On Radiating Waves Generated from Barotropic Shear Instability of a Western Boundary Current

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ABSTRACT

Results from barotropic stability analyses of strong westerly jets generally tend to conclude that those waves that can extract energy from the shear of the jet are largely trapped by the jet. Therefore, the available shear energy of the flow cannot be transmitted by propagating waves away from the strong shear zone. We show here that the situation is different if the jet is in the meridional instead of the zonal direction. In this case, unstable waves are generated that are able to propagate energy eastward even in the presence of a realistic dissipation. It is then possible for the shear energy of a western boundary current to be transported large distances into the interior of the ocean.

1. Introduction

Two recent papers by Talley (1983a, b) have explored the possibility that barotropic and/or baroclinic zonal jets have linearly unstable modes with a horizontally radiating structure. Such modes, even if slowly growing, would propagate energy away from the jet and in the ocean interior, and would provide a possible mechanism for the observed distribution of eddy energy in the ocean (see Schmitz et al., 1983, for a review of this problem).

Earlier studies (Pedlosky, 1977; Harrison and Robinson, 1979) had considered the radiation of energy in a basin with an imposed boundary forcing representing a meandering current, and Talley extended this line of work with a linear model explicitly including the unstable current itself.

The main result was that a barotropic eastward jet does not have the required kind of radiating instability. Modifications to the basic state, like baroclinicity either in the jet or in the far field, were needed to support radiating modes. The apparent reason for this failure is that an eastward jet has only unstable modes with eastward phase speed; modes with a negative (westward) phase speed cannot be ruled out a priori on the β-plane (see Pedlosky, 1979), but they are not found. The far field, on the other hand, has only free modes with westward phase speed (Rossby waves). The forcing is thus being applied in a frequency domain in which the far field cannot respond with wave motion, and the unstable modes are therefore strongly trapped near the source.

As an alternative to the introduction of baroclinicity either in the jet or in the far field, we observe that in the geophysical system under consideration the unstable current is not always, and never exactly, zonal. The Gulf Stream flows, for part of its path, in a predominantly meridional direction. A northward flowing jet is a possible idealization of such a current, and theoretical models of the ocean circulation show a northward flowing western boundary current. We then want to answer the following questions: Can a meridional jet on the β-plane have eastward radiating barotropic instability? How susceptible are those waves to the presence of damping? We will make use of a bottom Ekman friction consistent with that required by Stommel's model of the western boundary current (see Pedlosky, 1979) and we assume that the mechanism that dissipates the wave energy is the same responsible for the existence of the basic state current itself. We will not be concerned with mechanisms (viz. vorticity sources and sinks), other than Ekman pumping, that may be needed to make our basic state a steady solution of the barotropic vorticity equation. For mathematical convenience we will use a piecewise constant basic state velocity profile: it is assumed that it is maintained by some unspecified sources that enter the O(1) (basic state) equation, but not the O(ϵ) in a formal expansion in the amplitude of the perturbation. What is studied here is the linear stability of such a prescribed basic flow to barotropic disturbances.

2. The model

The problem that we consider is described by the barotropic vorticity equation

$$\frac{d}{dt}\zeta+\beta v=-\nu^*\zeta+\mathcal{F}, \tag{1}$$

where $\zeta=v_x-u_y$ and $\mathcal{F}$ is any forcing needed to make the basic state that we will choose a steady solution. The x-coordinate is in the eastward direction and the y-coordinate is northward; u and v are zonal and meridional velocity, respectively; β is the latitudinal variation of the Coriolis parameter and ν^* is a dissipation coefficient.

Consider a steady solution $V(x)$. For such a flow, Eq. (1) becomes

$$\beta V=-\nu^* V_x+\mathcal{F}, \tag{2}$$

which defines the forcing $\mathcal{F}$. We now examine small perturbations, denoted by primed quantities, to this steady solution. Let $v=V(x)+v'$, $u=u'$. Neglecting quadratic terms in the amplitude of the perturbation, Eq. (1) becomes, after subtraction of (2):

$$(\partial_t+V\partial_y)(v'_x-u'_y)+u'\partial_x V_x+\beta v'=-\nu^*(v'_x-u'_y). \tag{3}$$

The flow is nondivergent for the continuity equation $u'_x+v'_y=0$; so we can define a streamfunction ψ such that $u'=-\psi_y$, $v'=\psi_x$:

$$(\partial_t+V\partial_y)\nabla^2\psi-\psi_y V_{xx}+\beta\psi_x+\nu^*\nabla^2\psi=0. \tag{3'}$$

This equation has coefficients that are only functions of x, so the normal mode assumption can be made in the other direction

$$\psi=\mathrm{Re}\{\phi(x)e^{i(ly-\omega t)}\}.$$

With $c=\omega+i\nu^*/l$, the equation for ϕ is

$$\phi_{xx}+\frac{\beta}{il(V-c)}\phi_x-\left(\frac{V_{xx}}{V-c}+l^2\right)\phi=0. \tag{4}$$

The presence of the first-order term makes it difficult to extend the integral theorems (Rayleigh-Kuo, Miles-Howard, Fjortoft) that are known for the zonal case. It also introduces an asymmetry in the propagation properties of the solution, as we will see below.

Consider first the case with no friction. If the basic flow $V(x)$ is a jet on an infinite plane, i.e., assumes a constant far-field value V_1 for $x\to\pm\infty$, the solutions in the far field will be waves of zonal wavenumber k satisfying the usual relation for frequency and wavenumber that holds for Rossby waves

$$c-V_1=-\frac{\beta k}{l(k^2+l^2)}. \tag{5}$$

Here c is allowed to be complex and determined by explicitly solving the eigenvalue problem involving the shear flow. For fixed ω and l, Eq. (5) gives the zonal wavenumbers k as $x\to\pm\infty$.

$$k=\frac{\beta}{2l(V_1-c)}\pm\left(\frac{\beta^2}{4l^2(V_1-c)^2}-l^2\right)^{1/2}. \tag{6}$$

As boundary condition at $x\to\pm\infty$, we require that the solution goes to zero or that the group velocity is directed outward.

We can assume $V_1=0$ with no loss of generality. If we rewrite (5) with $V_1=0$ for its real and imaginary part we get

$$\omega_r=-\frac{\beta k_r}{|l^2+k^2|^2}(l^2+|k|^2);\quad \omega_i=-\frac{\beta k_i}{|l^2+k^2|^2}(l^2-|k|^2),$$

from which we see that only neutral waves are purely radiating, which is consistent with the physical argument that if a disturbance is growing in time and propagating, what is observed far from the source has been generated at an earlier time, and thus with a smaller amplitude, than what is currently observed near the source.

For all unstable waves we then require that the two roots (6) have opposite imaginary parts, so that one of them is acceptable in either limit $x\to\pm\infty$. That this is always the case is easily seen from Eq. (5) rewritten as a second-degree equation for k [the solutions of which are (6)]:

$$k^2+\frac{\beta}{\omega}k+l^2=0.$$

The product of the two roots of this equation is l^2, which is real, so that the two (complex) roots always have opposite phases, and so opposite imaginary parts.

For purely radiating solutions we consider the group velocity:

$$c_{gx}=\frac{\partial\omega}{\partial k}=\beta\frac{k^2-l^2}{(k^2+l^2)^2}.$$

From (6) we can evaluate k^2-l^2:

$$k^2-l^2=\frac{\beta}{2\omega^2}\left[1-\frac{4\omega^2l^2}{\beta^2}\pm\left(1-\frac{4\omega^2l^2}{\beta^2}\right)^{1/2}\right],$$

and since $0<1-(4\omega^2l^2/\beta^2)<1$ (where the lower limit is required to have real k and thus purely radiating waves), this is always positive for the upper sign and negative for the lower. Consequently, the group velocity is always eastward for the short (high wavenumber) wave and westward for the long wave, as is commonly known.

As discussed in Pedlosky (1965), the larger scales of motion, as those that propagate westward, are essentially unaffected by bottom friction and will be propagated into the interior.

On the other hand, the short Rossby waves that propagate energy eastward may become rapidly trapped when frictional dissipation is incorporated. We then want to see if *unstable* waves generated at a western boundary are able to overcome dissipation and propagate energy in the interior. With this in mind, we will solve the stability problem on a semi-infinite β-plane, for a rectangular jet located right at the western boundary:

$$V(x)=\begin{cases} V_0, & 0<x<x_0 \\ 0, & x>x_0. \end{cases}$$

With this basic state, $\phi \sim e^{ikx}$ everywhere and (6) gives the wavenumbers.

When we explicitly include dissipation, the growth rate is decreased by an amount ν^*, and all of the preceding discussion still holds with ω_i replaced by $\omega_i + \nu^*$. In this case, the solutions with $k_i = 0$ become damped in time with a decay rate $\omega_i = -\nu^*$.

For solutions with $\omega_i \geq 0$ we will find perturbations with a decaying (in space) envelope. We will observe the solution as a wave if the spatial decay is slow enough to allow for significant amplitude far from the source, and the wave will be able to radiate energy away from the source if, in addition to a positive group velocity, its growth rate is high enough to overcome frictional dissipation.

We now proceed to the explicit calculation of the eigenvalues. If the width of the jet x_0 is taken as a length scale and its velocity V_0 is a velocity scale, the only nondimensional parameter is $\hat{\beta} = \beta x_0^2/V_0$ and the equations in nondimensional form are

$$\begin{cases} \phi_{xx}-\dfrac{\hat{\beta}}{ilc}\phi_x - l^2\phi = 0, & x>1 \\ \phi_{xx}+\dfrac{\hat{\beta}}{il(1-c)}\phi_x - l^2\phi = 0, & 0<x<1. \end{cases}$$

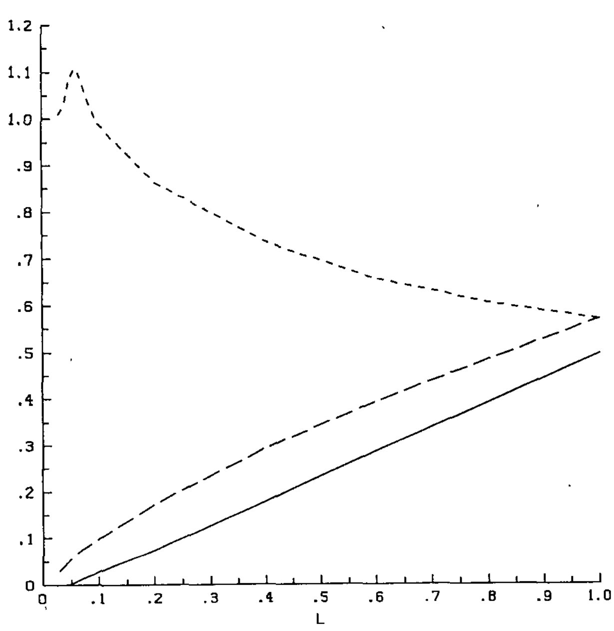


FIG. 1. Eigenvalues for $\hat{\beta} = 0.01$. Solid line: inviscid growth rate ω_i; long dashed line: frequency ω_r; short dashed line: meridional phase speed $c_r = \omega_r/l$.

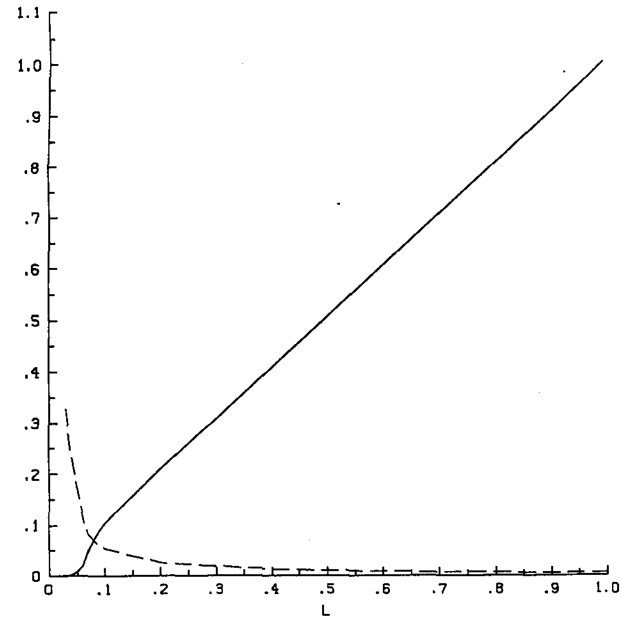


FIG. 2. Zonal wave numbers for $\hat{\beta} = 0.01$. Solid: imaginary part k_i—spatial decay rate; dashed: real part k_r—wave number of the oscillatory part of the solution.

As shown previously, there is always one solution satisfying the boundary conditions in the external region $x > 1$.

The solution is then

$$\begin{cases} \phi^{\mathrm{I}} = A(e^{ik_1^{(1)}x} - e^{ik_1^{(2)}x}), & 0<x<1 \\ \phi^{\mathrm{II}} = Be^{ik_2x}, & x>1. \end{cases} \qquad (7)$$

Here the condition of no normal flow $\phi = 0$ at the solid boundary $x = 0$ has been applied. The wavenumbers are

$$k_1^{(1,2)} = \frac{1}{2(1-c)}\{b \pm [b^2 - 4(1-c)^2l^2]^{1/2}\}$$

$$k_2 = \frac{1}{2c}[-b \pm (b^2 - 4c^2l^2)^{1/2}]$$

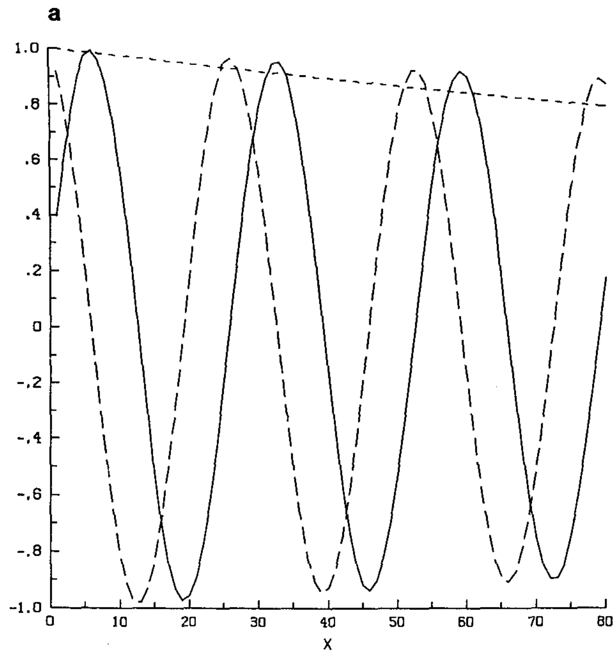
and $b = \hat{\beta}/l$ has been defined. The sign of k_2 must be chosen appropriately, while $k_1^{(1,2)}$ can be interchanged with no consequences.

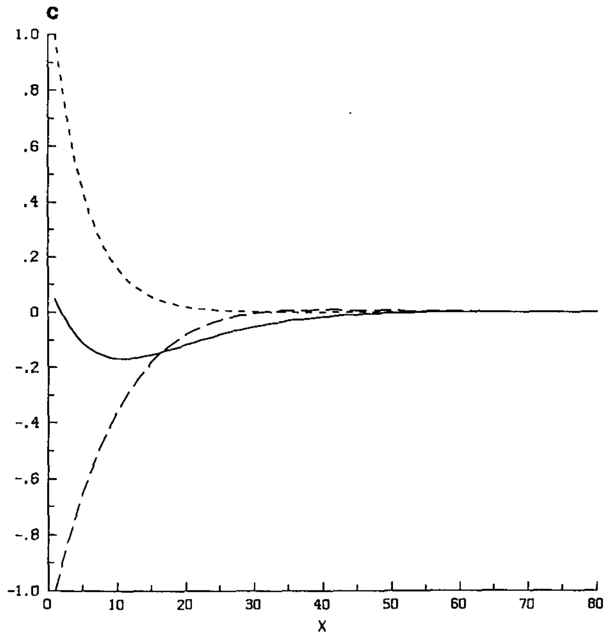
The solutions (7) need to be matched at the interface $x = 1$; the requirement of continuity of the displacement normal to the interface and integration of (4) across the interface give the matching conditions:

$$\Delta\left[\frac{\phi}{c-V}\right] = 0; \quad \Delta[(c-V)\phi_x + ib\phi] = 0,$$

where $\Delta[\cdot]$ indicates the jump that the quantity in brackets experiences across the interface.

The dispersion relation derived from those conditions is





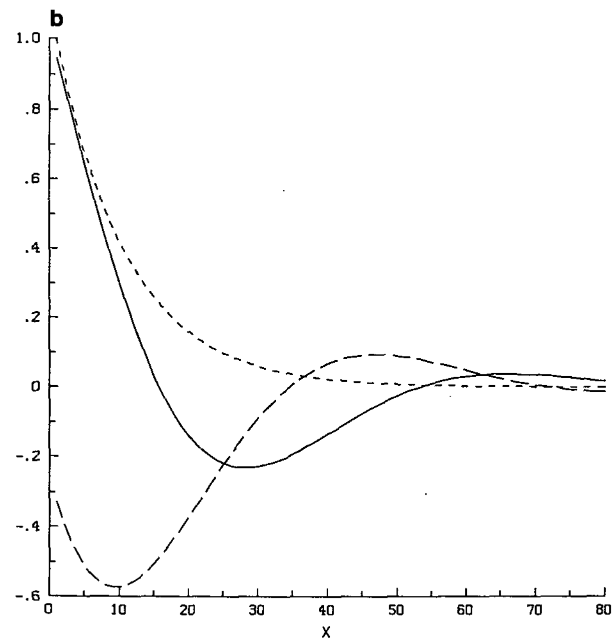


FIG. 3. Zonal structure of the eigensolutions. Solid: real part ϕ_r; long dash: imaginary part ϕ_i; short dash: $|\phi|^2$, which is proportional to perturbation kinetic energy. (a) $l = 0.04$, $\omega_r = 0.0412$, inviscid $\omega_i = 0.0002$, $c_r = 1.030$, $k_r = -0.2360$, $k_i = 0.0015$, zonal group velocity $c_{gx} = 0.1648$; (b) $l = 0.07$, $\omega_r = 0.0759$, $\omega_i = 0.0138$, $c_r = 1.084$, $k_r = -0.0838$, $k_i = 0.0485$, $c_{gx} = 0.5061$; (c) $l = 0.1$, $\omega_r = 0.0991$, $\omega_i = 0.0276$, $c_r = 0.9906$, $k_r = -0.0537$, $k_i = 0.1026$, $c_{gx} = 1.625$.

$$[(c-1)^2 k_1^{(1)} - c^2 k_2 - b]e^{ik_1^{(1)}} - [(c-1)^2 k_1^{(2)} - c^2 k_2 - b]e^{ik_1^{(2)}} = 0.$$

This has been solved numerically for some values of the parameters. Results are shown in the next section.

3. Results

The model has one unstable mode for meridional wavenumbers l larger than a cutoff value that is zero for $\hat{\beta} = 0$ and increases with increasing $\hat{\beta}$.

Figure 1 shows the eigenvalue ω and the meridional phase speed $c_r = \omega_r/l$ of the unstable mode for a typical value of $\hat{\beta}$ (=0.01), and Fig. 2 gives the zonal wavenumbers of those solutions. As $l \rightarrow \infty$, the limits $c \rightarrow (1/2, 1/2)$ and $k_2 \rightarrow (0, l)$ are reached. As l decreases, ω_r and ω_i both decrease but the meridional phase speed ω_r/l increases and reaches a maximum larger than the maximum basic state velocity before it starts decreasing again. We note that this never happens for the streamwise phase speed when the basic current is zonal, as shown by the semicircle theorem. There is no known extension of this for a meridional current. All these solutions have positive zonal group velocity $\partial\omega_r/\partial k_r$.

For those same waves with maximum meridional

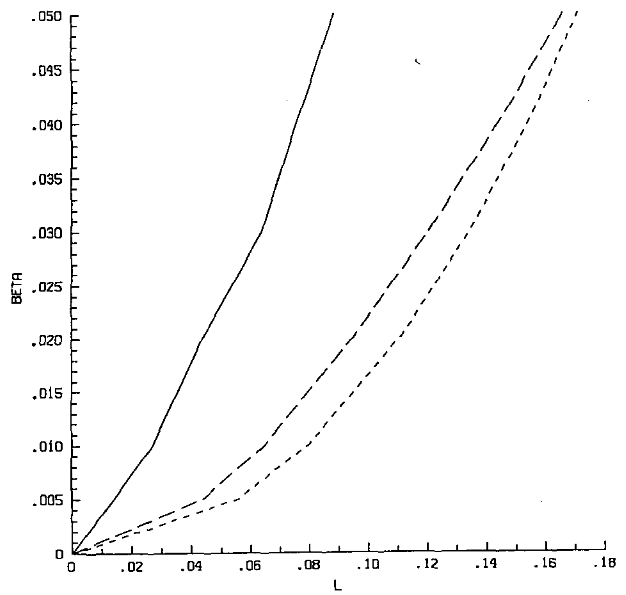


FIG. 4. Solid: inviscid marginal stability line—the region to the left of this curve is stable. Long dashed line: marginal stability for $\nu = \hat{\beta}$—to the left of this curve the inviscid growth rate is smaller than the dissipation so that the mode is decaying in time. Short dashed line: locus of the modes with $|k_r| = k_i$. We call "radiating" the region between the two latter curves, where $\omega_i + \nu \geq \hat{\beta}$ and $|k_r| \geq k_i$.

phase speed, the real part of the zonal wave number becomes larger than the imaginary part. This seems a reasonable minimal requirement for "radiating" modes. In the case shown, this happens near $l = 0.08$. This wave has a growth rate (inviscid) of 0.0188, so this value can be taken as an upper bound on the friction coefficient ν compatible with the presence of radiating modes: for larger ν all unstable ($\omega_i > 0$) modes are trapped ($k_i > |k_r|$); for smaller ν there is a small interval of wavenumbers (zonal and meridional) that have both $\omega_i > 0$ and $k_i < |k_r|$.

To estimate a typical value of ν we recall that the width of the boundary layer in Stommel's model is ν^*/β. That is the width of the jet that we called x_0. So $x_0 \sim \nu^*/\beta$; in nondimensional form $\nu \sim \hat{\beta}$. If we take $\nu = \hat{\beta}$, we see that for the case shown, all $l > 0.06$ are unstable, so the waves $0.06 < l < 0.08$ can be considered radiating modes.

It may be noted at this point that although in this model the group velocity needs only to be positive, in the real ocean there is a slow westward drift which feeds the western boundary current so that the group velocity should overcome this effect in order to have radiation of energy. The group velocity of our solutions in the radiating range goes from 0.34 at $l = 0.06$ to 0.88 at $l = 0.08$; these values seem high enough to satisfy the more stringent condition.

We show in Fig. 3 the zonal form of the solutions for three different wavenumbers l. Figure 3a is at $l = 0.04$, close to the inviscid marginal stability. The exact values of frequency, growth rate, phase speed, group velocity and zonal wavenumber are given in the figure caption. It is apparent that this is a wave structure with little spatial damping away from the source. On the other hand, its growth rate is very small, so that this solution is unstable only with very small or no viscosity. $l = 0.07$ (Fig. 3b) is within the radiating range: its growth rate is high enough to overcome a realistic dissipation (a dissipation consistent with the width of the boundary current as predicted in Stommel's model), while its spatial structure still gives a significant amplitude at a distance away from the source an order of magnitude larger than the boundary current. Finally, Fig. 3c is at $l = 0.1$; its growth rate is high, but so is k_i and the solution is bound to the source.

The behavior shown in detail for $\hat{\beta} = 0.01$ is typical of other values of this parameter. Figure 4 shows the upper and lower bounds of the interval of radiating modes (together with the inviscid marginal stability curve) on the $(l, \hat{\beta})$ plane. The upper bound is taken to be where $|k_r| = k_i$; the lower where $\omega_i + \nu = \hat{\beta}$. It can be seen that the interval is small and quickly disappears when $\hat{\beta}$ goes above 0.05. The radiating interval can be larger than the one shown for smaller ν and smaller or nonexisting for larger dissipation.

4. Summary

We have shown that a simple meridional current can have radiating barotropic instability for values of friction consistent with the width of the boundary current as predicted in Stommel's model. The radiating modes have long meridional wavelength and they have a meridional phase speed larger than the speed of the jet that generates them. We find radiating modes for values of the nondimensional parameter $\beta x_0^2/V_0$ (x_0 width of the jet, V_0 speed of the jet) smaller than 0.05 and a dissipation time $t_d = (\beta x_0)^{-1}$.

The model is too simplified to be directly compared with observations but is, in our view, sufficient to point out that the existence of barotropic radiating modes that has been ruled out in studies of westerly zonal jets still needs to be considered if the unstable current is not zonal.

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