

[1] Cartesian coordinates, with x eastward, y northward and z upward, are used here in the Introduction only for simplicity of presentation. In the main text, spherical coordinates, with λ being the longitude, φ the latitude, and with log-pressure in the vertical, will be used throughout.

[2] A formal linearization about the zonal mean horizontal flow $(\bar{u}, \bar{v})$ should give

$$w_f = \bar{u} \frac{\partial}{\partial x} h + \bar{v} \frac{\partial}{\partial y} h.$$

In calculations concerning large scale stationary waves in extratropical latitudes, the mean meridional flow $\bar{v}$ is usually assumed to be much smaller than the mean zonal flow $\bar{u}$, thus giving (1.2).

[3] The evaluation of the linearized expression for w_f at the sea-level is commonly taken as part of the linearization process, although this is not necessary for the purpose of linearization, unlike the case of say, water waves on an interface.

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$$w_f = \bar{u}\frac{\partial}{\partial x}h + v'\frac{\partial}{\partial y}h,$$

cannot be easily incorporated in a linear theory. (In fact, it is inconsistent with the procedure of linearization, which drops all quadratic terms involving perturbation quantities, h being treated as a small perturbation quantity also.) We will show that such a remedy is not necessary. The meridional advection term, if included consistently in both the flow around mountain and in the temperature advection, yields little net effect on the geopotential height for most practical cases (although w_f is altered with the additional terms included). This result seems to be consistent with the nonlinear calculations of Egger (1976) and Ashe (1979), which show less discrepancy between the nonlinear and linear topographic solutions than hypothesized by Saltzman and Irsch.

It would also appear that approximation b) may introduce a substantial error (this time an *underestimate* of the forcing), because in (1.2) it is the surface wind $\bar{u}(y, 0)$ that is used instead of the wind at the actual surface $\bar{u}(y, h)$, which is usually of a substantially higher value during winter. It was sometimes proposed that instead of (1.2), the following modified (but still linear) version should be used (see, e.g., Lindzen *et al.*, 1982):

$$w_f = \bar{u}(y, h)\frac{\partial}{\partial x}h \quad \text{at} \quad z = 0. \tag{1.3}$$

It will be shown here, however, that as far as the wave response in geopotential height is concerned, the lower boundary forcing is unaltered whether one uses (1.3) or (1.2).

2. The lower boundary conditions for the conservative case

The general nonlinear boundary condition at the lower surface is derived in the Appendix in terms of the geopotential height Φ in the log-pressure coordinate $[z^* \equiv H_0 \ln(p_{00}/p)]$ for a hydrostatic atmosphere. In this section, we shall study the case of steady stationary long waves in the absence of dissipation and surface diabatic heating. (The nonconservative case will be discussed in a later section.) The boundary condition becomes (see A14):

$$-\mathbf{u}_P\cdot\nabla\left\{\frac{H}{H_0^2N^2}\Phi_{z^*} - \frac{1}{g}\Phi\right\} = \mathbf{u}_P\cdot\nabla h$$

$$\text{at the lower boundary,} \tag{2.1}$$

where $\mathbf{u}_P$ is the "horizontal" velocity in pressure coordinates. For quasi-geostrophic motions of either class 1 or class 2 (Phillips, 1963), the "horizontal" velocities can be expressed in terms of the geopotential height as

$$\mathbf{u}_P = \mathbf{u}_g \equiv \frac{1}{f}\mathbf{k}^* \times \nabla\Phi, \tag{2.2}$$

where $\mathbf{k}^*$ is the unit vector in the z^*-direction, and f is the Coriolis parameter which is equal to $2\Omega \sin\varphi$ for class 2 and a constant $2\Omega \sin\varphi_0$ for the class 1 waves. For either case, Eq. (2.1) can be written as

$$J\left[\Phi, \left\{\frac{H}{H_0^2N^2}\Phi_{z^*} + h\right\}\right] = 0$$

$$\text{at the lower boundary,} \tag{2.3}$$

where $J[A, B] \equiv \mathbf{k}^*\cdot\nabla A \times \nabla B$.

Eq. (2.3) implies that the quantity in brackets is conserved on surfaces of constant Φ, and that along such a surface the temperature $T = \Phi_{z^*}H_0/R$ is lower at a higher elevation.

Before proceding further, let us write down the linearized form of (2.3) for comparison. Letting

$$\Phi = \bar{\Phi}(\varphi, z^*) + \Phi'(\lambda, \varphi, z^*) \tag{2.4}$$

and ignoring the quadratic terms in the primed quantities and in the product of h and a primed quantity, Eq. (2.3) becomes

$$U\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left\{\frac{H}{H_0^2N^2}\left[\Phi'_{z^*} - \frac{U_{z^*}}{U}\Phi'\right] + h\right\} = 0$$

at

$$z^* = 0, \tag{2.5}$$

where

$$U(\varphi, z^*) = -\frac{1}{fa}\frac{\partial}{\partial\varphi}\bar{\Phi}$$

is the mean zonal flow.

Integrating (2.5) along a longitude circle, one has

$$\frac{H}{H_0^2N^2}\left[\Phi'_{z^*} - \frac{U_{z^*}}{U}\Phi'\right] + h = G_0(\varphi)$$

at

$$z^* = 0, \tag{2.6}$$

with $G_0(\varphi)$ being the "constant" of integration. Taking the zonal mean of (2.6), the right-hand side is found to be[4]

$$G_0(\varphi) = \bar{h}(\varphi).$$

Letting

$$h'(\lambda, \varphi) \equiv h - \bar{h}(\varphi), \tag{2.7}$$

we arrive at the commonly used lower boundary condition for linear waves:

$$\frac{H}{H_0^2N^2}\left[\Phi'_{z^*} - \frac{U_{z^*}}{U}\Phi'\right] + h' = 0 \quad \text{at} \quad z^* = 0. \tag{2.8}$$

Eq. (2.8) is easy to apply because it involves a derivative in the z^* direction only, evaluated at a coordinate surface.

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[4] In linear calculations, $\bar{h}(\varphi)$ is usually taken to be zero.

3. Integration of the nonlinear condition

The nonlinear lower boundary condition (2.3) appears to be much more complicated. Apart from the fact that it is nonlinear, it also involves derivatives in all three dimensions, evaluated at a variable lower surface. Nevertheless, it turns out that Eq. (2.3) can be integrated once, similar to the linear case, but here in a horizontal direction perpendicular to lines of constant Φ. Since the lower surface is a streamline (surface) to the lowest order in Rossby number, such an integration can be performed (with an error the order of a Rossby number) while still staying on the lower boundary as required by (2.3). This yields:

$$\frac{H}{H_0^2N^2}\Phi_{z*} + h = G(\Phi) \quad \text{at the lower surface.} \tag{3.1}$$

To find $G(\Phi)$, Eq. (3.1) is evaluated at some point $\lambda = \lambda_0$ on the lower surface, usually taken to be far upstream of the topography. Using the superscript 0 to denote quantities at such an "upstream" location, Eq. (3.1) gives:

$$G(\Phi^{(0)}) = \frac{H}{H_0^2N^2}\Phi_{z*}{}^{(0)} + h^{(0)}. \tag{3.2}$$

Letting

$$U = -\frac{1}{f}\frac{\partial}{a\partial\varphi}\Phi^{(0)}, \tag{3.3}$$

one can rewrite Eq. (3.2) as

$$G(\Phi^{(0)}) = \frac{H}{H_0^2N^2}a\int^{\varphi}(-fU_{z*})d\varphi + h^{(0)}$$

$$= \frac{H}{H_0^2N^2}a\int^{\varphi}\left(\frac{U_{z*}}{U}\right)(-fU)d\varphi + h^{(0)}. \tag{3.4}$$

If the quantity (U_{z*}/U) is a constant at the lower surface upstream, the integration in (3.4) can be performed easily to yield:

$$G(\Phi^{(0)}) = \left(\frac{H}{H_0^2N^2}\right)\left(\frac{U_{z*}}{U}\right)\cdot\Phi^{(0)} + h^{(0)}. \tag{3.5}$$

Although the quantity (U_{z*}/U) is in general not a constant, it is not a bad approximation to assume it is. This is because the approximation, which is equivalent to assuming that $U(\varphi, z^*)$ is separable in its φ and z^* variables, is not required for the whole atmosphere, but only at the lower surface upstream ($z^* \simeq 0$). Also, the approximation on (U_{z*}/U) is used only for the meridional extent of the topography under consideration. We will comment on the general case in a moment; here we shall proceed with (3.5).

Taking $h^{(0)}$ to be zero or a constant, and substituting (3.5) into (3.1), one obtains, at the lower surface,

$$\left[\frac{H}{H_0^2N^2}\Phi_{z*} - \left(\frac{U_{z*}}{U}\right)_s\left(\frac{H}{H_0^2N^2}\right)_s\Phi\right] + h - h^{(0)} = 0, \tag{3.6}$$

where the subscript s is used to denote evaluation at the upstream lower surface (sea level).

Though no assumption concerning the amplitude of the wave has been made, Eq. (3.6) turns out to be linear. It is also remarkable that it is in practically the same form as the linearized version (2.8) after the upstream part is subtracted out of it. We therefore can conclude from this result that for the present case the incorporation of the full nonlinear flow around the mountain produces no significant difference compared to the linearized case, as far as the geopotential height of the stationary wave response to topographic forcing is concerned.

With regard to approximation b), it is seen from Eq. (3.6) that it *is* the upstream flow U *evaluated at sea level* that enters into the boundary condition, and *not* the mean flow evaluated at the top of the elevation $z = h$, as in (1.3).

Approximation c), which applies the lower boundary condition at sea level instead of the actual surface height, cannot be justified in general when the mountain height is not small. It is, however, justifiable for waves with long vertical wavelength L_v. The error introduced by evaluating the boundary condition at $z^* = 0$ instead of $z^* = h^* = O(h)$ is of the order of $|h/L_v|$. For forced stationary ultralong waves, L_v is of the order of 100 km (see Tung and Lindzen, 1979), and so that ratio is generally small.

Thus, the approximate lower boundary condition takes the following convenient form, assuming that H/N^2 at the height of the topography is not too different from its value at sea-level,

$$\frac{H}{H_0^2N^2}\left[\Phi'_{z*} - \left(\frac{U_{z*}}{U}\right)\Phi'\right] + h' = 0 \quad \text{at} \quad z^* = 0, \tag{3.7}$$

where the upstream part has been subtracted and

$$\Phi' \equiv \Phi - \Phi^{(0)}, \quad h' \equiv h - h^{(0)}.$$

Note that (3.7) is identical in form to the linearized version (2.8) provided that U in (2.8) is appropriately interpreted as the upstream flow.

In general, the "upstream" quantity $(U_{z*}/U)_s$ is unknown in a nonlinear calculation, and has to be determined by a separate set of equations for the "mean flow." This is especially true for the case of long waves on a sphere, for which there is no distinction between "upstream" and "downstream" flow fields. For the case where the disturbance is localized, the "upstream" quantity is then treated as known and prescribed.

Let us now return to the case not considered so far. It is the case where the quantity $(U_{z*}/U)_s$ has a strong dependence on latitude. This case can be relevant if the upstream flow has both a strong z^*- and a strong φ-dependence, *and* if over the meridional extent of the mountain of interest, the φ-dependence of U_{z*} and that of U differ considerably. For this case, the lower boundary condition (3.1) would be nonlinear and there would presumably be a significant effect produced by the flow around the mountain. This is the situation when comparison with the linearized formulae is not appropriate.

4. The nonlinear lower boundary condition in the presence of Ekman pumping

When the no-slip boundary condition is used, an Ekman boundary layer should be introduced. Taking h to be the height at *the top of the Ekman layer*, and h^* to be the corresponding "height" in log-p coordinate, we have from (A19) the following boundary condition for the interior (i.e., outside the Ekman boundary layer) flow:

$$J\left[\Phi, \left\{\frac{H}{H_0{}^2N^2}\Phi_{z^*} + h\right\}\right] = -fD_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi\right), \quad (4.1)$$

assuming steady flows. Subscript P in any derivative operator is used to denote that the operation is to be performed at constant pressure. Eq. (4.1) can again be integrated once along surfaces of constant Φ to yield:

$$\frac{H}{H_0{}^2N^2}\Phi_{z^*} + h - G(\Phi) = -B, \quad z^* = h^*, \quad (4.2)$$

where B is the integral of the Ekman damping term along the surfaces of constant Φ, and has the form

$$B \equiv \int_{-\infty}^{\xi} d\xi\left[D_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi'\right)\right]\Big/|\mathbf{u}_P|, \quad (4.3)$$

with (Φ, ξ) forming the orthogonal von Mises "streamline" coordinate (see, e.g., Kaplun 1967). Evaluating the "constant" of integration $G(\Phi)$ upstream as before, and using primes to denote deviation from upstream conditions, we have

$$\frac{H}{H_0{}^2N^2}[\Phi'_{z^*} - (U_{z^*}/U)\Phi'] = -h' - B$$

at

$$z^* = h^*. \quad (4.4)$$

The importance of the Ekman term depends crucially on whether or not the system is near resonance. It has been shown in Tung and Lindzen (1979) that the condition for resonance for the stationary waves is that the left-hand side of (4.4) vanishes, i.e.,

$$\Phi'_{z^*} - (U_{z^*}/U)_s\Phi' = 0, \quad (4.5)$$

If Ekman damping is absent, (4.4) then implies that at resonance the steady state wave has an infinite amplitude; this is true for nonlinear geostrophic waves as it is for linear waves. Therefore it is obvious that B, no matter how small it is, must be retained near resonance.

At "off-resonance," i.e., when the left-hand side of (4.4) is much larger in magnitude than the second term on the right-hand side of that equation, the primary balance is between the response on the left-hand side and the topographic forcing on the right-hand side. Thus approximately

$$\frac{H}{H_0{}^2N^2}[\Phi'_{z^*} - (U_{z^*}/U)_s\Phi'] = -h' \quad (4.6)$$

and our previous assessments on various linearization approximations still hold. Near resonance, (4.6) breaks down. The balance of terms is now between forcing and dissipation, i.e.,

$$h' = -B. \quad (4.7)$$

Or, in differentiated form, (4.7) is

$$\frac{1}{f}J[\Phi, h'] = -D_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi'\right). \quad (4.8)$$

Using again U to denote the upstream flow, one can rewrite (4.8) as

$$D_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi'\right) + \frac{1}{f}J[\Phi', h'] = -U\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}h'$$

at

$$z^* = h^*. \quad (4.9)$$

Eq. (4.9) yields the amplitude of the response to the topographic forcing $U(\partial/\partial x)h'$ by the *upstream* flow. Assessing the effects of approximations a), b) and c) mentioned in the Introduction, we note that by replacing $U(z^* = h^*)$ by $U(z^* = 0)$ in the forcing term [approximation b)], one in effect underestimates the forcing term. However, the error is small for resonant waves with long vertical wavelength. This can be seen as follows:

$$U(z^* = h^*)\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}h' \simeq [U(z^* = 0) + U_{z^*}(z^* = 0)h^*]\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}h'$$
$$= [U(z^* = 0) + h^*U(z^* = 0)\Phi'_{z^*}/\Phi']\frac{1}{a\cos\varphi}$$

when (4.5) is used. Thus the error introduced is

$$h^*\Phi'_{z^*}/\Phi' = O(h'/L_v), \quad (4.10)$$

i.e., of the order of the ratio between the height of the elevation and the vertical wavelength. This is small for ultralong stationary waves. Since (4.9) is linear, approximation c), which evaluates the lower boundary condition at $z^* = 0$ instead of $z^* = h^*$, is also seen to make a small error of the same order as (4.10).

The effect of approximation a) of neglecting contribution by flow around the mountain is more difficult to evaluate. Generally, since the sign of this term, the second term on the left-hand side of (4.9), is the same as the first term, the Ekman damping term (both being negative), we expect that the effect of approximation a) is to overestimate the response to topographic forcing, with an error of the order of

$|u'_P|/U$. This is small for linear waves, but may be order-one in nonlinear calculations.

5. Summary

We have found that for stationary topographically forced waves the usual practice of replacing the mountain uplift $w_f = \mathbf{u}_H \cdot \nabla h$ at the height of the mountain by $\bar{u}(z^* = 0)(\partial/\partial x)h$ evaluated at sea level does not alter significantly the lower boundary condition in terms of geopotential height provided that:

1) The vertical wavelength of the wave is much longer than the height of the mountain;
2) The vertical and horizontal shears of the upstream flow are not both strong; and
3) The wave is not near resonant.

At resonance, this approximation still holds if the amplitude of the wave is not too large, as for example in the case of moderately large Ekman damping. For the case when the resonant amplitude is large, the approximation still does not affect the *vertical structure* of the wave, which is determined by (4.5). However, the amplitude of the response tends to be overestimated by neglecting the flow over high mountains. This situation can be remedied by re-evaluating the amplitude using (4.9), which is linear and does not involve normal derivatives in the boundary.

Therefore it seems that the usual linearized boundary condition is adequate for most of the situations and does not seem to require *ad hoc* modifications of the type frequently suggested, as these modifications may introduce inconsistencies and hence greater errors. We should qualify our results discussed so far by remarking that we have not considered the ageostrophic effects of the flow, nor have we treated the effect of transient eddies. Lastly, we remark that though we have not discussed the effect of diabatic heating here, this can be easily incorporated [in, say, Eq. (4.9) by adding a term $-\kappa Q/H_0^2N^2$ to its right-hand side (see Eq. (A19)]. In general, when heating is not small, meridional mean and eddy advection terms may become important in the lower boundary condition. Eq. (1.2), which is intended for topographic forcing only, needs to be modified.

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APPENDIX

The Full Lower Boundary Condition

Here we derive the general lower boundary condition using the primitive equations in height or log-pressure coordinates.

Let the lower surface be describable *in height coordinates* (longitude, latitude, height) = (λ, φ, z), by

$$z - h(\lambda, \varphi) = 0. \tag{A1}$$

The corresponding description of the same surface in log pressure coordinates (λ, φ, z^*), is

$$z^* - h^*(\lambda, \varphi) = 0, \tag{A2}$$

where $z^* \equiv H_0 \ln(1000 \text{ mb}/p)$, $H_0 \approx 7.5$ km, p is the log-pressure coordinate (see Holton, 1975), and h^* is the "height" of the topography measured in log-pressure coordinates and may be a function of time because surface pressure is a dynamic variable.

1. Conservative case

Let $\mathbf{n}$ be the vector normal to the surface $z - h = 0$. It is obtainable by applying the gradient operator to both sides of (A1), yielding

$$\mathbf{n} = \mathbf{k} - \nabla_H h, \tag{A3}$$

where $\mathbf{k}$ is the unit vector in the z-direction and ∇_H is the horizontal gradient operator at constant height.

Similarly, the corresponding vector $\mathbf{n}^*$ normal to the surface $z^* - h^* = 0$ in log-pressure coordinates is

$$\mathbf{n}^* = \mathbf{k}^* - \nabla_P h^*, \tag{A4}$$

where $\mathbf{k}^*$ is the unit vector in the z^*-direction and ∇_P is the "horizontal" gradient operator at constant pressure.

Let us first consider the inviscid case. The appropriate boundary condition is that the fluid velocity normal to the surface should vanish, namely

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{at} \quad z = h \tag{A5}$$

or

$$\mathbf{u} \cdot \mathbf{n}^* = 0 \quad \text{at} \quad z^* = h^*, \tag{A6}$$

where $\mathbf{u}$ is the total fluid velocity vector; it is independent of the coordinate system used. Using (A3) and (A4), one can rewrite (A5) and (A6) as

$$w = \mathbf{u} \cdot \nabla_H h \quad \text{at} \quad z = h, \tag{A7}$$

$$w^* = \mathbf{u} \cdot \nabla_P h^* \quad \text{at} \quad z^* = h^*. \tag{A8}$$

The definitions for the vertical velocities w and w^* in height and log-pressure coordinates are

$$w \equiv \frac{d}{dt} z, \quad w^* \equiv \frac{d}{dt} z^*. \tag{A9}$$

Eq. (A7) is the usual tangency condition for the flow of an inviscid fluid over a rigid surface. The corresponding condition in log-pressure coordinates, Eq. (A8), is not easily usable in its present form because h^* is unknown.

Instead, in pressure coordinates one usually first expresses w^* in terms of w and then uses (A7). Using the identity (see Kasahara, 1974):

$$0 = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P z^*$$

$$= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_H z^* + \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P z \frac{\partial z^*}{\partial z},$$

we have, from (A9)

$$w^* = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_H z^* + w\frac{\partial}{\partial z} z^*$$

$$= \left[-\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P z + w\right]\frac{\partial z^*}{\partial z}.$$

Thus

$$w^* = \frac{H_0}{H}\left[w - \frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P \Phi\right], \quad \text{(A10)}$$

since for a hydrostatic atmosphere,

$$\frac{\partial}{\partial z} z^* = -\frac{H_0}{p}\frac{\partial p}{\partial z} = \frac{H_0}{H},$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P gz = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P \Phi,$$

the latter being a definition for the geopotential Φ. Now (A7) is, in pressure coordinates

$$w = \mathbf{u}\cdot\nabla_P h. \quad \text{(A11)}$$

This is because

$$\nabla_P h = \nabla_H h + \frac{\partial h}{\partial z}\nabla_P z,$$

$$\frac{\partial}{\partial z} h(\lambda, \varphi) = 0.$$

Therefore we can now replace (A8) by

$$w^* = \frac{H_0}{H}\left[\mathbf{u}\cdot\nabla_P h - \frac{1}{g}\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P \Phi\right]$$

at

$$z^* = h^*. \quad \text{(A12)}$$

We have not completely eliminated h^* in (A12), because the lower boundary still has to be specified at $z^* = h^*$. However, as we shall show, the location of the lower boundary can be approximated by $z^* \approx 0$ for waves with long vertical wavelengths, so that h^* does not appear any longer.

To express w^* in terms of the geopotential height, we use the following form of the thermodynamic equation (see Holton, 1975):

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P \Phi_{z^*} + N^2 w^* = \frac{\kappa Q}{H_0}, \quad \text{(A13)}$$

where

$$N^2(z^*) \equiv \frac{R}{H_0}\left(\frac{dT_e}{dz^*} + \frac{\kappa T_e}{H_0}\right), \quad \kappa \equiv \frac{R}{C_p}.$$

Taking the diabatic heating rate Q to be zero for the present case, one can rewrite (A12) as

$$-\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P\left[\frac{H}{H_0^2N^2}\Phi_{z^*} - \frac{1}{g}\Phi\right] = \mathbf{u}\cdot\nabla_P h$$

at

$$z^* = h^* \approx 0. \quad \text{(A14)}$$

2. Nonconservative case

In the presence of viscosity, the requirement of no-slip at the rigid surface introduces rapid variations of the fields with the vertical. When account is taken of the effect of this "planetary boundary layer," an Ekman pumping term is introduced in the vertical velocity at the top of the boundary layer. This pumping term has been given approximately by Charney and Eliassen (1949) to be

$$w_E = D_E \zeta, \quad \text{(A15)}$$

where ζ is the geostrophic vorticity at the top of the Ekman layer. The coefficient of eddy diffusion D_E has been assumed to be a constant in this simple model. The vertical velocity boundary condition (A7) should now be modified to

$$w = w_f + w_E \quad \text{at} \quad z = h, \quad \text{(A16)}$$

where w_E is given by (A14) and w_f is the part of the vertical velocity caused by the topographical uplift, i.e.,

$$w_f = \mathbf{u}\cdot\nabla_H h. \quad \text{(A17)}$$

In all these expressions and in those to follow, h is understood to stand for the height at the top of the Ekman boundary layer, i.e., the height of the topography plus the boundary layer thickness.

Incorporating the above Ekman pumping term and the surface diabatic heating term neglected in Section 1 above, we find that (A14) is modified to

$$-\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)_P\left[\frac{H}{H_0^2N^2}\Phi_{z^*} - \frac{1}{g}\Phi\right]$$

$$= \mathbf{u}\cdot\nabla_P h + D_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi\right) - \frac{\kappa H}{H_0^2N^2}Q$$

at

$$z^* = h^* \approx 0. \quad \text{(A18)}$$

For geostrophic flows, the horizontal velocity is expressible, to the lowest order in Rossby number, as

$$\mathbf{u}_P = \frac{1}{f}\mathbf{k}^* \times \nabla\Phi.$$

Eq. (A18) becomes, in terms of Φ only,

$$-\frac{\partial}{\partial t}\left(\frac{H}{H_0^2N^2}\Phi_{z^*}-\frac{1}{g}\Phi\right)$$
$$=\frac{1}{f}J\left[\Phi,\left\{\frac{H}{H_0^2N^2}\Phi_{z^*}+h\right\}\right]$$
$$+D_E\nabla_P\cdot\left(\frac{1}{f}\nabla_P\Phi\right)-\frac{\kappa H}{H_0^2N^2}Q$$

at

$$z^* = h^* \approx 0, \tag{A19}$$

where

$$J[A, B] \equiv \mathbf{k}^*\cdot\nabla A\times\nabla B.$$

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