A Theory of Stationary Long Waves. Part I: A Simple Theory of Blocking

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ABSTRACT

A theory is presented that attempts to explain the tropospheric blocking phenomenon caused by the resonant amplification of large-scale planetary waves forced by topography and surface heating. It is shown that a wave becomes resonant with the stationary forcings when the wind condition in the lower atmosphere is such that the phase speed of the wave is reduced to zero. The resonant behavior of the wave in the presence of Ekman pumping and other damping mechanisms is used to account for the time amplification of the pressure ridges that is an essential part of the blocking phenomenon. This same time behavior also allows the waves to interact with the mean flow in the stratosphere, possibly initiating major sudden warmings. Such a situation was, in fact, *assumed* by Matsuno (1971) in the lower boundary of his stratospheric model of sudden warming.

The basis for reviving the classical normal-mode theory (when faced with the difficulties associated with the zero-wind line) is presented in Part III. The present Part I serves as an introduction to the three-part series and discusses, with the aid of a simple mathematical model, the relevant physical mechanisms involved. Though the theory of resonant Rossby waves is a classical one, the contribution of the present papers is in pointing out that the theory offers, despite the difficulties and controversies associated with it, a viable mechanism that may be the cause of some prominent physical phenomena in the atmosphere.

1. Introduction and discussion

This is the first of a series of papers devoted to a study of stationary long waves in the atmosphere. Despite the importance of these waves in influencing large-scale weather and climate, no consistent theory exists that can satisfactorily account for many aspects of the wave behavior. One of the important but as yet unresolved questions concerning the stationary waves is: Why do certain large-scale waves amplify as they are observed to do during some winters?

The severe winter conditions experienced in the United States in 1977 are believed to be the results of a stationary high-pressure system which amplified near the east coast of the Pacific. The situation in the winter of 1977, though unusual, is not unique. A similar weather pattern also occurred during a previous record-breaking cold winter in 1963, when the United States experienced numerous outbreaks of cold arctic air with accompanying heavy snows and blizzards in the Northeast and persistent droughts in the West, while Alaska was abnormally warm. The weather patterns of these two "abnormal" winters in the United States are strikingly similar and both can be attributed to the same cause—a persistent and "enormously amplified ridge in the eastern Pacific" (O'Connor, 1963). The blocking ridge, as it is called by meteorologists, is an amplified stationary high-pressure center that blocks the normal passage of the westerlies and the accompanying weather. When a block forms near the west coast, it diverts to the north the warm moist air from the Pacific, causing drought in the western states and high temperatures in the arctic region, while the anticyclonic flow associated with the high guides cold polar air to the south over most of the states east of the Rockies. Sometimes the jet "meanders" as far south as Florida, causing citrus damage, as reported for both winters.

Simultaneously with the large-scale blocking activity in the troposphere, a sudden warming was observed in the stratosphere in both winters. *Amplifying* stationary waves were recorded to precede the warming events. Sudden warming is a stratospheric phenomenon that occurs during some winters in high-latitude regions. It evolves in the darkness of the polar night with no apparent external source of heating. Yet in a major warming event the mean temperature of the stratosphere is increased at a spectacular rate of up to 10°C day^{-1} and in less than a week the normal north-south temperature gradient (with warmer temperatures near the equator and colder ones at the North Pole) is reversed. The strong westerly circumpolar jet stream that existed before the warming is com-

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pletely destroyed and sometimes even reverses its direction.

The simultaneous occurrences of both phenomena are probably not mere coincidences. It seems that during every sudden warming occurrence in the stratosphere, a large-scale blocking ridge can always be found near the surface level. Such coincidence has been noted previously for some cases by researchers studying the sudden warming phenomenon (Murakami, 1965; Julian and Labitzke, 1965; Labitzke, 1965). That it seems to be true, in general, can be verified by checking the weather records for the periods when a major warming is taking place in the stratosphere.

More than one blocking high can sometimes occur simultaneously over different regions. For example, simultaneously with the blocking activity over the North American continent another ridge prevailed and amplified between Iceland and Britain during January 1963. The disturbance roughly resembles a wavenumber 2 pattern with two highs and two lows around a longitude circle. In the stratosphere the warming event that was taking place at the same time also had a wavenumber 2 character (Finger and Teweles, 1964).

There are also years when pronounced blocking occurrences in the troposphere are not accompanied by warming events in the stratosphere. An example is the winter of 1968–69 when ridges amplified simultaneously over the United States, Europe and Asia (a wavenumber 3 pattern), while no significant warming in the stratosphere was recorded. In fact, it is well known that blocking activity in the troposphere is a much more frequent phenomenon and can occur during all four seasons (Namias, 1964; Sumner, 1954; Brezowsky *et al.*, 1951; Rex, 1951). The disturbance can often have a higher wavenumber character. The absence of high wavenumber disturbances in the stratosphere can be attributed to the "filtering" effect of the atmosphere, which traps the shorter wavelength disturbances and prevents them from reaching the stratosphere (Charney and Drazin, 1961). The higher frequency of occurrence of the blocking phenomenon, as compared to the stratospheric warming events, may suggest that it is easier to excite the shorter wavelength disturbances than the larger ones.

In this paper we present a theory that attempts to explain blocking phenomena as caused by the resonance of planetary-scale waves forced by topography and land-sea differential heating. Possible relations to sudden warmings are also discussed. The reasons for focusing our attention on resonant waves instead of other mechanisms are twofold: first, it is generally observed that preceding the onset of warmings, planetary waves amplify in time in the stratosphere and these amplifying waves can be traced all the way down to sea level (Muench, 1965). Second, in order for the wave disturbances to alter the mean flows and temperature, wave-mean interactions must take place. But it is well known that no such interactions are likely if the waves are periodic or steady (Dickinson, 1969). This fact, however, does not preclude the possibility that the phenomena are caused by *unstable* waves, but previous attempts at explaining the warming as caused by barotropic and baroclinic instabilities of the stratospheric jet are largely unsatisfactory. There is some indication that the jet is stable on the time scales relevant to the problem (McIntyre, 1972), and for the cases where the necessary conditions for instabilities seem to have been met, it is found (Charney and Stern, 1962) that the stratospheric jet stream in the Southern Hemisphere should be more unstable by the same criteria, and yet no major warming events have been observed there. Indeed, as we will show in a separate paper where the necessary and sufficient conditions for instability are derived, the polar night jet is *not* unstable to stationary disturbances.

One major difference between the Northern and Southern Hemispheres is the degree of inhomogeneity of the earth's surface. Large-scale topographical disturbances are generated by wind blowing over major mountains and oceans. A large component of the energy of the disturbances lies in the wavenumber 1 and 2 regime. Cooling over land masses and warming over the oceans are also capable of producing large-scale disturbances of comparable magnitude, with possibly a difference in phase compared to that of topographic origin. The wave disturbances generated by the combined effects of topography and differential heating propagate their energy upward and, during winter when the mean wind is favorable for vertical propagation, the longest waves can penetrate into the stratosphere. It is generally agreed that the distortions of the circumpolar jet which are observed in the stratosphere during winter are due to the stationary long waves propagated up from below.

That the warming events in the stratosphere are probably caused by waves generated in the troposphere is revealed by several studies of the energetics of the stratosphere. It is found that preceding the warming events, unusually large fluxes of wave energy are transferred to the stratosphere from below and represent a main source of energy for the whole event (Reed *et al.*, 1963; Murakami, 1965; Muench, 1965; Julian and Labitzke, 1965; etc.). Early researchers were misled by the apparent downward propagation of the warming from above the highest observational level to search for an upper energy source. Recent observational studies show that the initial phase of warming probably occurs simultaneously at all levels, but the effects begin to be more pronounced at higher levels, probably due to the lower density of the atmosphere there. Moreover, numerical simulations, notably that due to Matsuno (1971), have been rather successful in reproducing many of the observed features of a warming event by specifying an observed wave forcing at the bottom of the stratosphere as a boundary condition. These experiments confirm the notion suggested by observations that the warming is forced by waves from below the stratosphere.

As pointed out by Matsuno, an important feature of the forcing function that he used (based on the observations of Hirota) is that of *wave amplification* preceding the onset of warming. It is the convergence of heat and momentum fluxes of this "transient" wave that is responsible for producing an induced meridional circulation through which the Coriolis torque acts to decelerate the westerly jet. Since the waves have larger amplitudes at greater heights due to the density effect, the deceleration of the mean flow is greatest at the higher levels, where the jet is first destroyed. The appearance of an easterly flow above the westerly produces a critical level where the mean wind speed matches the phase speed of the (stationary) wave. Here the drastic dissipation of the easterly momentum of the wave is responsible for the suddenness of the destruction of the westerly jet and the accompanying large increase in temperature. The easterly region descends as the westerly wind is destroyed. This sequence of events would not have been possible without the initial amplification of the wave present at the lower boundary (300 mb) of Matsuno's model. Since the presence and the behavior of this wave has not yet been accounted for, sudden warming remains in this sense an unexplained phenomenon, though Trenberth (1973) produced some time amplifying wave features in his nonlinear numerical model with annual heatings. Why certain waves are selectively amplified preceding a warming event seems to be a key question in the search for the real cause of the phenomenon.

In our theory of resonant Rossby waves, wave amplification occurs when the flow in the atmosphere is such that free-traveling Rossby waves of a certain zonal wavenumber and meridional structure are rendered stationary with respect to the surface of the earth, and hence can resonantly interact with the forcings of topography and differential heating which are stationary. Such a mechanism seems to be tentatively supported by recent satellite observations (Quiroz, 1975) showing that preceding the warming the traveling wave system slows down and finally coalesces with the standing wave. The resultant system then amplifies, as indicated by its increase in radiance. When a planetary wave becomes resonant, it manifests itself as an amplifying block in the troposphere and simultaneously as a cause for the sudden warming in the stratosphere where the temperature and flow can be altered more drastically due to the diminished density. The persistence of the blocking is determined by how long the mean flow remains near the state that makes a particular wave resonant. On the other hand, once a critical level is produced through the interaction of the amplifying wave with the mean flow, the further evolution of the warming event is no longer dependent on the waves being resonant. That blocking is a more frequent phenomenon can be accounted for by the fact that shorter waves (with zonal wavenumber 3, 4, 5 and up) are easier to resonate. But these waves are trapped in the troposphere and thus are not capable of influencing the circulation in the stratosphere. The theory will show that the resonant conditions for the longest waves (with wavenumbers 1 and 2) cannot be met by the climatological state of the winter atmosphere; this accounts for the infrequency of the sudden warming phenomenon. It takes abnormal conditions both in the troposphere and the stratosphere to make these waves resonant. Since a major part of the forcings for the planetary waves is in these long scales (Eliasen and Machenhauer, 1965), the long waves, once excited, would produce pronounced disturbances with amplitudes larger than those produced by the shorter waves.

What are the "abnormal" conditions for the atmosphere preceding a warming event? From the limited observations available it seems that westerlies in the stratosphere generally reach unusually high intensities before a warming. Johnson (1969) reports a 5-day mean wind speed of nearly 60 m s^{-1} at the 10 mb level along 60°N preceding the 1967–68 warming, a speed 25% greater than the speeds for the same period during any of the "normal years" from 1964 to 1966. Preceding the 1963 warming, winds exceeding 200 kt at the 10 mb level extending from middle latitudes northward to the arctic are reported by Finger and Teweles (1964). Four days before the 1957 warming it is reported that the wind above Goose Bay reached 225 kt at the 28 km level (Craig and Hering, 1959). Increase in the stratospheric wind speed per se is not necessarily a favorable condition for the resonance of long waves. In order for resonance to occur, the winds in the stratosphere have to be (among other things) such that these long waves become evanescent above the middle stratosphere. The reason for this condition is physically clear. If the waves are not trapped, their energy will be radiated away to the top of the atmosphere and no buildup of wave energy necessary for resonance is possible. If the waves become evanescent at too high a level, the increased damping of the waves at high altitudes will dissipate the waves and therefore also prevents their resonance.

Simple analytic studies (Charney and Drazin, 1961) suggest that waves are trapped by strong westerly winds. This is not always true in the real atmosphere where vertical shears of the mean wind seem to play a rather important role in determining whether a wave is evanescent or propagating. It can be shown that positive shears such as those found at the base of the jet maximum enhance the propagability of a wave, while negative shears, found usually above the jet cores, tend to make a wave evanescent. The long waves under consideration would be more effectively trapped if the stratospheric jet maximum were to descend to the middle stratosphere, so that there would exist an extended region of negative shears above approximately the 40 km level. Is this the condition that pervails before the warmings? No definite conclusion can be drawn from the limited stratospheric data available. The increase in wind speeds recorded for heights below 30 km can be interpreted either as due to the presence of the jet maximum at around 35–40 km, or just as a manifestation of a general increase in wind strength at all heights while the jet core stays at the usual position of around 65 km. More observations are needed to distinguish between these two cases. However, there does exist a piece of data that seems to indicate that the descent of the jet is the case. Quiroz *et al.* (1975) recently have found that a couple of days before the warming event of 1973, the jet core descended to the 35 km level along the 90°E meridian, with the wind strength at the core exceeding 100 m s^{-1}. Zonal averages of the data for the same data based on measurements at four meridians show a weaker jet (~40 m s^{-1}), but the position of the core remains at the 35 km level. We have found that the negative shear region above the jet core traps the waves to below 35 km. This appears to be a favorable condition for resonance.

As mentioned above, the wind condition in the atmosphere appears to determine whether a certain wave becomes resonant. When a wave does become resonant, the amplitude that it can attain through resonant growth is determined by the amount of damping that is present and also by the magnitude of the forcing in that particular wave component. In an inviscid linear model, a resonant wave would grow linearly in time to an infinite magnitude. It is shown in the present study that in the presence of damping mechanisms, such as Ekman pumping, Newtonian cooling and various diffusions, the growth of a resonant wave is no longer linear in time and the amplitude gradually tapers off to a maximum value determined by the shortest damping time scale in the atmosphere. It is found that the amount of wave amplification usually observed before a large-scale blocking event is consistent with an Ekman pumping scale of about 5–6 days.

If the forcings for atmospheric waves were to remain the same from year to year, then one could reasonably expect that the *maximum* amplitude attainable by a wave through resonant growth to be about the same during sudden warming winters. This turns out to be true for a majority of these winters. However, there remain a few highly abnormal winters (the winter of 1977 is among them) throughout which greatly heightened wave activities are observed (Quiroz, 1977, and private communication). Not only do waves attain higher maximum amplitudes during the "resonant" episodes (there can be a number of these wave amplifications during a single winter) but even the general background wave amplitudes are larger than in other winters. This we can attribute only to an increase in the wave forcings during that particular year. The topographic forcings are not expected to change from year to year because the continental elevations do not change and the averaged westerly surface wind magnitudes do not seem to have much variability from winter to winter. It may be a different story for the thermal forcings arising from land-sea temperature contrasts (sea warms and land cools the air during winters). It is known that large ocean warm currents occasionally meander and thus change the sea surface temperature distribution. During late fall and winter of 1976–77, it is reported that the waters off the west coast of North America were abnormally warm, while over a large area in the central and northern Pacific the sea surface temperature was abnormally cold (a 3°C anomaly was recorded). How this abnormal sea surface temperature distribution affects planetary waves is still unclear. Early studies of Sankar-Rao (1965) and Sankar-Rao and Saltzman (1969) point to the important role played by the differential heating. They showed with their two-layer models that *without* the heating, the forcing due to elevations of mountains and oceans alone produces planetary waves with amplitudes close to the observed, but the locations of the highs and lows are out of phase compared to the real atmosphere. Only when the effect of differential heating is incorporated did they obtain reasonable agreement in the phase. A recent work of Bates (1977) also points to the sensitivity of the atmosphere to the location of the surface heating. In this paper we compare the observed wave amplitude in the lower atmosphere in a "normal" winter with that estimated using topographic forcing alone, and attribute the difference to the thermal effects. We infer from the results that during a normal year there exist in the long wave components significant destructive interferences between the forcing due to topography and that due to land-sea differential heating. Therefore, it is not unreasonable to expect that when the usual sea surface temperature distribution is altered (as apparently was the case preceding the winter 1976–77), the two kinds of forcings may act in harmony to reinforce instead of cancel each other, and thus produce planetary waves of unusually high intensity. As a result, a wind condition that otherwise would produce a minor warming in a normal year may cause a major warming event in the presence of such large combined forcings. This observation, if proven correct, could have a predictive value in the sense that if an abnormal ocean surface temperature distribution, with warm currents near continental land masses and cold waters over large areas in the middle oceans, is observed in the fall, one can reasonably state that the *likelihood* for the occurrences of major large-scale blocking and warming events during that winter[1] is higher than during normal years.

This presentation will be divided into three parts. In Part I, the present paper, the general ideas are illustrated with a simple uniform wind model. The emphasis is on the temporal behavior of the resonant Rossby waves forced by the topography and land-sea differential heating in the presence of various damping mechanisms. Part II deals with the effects of vertical shears and includes simple numerical experiments in which the wind profiles are varied in a number of physically possible ways and the wave response studied in search of the most favorable configuration for resonance. The effects of the presence of meridional shears and spherical geometry are studied in Part III. The problem concerning the existence of normal modes in the presence of the zero-wind line is investigated. It is shown there that the waves are "reflected" instead of absorbed by critical layers, given realistic values for the ratio of viscosity and nonlinearity that are relevant to the earth's atmosphere. The quantization of the waves in the presence of critical surfaces is also obtained.

In the remaining portion of Part I, a simple model is presented to illustrate the temporal behavior of resonant Rossby waves forced by topography and land-sea differential heating. To gain a better understanding of the essential physical processes involved, some simplifying assumptions are made in order to obtain explicit solutions. The β-plane approximation is used to simplify the spherical geometry, and the zonal wind is assumed to be spatially uniform but is allowed to vary in time. Only quasi-geostrophic disturbances are considered in this paper. These approximations will be relaxed in the subsequent papers, where the effects of shear and spherical geometry will be considered. In Sections 2 and 3 the initial value problems will be considered, and it is shown that as the zonal wind is varied in time, a Rossby wave becomes resonant when the wind speed reaches the Rossby-Haurwitz phase speed for that particular wave mode. When this happens, the free-traveling wave is rendered stationary with respect to the earth and can thus be in phase with the stationary forcings of topography and differential heating. The resonant speeds for various wave modes are calculated in Section 4. It is seen that the resonant speed requirements for the shorter waves are low and easily attainable with the normal conditions of the atmosphere. On the other hand, the resonant speeds for large-scale waves with zonal wavenumbers 1 and 2 are relatively high and are not usually attainable. This result may explain why the warming event is not a frequent occurrence. A majority of the tropospheric blocking occurrences are caused by resonant waves with higher wavenumbers ($s = 3$, 4 and 5), but their influences are not felt in the stratosphere, for these shorter waves are usually trapped in the lower atmosphere. In Sections 5 and 6 the amplitudes of the combined forcing due to topography and land-sea differential heating are estimated. It is found that the longer waves, though more difficult to resonate, produce much larger amplitudes once they become resonant. The results from these sections also point to the likelihood that the forcings due to continental elevations and land-sea differential heating have comparable magnitude and are usually out of phase during normal years.

The resonant waves amplify roughly linearly in time until either the zonal wind moves off the resonant value (either due to other external circulation mechanisms or due to interaction with the waves), or an equilibrium is reached with the damping mechanisms that are present in the system. Damping due to Ekman pumping and other mechanisms is considered in Section 7. The time variations of the heights of isobaric surfaces are calculated in Section 8, and for wavenumber 2 the results are compared with the "lower boundary forcing function" used by Matsuno (1971) in his numerical model. Good agreement is found. Thus we may have an explanation of the origin of the wave that forces the sudden warming event in Matsuno's model. In Section 9 sensitivity of the wave response to changes in wind condition from the exact resonant configuration is studied. In Section 10 we point out that resonant stationary waves are also capable of producing large meridional heat fluxes, which have traditionally been attributed to "baroclinic eddies."

2. The initial value problems

Assuming geostrophy[2] for the large-scale waves that we are interested in, the eastward and north-

[1] The time scale of ocean temperature variability is rather slow compared to the atmospheric time scales, and thus it is reasonable to expect the anomalous fall ocean conditions would persist to winter.

[2] Strictly speaking, both the β-plane and gesotrophic assumptions are not valid approximations for the long waves under consideration and instead the primitive equations on a sphere

ward velocities can be expressed as (see Appendix A for list of symbols):

$$\left.\begin{aligned} u &= \bar{u} + u' \\ v &= v' \end{aligned}\right\}, \tag{1}$$

with

$$u' = -\frac{1}{f}\frac{\partial}{\partial y}\Phi', \quad \bar{u} = -\frac{1}{f}\frac{\partial}{\partial y}\bar{\Phi}$$

and

$$v' = \frac{1}{f}\frac{\partial}{\partial x}\Phi',$$

where Φ' is the perturbation geopotential and $f = 2\Omega \sin\varphi \approx f_0 + \beta y$ is the Coriolis parameter. In $\log p$ coordinates, where

$$z^* \equiv \ln\left(\frac{p_0}{p}\right),$$

the governing equation for Φ' is the following conservation equation for potential vorticity (see Charney, 1973):

$$\left\{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right] + \beta\frac{\partial}{\partial x}\right\}\Phi' = 0. \tag{2}$$

In obtaining Eq. (2), it is assumed that the mean zonal flow is a function of time t only, i.e., $\bar{u} = \bar{u}(t)$.

The lower boundary condition is specified on the vertical velocity by the tangency condition

$$w = \bar{u}\frac{\partial}{\partial x}h(x,y) \quad \text{at} \quad z^* \approx 0, \tag{3}$$

where h is the surface elevation, a Fourier component of which can be written as

$$h(x,y) = h_0 e^{ikx} \sin[l(y_p - y)],$$

where y_p is the location of the northern polar boundary. To express w in terms of Φ', one proceeds by using the definitions

$$w^* \equiv \frac{d}{dt}z^*, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$

the hydrostatic condition $(\partial/\partial z)z^* = g/RT \equiv 1/H$ and $dz^* = -(1/gH)d\Phi$. After linearizing these give

$$w^* = -\frac{1}{gH}\frac{\partial}{\partial t}\Phi' + \frac{1}{H}w, \tag{4}$$

should be used. However, the temporal and vertical structures of the wave are not expected to be qualitatively affected (see Part III for a more detailed discussion), and therefore the present simple model suffices for our purpose here.

but w^* and Φ' are also related through the energy equation

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\Phi'_{z^*} + Sw^* = 0. \tag{5}$$

Substituting (5) into (4) then yields the desired relation:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{H}{S}\Phi'_{z^*}\right) - \frac{1}{g}\frac{\partial}{\partial t}\Phi' = -w. \tag{6}$$

The simplest initial value problem that can be posed seems to be a switch-on problem with $\bar{u}(t)$ taking the form $\bar{u}(t) = U \cdot H(t)$, where U is a positive constant and $H(t)$ is the Heaviside unit step function given by

$$H(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t < 0. \end{cases}$$

The set (2), (6) and (3) then defines the problem. A similar switch-on problem has been considered by Clark (1972, 1974). Although his boundary condition is not appropriate for our problem of topographic forcings [Clark, in our Eq. (3), used w^* instead of w], some of his results concerning the time behavior of the solutions can be shown to be still applicable to our present problem.

It is shown in Appendix B that the solution for w consists of two parts: one part is the transient waves introduced by the discontinuity in forcing at $t = 0$. These waves represent the atmosphere's adjustment to a varying forcing and decay in time as

$$w_{\text{transient}} \sim O(t^{-5/6}) \tag{8}$$

[see also Clark (1972), Eq. (30)]. The remaining part of the solution is the forced wave

$$w_0 = ikh_0 UH(t)[e^{ikx}\sin l(y_p - y)] \times \exp[(z^*/2) - b_1 z^*], \tag{9}$$

where

$$b_1 = \left\{\frac{1}{4} - \frac{S}{f_0^2}\left[\frac{\beta}{U} - (k^2 + l^2)\right]\right\}^{1/2} \tag{10}$$

is defined so that $\mathrm{Re}\,b_1 > 0$ or $\mathrm{Im}\,b_1 < 0$ to satisfy the boundedness or radiation at infinity. Eq. (9) alone satisfies the boundary condition (3) and is the long-time solution in the sense that it will be the only remaining term in the solution when the transients have decayed to insignificant values after a sufficiently long time.

For a moment we neglect the transient waves and discuss the forced wave in more detail. The forced part of Φ' corresponding to w_0 can be found by substituting (9) into (6) to yield

$$\Phi'_0 = \frac{-h_0[e^{ikx}\sin l(y_p - y)]\exp[(z^*/2) - b_1 z^*]}{\left[\frac{H}{S}\left(\frac{1}{2} - b_1\right)\right]}.$$

To this one can always add a free-traveling solution satisfying Eq. (2) and $w = 0$ at the lower boundary. Thus, we can rewrite Φ_0' in the more general form

$$\Phi_0' = -\hbar_0[e^{ikx}\sin l(y_p - y)]e^{z^*/2}\frac{[e^{-b_1z^*} - e^{-ikct}e^{-b_2z^*}]}{\left[\frac{H}{S}\left(\frac{1}{2} - b_1\right)\right]} + D[e^{ikx}\sin l(y_p - y)]e^{-ikct}e^{-b_2z^*}e^{z^*/2}, \quad (11)$$

where c is the phase speed of the traveling wave satisfying the Rossby-Haurwitz dispersion formula

$$c - U = -\frac{\beta}{(k^2 + l^2) + (f_0^2/S)(1/4 - b_2^2)}, \quad (12)$$

where

$$b_2 = \frac{1}{2} + \frac{S}{gH}\frac{c}{U - c}.$$

Resonance occurs when

$$\left.\begin{array}{l} b_1 \to 1/2 \\ b_2 \to b_1 \\ c \to 0 \end{array}\right\}. \quad (13)$$

That is, the free traveling waves become stationary with respect to the forcing and possess the same vertical structure as that of the forced wave when resonance occurs. It is seen that to have resonance, the zonal wind must reach the critical (resonant) value given by

$$U_r = \frac{\beta}{k^2 + l^2}. \quad (14)$$

When this happens, the phase speed of the traveling wave is reduced to zero by virtue of the dispersion relation (12).

The behavior of Φ_0' at resonance is found [by taking the limits (13) using l'Hospital's rule] to be

$$\Phi_0' = \frac{-\hbar_0[e^{ikx}\sin l(y_p - y)]}{\left[1 + \frac{f_0^2}{gH(k^2 + l^2)}\right]} \times \left[\frac{S}{H}z^* - \frac{ikU_rf_0^2}{(k^2 + l^2)H}t\right] + D[e^{ikx}\sin l(y_p - y)]. \quad (15)$$

Note that the resonant solution in the present uniform wind model is *barotropic*; w_0 does not depend on z^*, while Φ_0' varies only linearly with height. The resonant solution also grows linearly in t, with $t = 0$ being defined here as the instant when U reaches U_r.

An undesirable aspect of the switch-on model is that the transients dominate over the forced solution described above until after more than a week. The large magnitude of the transient waves is *artificially* produced by the discontinuous time variation of the zonal wind in the model and it can be reduced to insignificant values if the wind variation becomes smooth, as will be shown in Section 3. Since these transient waves can interact with the mean flow, even in the absence of resonance, it is rather misleading to generate "warming events" in numerical models using this artificial switch-on mechanism alone, as in Geisler (1974).

In order to concentrate one's attention on the more systematic time behavior of the resonant waves (not masked by the presence of transients which are difficult to model realistically) a two-timing model in which the wind is varied smoothly will be presented in the next section.

3. Two-timing formalism

In this section the case where the zonal wind is a slow function of time is considered. The resonant value U_r of the zonal wind speed is approached slowly and smoothly in a time interval t_r. The reciprocal of the approach time gives a measure of the "slowness" of the zonal wind variation. Accordingly, the small parameter of the problem is defined as

$$\epsilon = \frac{1}{(kU_rt_r)}, \quad 0 < \epsilon \ll 1.$$

A slow and a fast time are also defined

$$\left.\begin{array}{l} \tilde{t} = \epsilon t \\ t^* = \int_0^t c(\tilde{t})dt \end{array}\right\}, \quad (16)$$

where

$$c(\tilde{t}) = c_0(\tilde{t}) + \epsilon c_1(\tilde{t}) + O(\epsilon^2) \quad (17)$$

is to be determined in the process of solution. Without the expansion for c, it is found that there would not be enough free parameters to suppress the secular terms *and* the singularity in the solution at resonance. The zonal wind is assumed to be a function of the slow time only, i.e., $\bar{u} = \bar{u}(\tilde{t})$. Instead of expanding the solutions in an asymptotic series in powers of ϵ, we define another small parameter

$$\alpha(\tilde{t}) = \frac{\bar{u}(\tilde{t}) - U_r}{U_r}\epsilon \quad (18)$$

and expand the solutions in powers of $\alpha(\tilde{t})$. Though the resulting asymptotic formulation in terms of a variable parameter is more complicated, it nevertheless offers the advantage that at resonance all the higher order terms disappear and the zeroth-order term becomes the exact solution of the problem.

We expand w and Φ' as

$$\left.\begin{aligned} w &= w_0 + \alpha w_1 + \alpha^2 w_2 + O(\alpha^3) \\ \Phi_0' &= \Phi_0' + \alpha\Phi_1' + \alpha^2\Phi_2' + O(\alpha^3) \end{aligned}\right\}. \quad (19)$$

Since

$$\frac{\partial}{\partial t} = \frac{\partial \tilde{t}}{\partial t}\frac{\partial}{\partial \tilde{t}} + \frac{\partial t^*}{\partial t}\frac{\partial}{\partial t^*}$$

$$= \epsilon\frac{\partial}{\partial \tilde{t}} + [c_0 + \epsilon c_1 + \ldots]\frac{\partial}{\partial t^*},$$

one has

$$\frac{\partial}{\partial t}\Phi' = c_0\frac{\partial}{\partial t^*}\Phi_0'$$

$$+ \epsilon\left[\frac{\alpha}{\epsilon}c_0\frac{\partial}{\partial t^*}\Phi_1' + c_1\frac{\partial}{\partial t^*}\Phi_0' + \frac{\partial}{\partial \tilde{t}}\Phi_0'\right] + O(\epsilon^2).$$

To the lowest order, the equations become

$$\left\{\left[c_0(\tilde{t})\frac{\partial}{\partial t^*} + \bar{u}(\tilde{t})\frac{\partial}{\partial x}\right] \times \left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right] + \beta\frac{\partial}{\partial x}\right\}\begin{Bmatrix} w_0 \\ \Phi_0' \end{Bmatrix} = 0. \quad (20)$$

Treating t^* and $\tilde{t}$ as independent variables, as usually done in the two-timing formalism, gives the following solutions to (20) [cf. Eqs. (9) and (11)]:

$$w_0 = ik\hbar_0\bar{u}(\tilde{t})[e^{ikx}\sin l(y_p - y)] \times \exp\{[½ - b_1(\tilde{t})]z^*\}, \quad (21)$$

where

$$b_1(\tilde{t}) = \left\{\frac{1}{4} - \frac{S}{f_0^2}\left[\frac{\beta}{\bar{u}(\tilde{t})} - (k^2 + l^2)\right]\right\}^{1/2},$$

$$\Phi_0' = -[e^{ikx}\sin l(y_p - y)]\left\{\hbar_0\frac{\exp\{[½ - b_1(\tilde{t})]z^*\}}{(H/S)[½ - b_1(\tilde{t})]} + A_0(\tilde{t})e^{-ikt^*}\exp\{[½ - b_2(\tilde{t})]z^*\}\right\}, \quad (22)$$

where

$$c_0(\tilde{t}) - \bar{u}(\tilde{t}) = -\frac{\beta}{(k^2 + l^2) + (f_0^2/S)[(1/4) - b_2^2(\tilde{t})]}, \quad (23)$$

$$b_2(\tilde{t}) = \frac{1}{2} + \frac{S}{gH}\frac{c_0(\tilde{t})}{\bar{u}(\tilde{t}) - c_0(\tilde{t})}. \quad (24)$$

An unusual feature of the present two-timing solutions is that the amplitude of w_0 in (21) is completely determined by the lower boundary condition, and the function $A_0(\tilde{t})$ in Φ_0' is not really "free" to be used for the suppression of secular terms in higher order solutions. $A_0(\tilde{t})$ must be used in this zeroth-order solution to suppress the singularity at resonance and is found to be of the form

$$A_0(\tilde{t}) = -\frac{\hbar_0}{(H/S)[½ - b_1(\tilde{t})]}, \quad (25)$$

plus perhaps an arbitrary, free wave amplitude $D(\tilde{t})$, but since the governing equations are linear, the free wave can be considered separately.

The equations to the next order are

$$\left\{\left[c_0(\tilde{t})\frac{\partial}{\partial t^*} + \bar{u}(\tilde{t})\frac{\partial}{\partial x}\right]\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right] + \beta\frac{\partial}{\partial x}\right\}\begin{Bmatrix} w_1 \\ \Phi_1' \end{Bmatrix}$$

$$= -\frac{\epsilon}{\alpha}\left(c_1(\tilde{t}) + \frac{\partial}{\partial t^*}\right)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right]\begin{Bmatrix} w_0 \\ \Phi_0' \end{Bmatrix}. \quad (26)$$

Here Φ' has a secular term in the fast time t^* forced by $A_0(\tilde{t})$. To suppress it, we make use of the still arbitrary function $c_1(\tilde{t})$ and require that

$$ikc_1(\tilde{t})e^{-b_2(\tilde{t})z^*}e^{-ikt^*}\left[-(k^2 + l^2) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right]A_0(\tilde{t})$$

$$+ e^{-ikt^*}\frac{\partial}{\partial \tilde{t}}\left\{\left[-(k^2 + l^2) + \frac{f_0^2}{S}\left[b_2^2(\tilde{t}) - \frac{1}{4}\right]\right]A_0(\tilde{t})e^{-b_2(\tilde{t})z^*}\right\} = 0. \quad (27)$$

From (27), one finds

$$c_1(\tilde{t}) = -\frac{(\partial/\partial\tilde{t})A_0}{ikA_0} - \frac{1}{ik}\frac{\partial}{\partial \tilde{t}} \times \ln\left\{\left[-(k^2 + l^2) + \frac{f_0^2}{S}\left(b_2^2 - \frac{1}{4}\right)\right]e^{-b_2 z^*}\right\}$$

$$= \frac{iS\beta(\partial\bar{u}/\partial\tilde{t})/\bar{u}(\tilde{t})^2}{f_0^2 k 2b_1(\tilde{t})[½ - b_1(\tilde{t})]} - \frac{i}{k}\left\{\left[\frac{S}{gh} + \frac{2f_0^2}{\beta gH} \times (\bar{u}(\tilde{t}) - c_0(\tilde{t}))\right]\frac{\partial}{\partial \tilde{t}}\left(\frac{c_0(\tilde{t})}{\bar{u} - c_0}\right)\right\}. \quad (28)$$

In a similar way, the secular terms in still higher order equations can be suppressed, making use of the free functions $c_i(\tilde{t})$. We are not particularly interested in the details of the higher order solutions, other than the fact that they are nonsecular, thus ensuring the validity of our expansion procedure.

At resonance, $\bar{u}(t) \to U_r$, $\alpha(\tilde{t}) \to 0$, all higher order terms vanish and the solutions become, for $t \geq t_r$,

$$w \to w_0 = ik\hbar_0 U_r[e^{ikx} \sin l(y_p - y)],$$

$$\Phi' \to \Phi_0' = -\frac{\hbar_0[e^{ikx} \sin l(y_p - y)]}{\left[1 + \frac{f_0^2}{gH(k^2 + l^2)}\right]} \times \left[\frac{S}{H} z^* - \frac{ikU_r f_0^2}{H(k^2 + l^2)}(t - t_r)\right]. \quad (29)$$

There are no transient waves in (29). This is due to the fact that the zonal wind varies slowly and smoothly in time. In particular, the finiteness of $c_1(\tilde{t})$ in (28) requires the approach to U_r to be so slow that

$$\frac{d\bar{u}(\tilde{t})/d\tilde{t}}{[\tfrac{1}{2} - b_1(\tilde{t})]}$$

remains nonsingular at resonance when $\tfrac{1}{2} - b_1(\tilde{t}) \to 0$.

4. The resonant wind speed

Results from the last section show that a Rossby wave with zonal wavenumber k and meridional wavenumber l becomes resonant with the stationary forcings when the speed of the zonal wind reaches the value

$$U_r(k,l) = \frac{\beta}{k^2 + l^2} = \frac{2\Omega a \cos^3\varphi_0}{s^2 + m^2}. \quad (30)$$

This is one of the "selection principles" in determining which wave will be preferentially excited. In (30) s is the dimensionless zonal wavenumber; we are interested in the planetary-scale waves with $s = 1, 2, 3$ and 4. And $m \equiv l(a \cos\varphi_0)$ [$\varphi_0 = \pi/4$ in the midlatitude β-plane approximation] is the nondimensional meridional wavenumber; its values depend on the particular meridional quantization rule used. For uniform zonal winds the wave solution can be either symmetric or antisymmetric about the equator. The antisymmetric modes are obtained by requiring Φ' to vanish at $y \doteq 0$, the equator. This yields the following quantization rule:

$$l = l_n^{(a)} = \frac{n\pi}{y_p} = \frac{2n}{a} = \frac{(n' + 1)}{a} \quad \begin{cases} n = 1, 2, 3, \ldots \\ n' = 1, 3, 5, \ldots . \end{cases} \quad (31)$$

The symmetric modes are obtained by requiring $(\partial/\partial y)\Phi'$ to be zero at $y = 0$, giving

$$l = l_n^{(s)} = \frac{(\tfrac{1}{2} + n)\pi}{y_p} = \frac{(2n + 1)}{a} = \frac{(n' + 1)}{a} \quad \begin{cases} n = 0, 1, 2, 3, \ldots \\ n' = 0, 2, 4, \ldots . \end{cases} \quad (32)$$

In many previous studies using the simple uniform wind model, walls are placed at the equator and the pole as side boundary conditions. As a result, only the antisymmetric modes given by (31) are present in those models. It turns out, however, that in the real atmosphere in the presence of a zero-wind line in the equatorial region, the stationary normal mode waves resemble the symmetric modes [Eq. (32)] more than the antisymmetric modes commonly used. It will be shown in Part III, where the effects of meridional shears of the zonal wind are discussed, that the presence of the zero-wind line significantly affects the character and quantization of the wave modes in the westerly region. If the zero-wind line totally absorbs all incident waves as in Dickinson (1968), no normal mode waves can form, and therefore the existence of the resonant waves discussed here is very much in doubt. However, as will be shown in that paper, for conditions relevant to the real atmosphere, the nonlinearity of the waves seems to dominate over the effects of viscosity in the critical layer, and as a result, the stationary waves are most likely "reflected" rather than absorbed at the zero-wind line. The resulting quasi-normal modes are quantized so that there are an odd number of quarter-wavelengths between the critical surface and the North Pole. In the present simple model, the presence of the zero-wind line and its reflecting nature can be *simulated*[3] by using a Helmholtz type wind profile

$$\bar{u} = \begin{cases} U, & y > y_c \\ -U, & y < y_c \end{cases} \quad (33)$$

where U is a positive constant and y_c the location of the zero-wind line. In the westerly region ($y > y_c$), the meridional wave structure that satisfies the northern boundary condition is $\sin[l(y_p - y)]$. The quantization rule that gives an odd number of quarter wave-lengths between the critical surface y_c and the pole is

$$l_n(y_p - y_c) = (\tfrac{1}{2} + n)\pi, \quad n = 0, 1, 2, 3. \ldots \quad (34)$$

[3] The zero-wind line in the Helmholtz profile is perfectly reflecting due simply to the infinite shear of a shear layer of zero width.

For the case when the zero wind line is located at the equator (i.e., $y_c = 0$), the quantization turns out to be the same as that for the symmetric modes given by Eq. (32) in the uniform wind profile.

In Table 1 the resonant wind speeds for various modes with $s = 1$ to 4 and $n = 0$ to 4 are listed. The commonly used antisymmetric modes with l_n given by Eq. (31) are listed in the third column, while in the fourth column we list the resonant speeds for the more relevant quantizations given by the symmetric modes (32) or (34) with $y_c = 0$. The case of critical surface quantization with $y_c = (\pi/6)a$ is in the last column. It can be seen in Table 1 that, depending on the values of the zonal wind speed $\bar{u}$, different wave modes will be resonantly excited. It is important to note that the shorter waves (i.e., waves with larger values of s and n) require lower wind speeds to be excited. It is more difficult to resonate the longer waves but, as will be shown in a later section, they would have larger amplitudes once excited. One should also note that for a wave mode with a given number of nodes in the meridional domain, the resonant wind speed is lowered when the position of the zero-wind line shifts northward. Such a change, though not strictly necessary, may be important since it can lower an unattainably high resonant wind speed requirement to a more realistic value.

5. Continental elevations

The topographic elevations of continents and oceans on the earth's surface are recorded in topographic maps and are usually analyzed and expressed for meteorological purposes in terms of spherical harmonics in the form

$$\begin{aligned} h(\lambda,\varphi) &= \sum_{s=0}^{\infty}\sum_{n'=0}^{\infty} [h_{s,n'}^{(1)} \cos(s\lambda) + h_{s,n'}^{(2)} \sin(s\lambda)] P_{s+n'}^{s}(\mu) \\ &= \sum_{s=0}^{\infty}\sum_{n'=0}^{\infty} A_{s,n'} \cos[s(\lambda - \delta_{s,n'})] P_{s+n'}^{s}(\mu), \end{aligned} \tag{35}$$

where λ is the longitude, φ the latitude, and $\mu = \sin\varphi$. For n' an odd integer, the associated Legendre function $P_{s+n'}^{s}(\mu)$ is antisymmetric about $\mu = 0$, the equator, and symmetric if n' is an even integer. The values of the cosine and sine amplitudes, $h_{s,n'}^{(1)}$ and $h_{s,n'}^{(2)}$, taken from Sankar-Rao (1965), are listed in Table 2. The total amplitudes

$$A_{s,n'} \equiv [(h_{s,n'}^{(1)})^2 + (h_{s,n'}^{(2)})^2]^{1/2} \tag{36}$$

are calculated and also listed.

On a β-plane, the meridional function that corresponds to the associated Legendre function is[4] $\sin l_n(y_p - y)$, where $l_n = (n' + 1)/a, n' = 0, 1, 2, 3, \ldots$. Then (35) becomes, on a β-plane

$$\begin{aligned} h(\lambda,y) &= \sum_{s=0}^{\infty}\sum_{n'=0}^{\infty} [h_{s,n'}^{(1)} \cos(s\lambda) + h_{s,n'}^{(2)} \sin(s\lambda)] \sin l_n(y_p - y) \\ &= \sum_{s=0}^{\infty}\sum_{n'=0}^{\infty} A_{s,n'} \cos[s(\lambda - \delta_{s,n'})] \times \sin l_n(y_p - y). \end{aligned} \tag{37}$$

TABLE 1. Values of the resonant wind speed $U_r = \Omega a[\sqrt{2}(s^2 + m^2)]^{-1}$ (m s^{-1}) in the uniform wind model.

s	n	Antisymmetric modes	Symmetric modes or Eq. (34) with $y_c = 0$	Eq. (34) with $y_c = (\pi/6)a$
1	0	—	310.3	219.0
1	1	155.1	84.6	41.8
1	2	51.7	34.5	16.0
1	3	24.5	18.3	8.3
1	4	14.1	11.2	5.1
2	0	—	103.4	90.8
2	1	77.6	54.8	32.9
2	2	38.8	28.2	14.5
2	3	21.2	16.3	7.9
2	4	12.9	10.5	4.9
3	0	—	49.0	46.0
3	1	42.3	34.5	24.3
3	2	27.4	21.6	12.5
3	3	17.2	13.9	7.3
3	4	11.4	9.4	4.6
4	0	—	28.2	27.2
4	1	25.9	22.7	17.8
4	2	19.4	16.3	10.6
4	3	13.7	11.5	6.5
4	4	9.7	8.2	4.3

6. Forcings due to the combined effects of topography and heating

Compared to topography, the forcing due to land-sea differential heating is more difficult to determine. It depends on surface winds and on surface temperature distributions, and hence varies from season to season and also from year to year. Although the response of a wave to the heating is not yet clearly understood, a number of parameterizations have been devised to account for the effects of land-sea differential heating on the waves (Döö, 1962; Sankar Rao and Saltzman, 1969). A different, and particularly naive approach is adopted here. We assume our model is sufficient to yield the re-

[4] It seems that Sankar-Rao (1965) used *normalized* Legendre functions, i.e.,

$$\int_{-1}^{1} [P_{s+n'}^{s}(\mu)]^2 d\mu = 1.$$

The sine function has the same normalization, *viz.*,

$$\int_{-1}^{1} \sin^2 l_n(y_p - y) d(y/\tfrac{1}{2}\pi a) = 1.$$

TABLE 2. Coefficients of cosine and sine spherical harmonics of continental elevations (m).

			$h^{(c)}_{s,s+n'}$		$h^{(s)}_{s,s+n'}$		$\{h^{(c)2} + h^{(s)2}\}^{1/2}$	
s	n	n'	Antisymmetric	Symmetric	Antisymmetric	Symmetric	Antisymmetric	Symmetric
1	0	0		172.1		205.0		267.7
	1	1	−3.7		150.9		150.9	
		2		−60.3		112.3		127.5
	2	3	−50.5		−249.4		254.5	
		4		30.3		13.1		33.0
	3	5	47.9		−255.5		260.0	
		6		102.7		141.6		174.9
	4	7	−1.8		−76.0		76.0	
		8		89.4		170.9		192.9
2	0	0		−159.1		48.8		166.3
	1	1	−209.4		122.6		242.7	
		2		−191.9		19.9		192.8
	2	3	10.3		−38.3		39.7	
		4		79.1		−42.4		89.7
	3	5	178.1		−33.7		181.3	
		6		7.4		−48.3		48.7
	4	7	6.5		−25.1		25.9	
		8		−121.0		−17.7		122.3
3	0	0		10.2		96.5		97.0
	1	1	109.1		−157.0		191.2	
		2		53.5		−110.4		122.7
	2	3	37.3		−3.9		37.5	
		4		55.6		51.8		76.0
	3	5	10.0		44.4		45.5	
		6		−9.3		22.5		24.3
	4	7	−36.9		−123.5		128.9	
		8		4.8		−62.5		62.7
4	0	0		23.5		184.5		186.0
		1	174.8		−99.1		200.9	
		2		132.4		−44.7		139.7
	2	3	5.9		8.1		10.0	
		4		−53.5		2.1		53.5
	3	5	−97.1		31.5		102.1	
		6		−83.9		38.6		92.4
	4	7	6.9		−26.8		27.7	
		8		−3.2		5.3		6.2

sponse to stationary forcing with modest accuracy, and, therefore, that we can infer a realistic lower boundary forcing (in the form of h_0) by requiring calculated and observed stationary waves to agree. The forcing obtained this way contains the effects of both topographic elevation and land-sea differential heating.[5] Using results obtained this way and comparing with those of the last section, one hopes to obtain a qualitative feel of the relative magnitudes and phases of the two kinds of wave forcings *during a normal year.*

The data are taken from Eliasen and Machenhauer (1969) for large-scale wave motions over the whole earth during October 1957, the International Geophysical Year. They list, in their Table 1, the mean amplitudes of geopotential height $[(1/g)\Phi']_{s,n'}$ of the 500 mb surface. To deduce h_0, the following admittedly crude formula for forced waves is used:

$$\left(\frac{1}{g}\Phi'\right)_{s,n'} = \frac{(h_0)_{s,n'}}{g}\frac{\exp(\frac{1}{2}\ln 2)}{(H/S)(\frac{1}{2} - b_1)}\exp(-b_1 \ln 2), \quad (38)$$

where

$$b_1 = -i\left\{\frac{S}{f_0^2}\left[\frac{\beta}{\bar{u}} - (k^2 + l^2)\right] - \frac{1}{4}\right\}^{1/2}.$$

From (37) one can deduce that

$$|h_0|_{s,n'} = |(g^{-1}\Phi')_{s,n'}|(2\kappa)^{-1/2}(gH)^{1/2}/\Omega a \times \left[\frac{\Omega}{\bar{\omega}} - (s^2 + m^2)\right]^{1/2}, \quad (39)$$

[5] The "forcing" obtained this way actually contains all thermal effects. We here loosely call it the surface heating, since it is deduced from waves in the lower atmosphere.

where $\bar{\omega}$ is the angular frequency of the zonal flow. Eliasen and Machenhauer did not give a value for $\bar{\omega}$ for October 1957, but from their earlier work (Eliasen and Machenhauer, 1965), a mean average value of $\bar{\omega} = 0.0225\Omega$ seems reasonable. Using this value for the mean wind, the equivalent topographic forcing $|h_0|_{s,n'}$ is calculated from Eq. (39) and listed in Table 3, together with the observed values for the amplitudes $|(1/g)(\Phi')_{s,n'}|$, taken from Eliasen and Machenhauer. Note that Eqs. (38) and (39) are valid only for waves that propagate at 500 mb, and are not applicable for waves that are trapped below 500 mb; for the latter, a separate calculation is needed. However, such a calculation may not be too meaningful due to the observational uncertainties for these short waves.

TABLE 3. Amplitudes (m) of the combined topographic and thermal forcings as deduced from the observed wave amplitudes given by Eliasen and Machenhauer (1969).

			$\lvert g^{-1}(\Phi')_{s,n'}\rvert$		$\lvert(h_0)_{s,n'}\rvert$	
s	n	n'	Antisymmetric mode	Symmetric mode	Antisymmetric mode	Symmetric mode
1	0	0		18		91
	1	1	19		94	
		2		20		96
	2	3	19		88	
		4		17		73
	3	5	16		62	
		6		13		43
	4	7	20		51	
		8		14		19
2	0	0		7		35
	1	1	10		48	
		2		22		101
	2	3	14		62	
		4		25		102
	3	5	15		55	
		6		13		41
	4	7	11		25	
		8		9		—
3	0	0		6		27
	1	1	11		49	
		2		13		55
	2	3	10		41	
		4		17		63
	3	5	11		35	
		6		11		28
	4	7	9		12	
		8		9		—
4	0	0		7		28
	1	1	10		40	
		2		14		53
	2	3	15		52	
		4		14		43
	3	5	12		30	
		6		12		18
	4	7	7		—	
		8		9		—

Bearing in mind that the forcing due to land-sea differential heating may change from year to year, one should not place too much emphasis on the numerical values of the forcings listed in Table 3. Nonetheless, the following general characteristics of the combined forcing should be noted, since we find these gross features to be present also in other years:

1) Thermal forcing has amplitudes comparable to those of continental elevations. This conclusion can be deduced from the fact that the amplitudes of the combined forcing are substantially different from those due to topographic forcing alone. Therefore, in a complete treatment of forced atmospheric waves, the effects of land-sea differential heating cannot be ignored.

2) Thermal forcing is usually out of phase with that of topography. For the longer waves the amplitudes due to the combined forcings are substantially lower than those due to the topographic forcing alone. This fact can be explained as arising from "destructive interference" of the thermal and topographic forcings. The possibility exists that stationary waves with unusually large amplitudes may be excited during some rare occasions when the large-scale forcings of both forms are approximately in phase. This may be the case during January 1977, when blockings and warming of unusual intensity occurred after abnormal ocean temperature distributions have been observed.

7. Resonance in the presence of Ekman pumping and other damping mechanisms

When damping is present in the system, resonance in the usual sense (of indefinite linear time amplification) can no longer occur. When the amount of damping is large (i.e., short damping time scale), little or no time amplification of the wave occurs. However, when the damping time scale is long, wave amplification will still occur, but the time behavior is more complicated than the linear amplification of the inviscid case. The time behavior of the quasi-resonant planetary waves in the presence of damping will be shown to be analogous to that of a damped harmonic oscillator in mechanics. We will also show that the observed amplifications can be obtained in the presence of dampings with time scales as short as, and in some cases shorter than, a week.

For large-scale planetary waves, the relevant damping mechanisms seem to be Ekman pumping, Newtonian cooling and turbulent eddy diffusions. Ekman pumping will be discussed first since it seems to have the shortest damping time scale.

The equation for the conservation of vorticity can be written as

$$e^{-z^*}\left\{\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\beta\frac{\partial}{\partial x}\right\}\Phi' = f_0^2\frac{\partial}{\partial z^*}(e^{-z^*}w^*). \quad (40)$$

Using Eq. (4) to rewrite the right-hand side of Eq. (40) yields

$$\left[\left(\frac{\partial}{\partial t}+\bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\beta\frac{\partial}{\partial x}+\frac{f_0^2}{gH}\frac{\partial}{\partial t}\frac{\partial}{\partial z^*}\right](e^{-z^*}\Phi') = \frac{f_0^2}{H}\frac{\partial}{\partial z^*}(e^{-z^*}w). \quad (41)$$

Let $\bar{u}(t)$ slowly approach the resonant value $U_r = \beta/(k^2+l^2)$ in the manner described in Section 3. When $\bar{u} = U_r$, the following balance of terms exists:

$$\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi'+\beta\frac{\partial}{\partial x}\Phi' = 0.$$

Thus Eq. (41) reduces to

$$\left[-(k^2+l^2)+\frac{f_0^2}{gH}\frac{\partial}{\partial z^*}\right]\frac{\partial}{\partial t}(e^{-z^*}\Phi') = \frac{f_0^2}{H}\frac{\partial}{\partial z^*}(e^{-z^*}w),$$

which, when integrated on both sides with respect to z^* from z_1^*, the top of the planetary boundary layer, to ∞, yields

$$(k^2+l^2)\frac{\partial}{\partial t}\int_{z_1^*}^{\infty}(e^{-z^*}\Phi')dz^* + \frac{f_0^2}{gH}\frac{\partial}{\partial t}[\exp(-z_1^*)\Phi'(z_1^*)] = \frac{f_0^2}{H}w(z_1^*)\exp(-z_1^*). \quad (42)$$

At z_1^*, the vertical velocity w consists of two parts:

$$w(z_1^*) = w_f + w_e,$$

where

$$w_f = ik\bar{u}\hbar_0[e^{ikx}\sin l(y_p-y)]$$

is caused by surface forcing. The other part, w_e, is related to Ekman pumping in the planetary boundary layer and is given approximately by (Charney and Eliasen, 1949)

$$w_e = \frac{H}{f_0^2}a_e\nabla^2\Phi'(z_1^*), \quad (43)$$

where

$$a_e = \frac{\sin 2\alpha}{\sqrt{2}}\frac{(Kf_0)^{1/2}}{H}. \quad (44)$$

In Eq. (44), α is the angle between isobars and the surface wind, and K is the eddy-diffusivity. Eq. (42) now becomes

$$(k^2+l^2)\frac{\partial}{\partial t}\left[\exp z_1^*\int_{z_1^*}^{\infty}(e^{-z^*}\Phi')dz^*\right] + \frac{f_0^2}{gH}\frac{\partial}{\partial t}\Phi'(z_1^*)+(k^2+l^2)a_e\Phi'(z_1^*) = \frac{f_0^2}{H}w_f. \quad (45)$$

If one neglects, for a moment, the z^* dependence of Φ', then Eq. (45) can be written in a form similar to the equation for a forced damped oscillator, i.e.,

$$\frac{\partial}{\partial t}\Phi'+\tilde{a}_e\Phi' = \frac{f_0^2/H}{\left[(k^2+l^2)+\frac{f_0^2}{gH}\right]}w_f, \quad (46)$$

where

$$\tilde{a}_e \equiv \frac{(k^2+l^2)}{\left[(k^2+l^2)+\frac{f_0^2}{gH}\right]}a_e \quad (47)$$

can be identified as the rate of damping.

Eq. (46) can be solved to yield the time-dependent solution

$$\Phi' = \frac{(f_0^2/H)w_f}{\left[(k^2+l^2)+\frac{f_0^2}{gH}\right]} \times \left\{\left[\frac{1}{\tilde{a}_e}(1-\exp(-\tilde{a}_e t))\right]+C\exp(-\tilde{a}_e t)\right\}. \quad (48)$$

It is seen that the damping time scale is given by

$$T_e = \frac{1}{\tilde{a}_e} = \frac{1}{a_e}\left[1+\frac{(\Omega a)^2}{gH}(s^2+m^2)^{-1}\right]. \quad (49)$$

For t/T_e small, Eq. (48) is

$$\Phi' = \frac{(f_0^2/H)w_f}{\left[(k^2+l^2)+\frac{f_0^2}{gH}\right]}(t+C). \quad (50)$$

Thus, at least initially, the time behavior is the same as that of inviscid resonance, and by comparing with Eq. (29), C can be seen to be

$$-\frac{z_1^*(k^2+l^2)\frac{S}{H}}{ikU_r\frac{f_0^2}{H}}.$$

For large time, however, the maximum amplitude that can be reached in the presence of damping is

$$(\Phi')_{\max} = \frac{(f_0^2/H)w_f}{\left[(k^2 + l^2) + \dfrac{f_0^2}{gH}\right]} T_e, \quad (51)$$

which is the same as that reached by linear growth at $z^* = 0$ in a time T_e. When the z^* dependence of Φ' is included, the exact solution to Eq. (45) becomes

$$\Phi'(z^*,t) = \frac{(f_0^2/H)w_f}{\left[(k^2 + l^2) + \dfrac{f_0^2}{gH}\right]} \times \left\{\left[\frac{1}{\hat{a}_e}\left(1 - e^{-\hat{a}_e t} \exp[(\tfrac{1}{2} - \mu)(z^* - z_1^*)]\right)\right] + C \exp[(\tfrac{1}{2} - \mu)(z^* - z_1^*) - \hat{a}_e t]\right\}, \quad (52)$$

where

$$\mu = \left[\frac{1}{4} + \frac{\hat{a}_e(k^2 + l^2)}{(\hat{a}_e - ik\bar{u})(f_0^2/S)}\right]^{1/2},$$

$$\hat{a}_e = a_e\left[\frac{1}{(\tfrac{1}{2} + \mu)} + \frac{f_0^2}{gH(k^2 + l^2)}\right]^{-1}.$$

The temporal behavior of (52) is almost identical to that of the approximate solution (48). In particular, the initial response at $z^* = z_1^*$ is linear [cf. Eq. (50)], i.e.,

$$\Phi'(z_1^*,t) = \frac{(f_0^2/H)w_f}{\left[(k^2 + l^2) + \dfrac{f_0^2}{gH}\right]} \times \left\{\frac{\left[1 + \dfrac{f_0^2}{gH(k^2 + l^2)}\right]}{\left[\dfrac{1}{(\tfrac{1}{2} + \mu)} + \dfrac{f_0^2}{gH(k^2 + l^2)}\right]} t + C\right\}. \quad (53)$$

And, for large time, the maximum amplitude that can be reached is again

$$(\Phi')_{\max} = \frac{(f_0^2/H)w_f}{\left[(k^2 + l^2) + \dfrac{f_0^2}{gH}\right]} T_e,$$

which is exactly the same as Eq. (51). Note that this maximum amplitude is independent of height and is the same as the amplitude at the surface that can be reached by the inviscid resonant solution in a time equal to the damping time scale, $T_e \equiv 1/\hat{a}_e$.

In Fig. 1 the time behavior of the solution as given by Eq. (48) is shown schematically, and in Tables 4 and 5 the amplitude of $(1/g)\Phi'$ given by (53) for $t = 0$ and at $z^* = z_1^*$ are listed for various wave modes using $a_e = 1/(5 \text{ days})$ (Charney and Eliasen, 1949). Also listed are the maximum amplitudes as $t \to \infty$. In Table 4 topographic forcing with amplitudes as given in Table 2 is used, while in Table 5 the combined topographic and thermal forcing is used. Note the following general features:

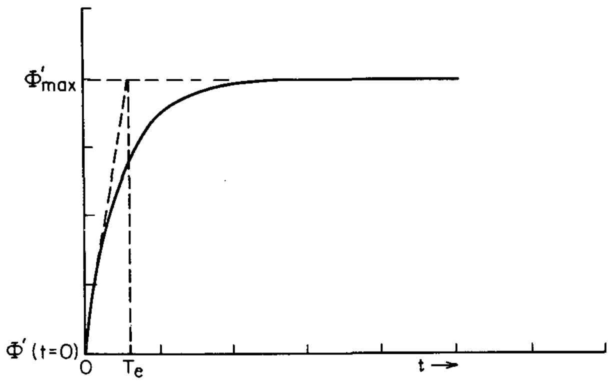

FIG. 1. Schematic diagram of the time behavior of the resonant solution in the presence of damping with a time scale T_e.

(i) The maximum amplitudes reachable in the presence of Ekman damping decrease rather rapidly with increasing meridional wavenumber. Due to their small amplitudes, the waves with meridional wavenumber $n > 1$ can almost be discounted. This seems to be another "selection principle" working against waves with short meridional scales.

(ii) The $(s,n) = (1,0)$ and $(s,n) = (2,0)$ modes have very large maximum amplitudes, but as mentioned before, these two modes cannot be excited. Focusing only on the symmetric waves, which have the proper quantization even when critical levels are present, the next mode for wavenumber 1 is $(s,n) = (1,1)$, with a maximum amplitude of 389 m with topographic forcing and 293 m with combined forcing. For wavenumber 2, the corresponding amplitudes for the mode $(s,n) = (2,1)$ are 493 and 258 m. The amplitudes are of the correct order of magnitude as observed. A more detailed comparison will be given in the next section.

(iii) Ekman pumping damps shorter waves more than the longer waves.

When the damping time scale T_e is longer than one week, it should be replaced by other damping time scales. We will discuss this more in the following paragraph.

a. Newtonian cooling

Newtonian cooling damps vertically propagating waves and its damping time scale varies depending

TABLE 4. List of the calculated amplitudes of geopotential heights due to topographic forcing alone, $A(z^*,t) = g^{-1}\Phi'(x,y,z^*,t)/[e^{ikx}\sin l(y_p - y)]$ at $z^* = z_1^* = 1/7.5$ and $t = 0$, and also at $t \to \infty$ for any z^*.

s	n	n'	T_e (days)	$A(z_1^*, t = 0)$ (m)	$A(z^*, t \to \infty)$ (m)
1	0	0	14.8	3.4	10 991
	1	1	9.9	2.9	1549
		2	7.7	3.2	389
	2	3	6.6	7.3	290
		4	6.1	1.0	17
	3	5	5.8	8.6	67
		6	5.6	6.0	25
	4	7	5.4	2.7	6.4
		8	5.4	6.9	10
2	0	0	8.3	3.8	1517
	1	1	7.5	6.2	1246
		2	6.7	5.5	493
	2	3	6.2	1.2	51
		4	5.9	2.9	61
	3	5	5.7	6.1	69
		6	5.5	1.7	11
	4	7	5.4	0.9	3.7
		8	5.3	4.4	11
3	0	0	6.6	2.8	298
	1	1	6.3	5.7	438
		2	6.1	3.8	187
	2	3	5.9	1.2	36
		4	5.7	2.5	46
	3	5	5.5	1.6	17
		6	5.4	0.9	6
	4	7	5.4	4.6	21
		8	5.3	2.3	7
4	0	0	5.9	6.0	253
	1	1	5.8	6.5	229
		2	5.7	4.7	123
	2	3	5.6	0.3	6.4
		4	5.5	1.8	24
	3	5	5.4	3.6	33
		6	5.4	3.3	21
	4	7	5.3	1.0	4.4
		8	5.3	0.2	0.7

on the vertical wavelength of the waves.[6] The damping time scale due to Newtonian cooling can be shown to be

$$T_N = \frac{1}{a_N}\left[1 + \frac{(k^2 + l^2)}{(f_0^2/S)(\tfrac{1}{4} + \lambda^2)}\right], \qquad (54)$$

where λ is the vertical wavenumber, and is related to the vertical wavelength L by $\lambda = 2\pi H/L$. a_N is the coefficient of Newtonian cooling; commonly used values for $1/a_N$ are of the order of four weeks, though values as short as one week have been used by some authors occasionally.

From (54) it is seen that in the lower atmosphere, where the long waves are usually vertically propagating, the term in brackets is greater than 1; hence $T_N > 1/a_N \sim O$ (4 weeks). This is longer than the damping time scale due to Ekman pumping considered earlier and can therefore be neglected.

b. Horizontal diffusion

It can be shown that the damping time scale due to horizontal diffusions with an eddy diffusivity ν is given by

$$T_\nu = \frac{1}{\nu(k^2 + l^2)} = \left[\left(\frac{2\nu}{a^2}\right)(s^2 + m^2)\right]^{-1}.$$

Even for a rather high value of $\nu \approx 10^9$ cm^2 s^{-1}, we have $T_\nu \approx 2000$ days$/(s^2 + m^2)$. Therefore, damping

TABLE 5. As in Table 4, except that the combined topographic and thermal forcings are used.

s	n	n'	$A(z_1^*, t = 0)$ (m)	$A(z^*, t \to \infty)$ (m)
1	0	0	1.2	3736
	1	1	1.8	965
		2	2.4	293
	2	3	2.5	100
		4	2.3	37
	3	5	2.0	16
		6	1.5	6.1
	4	7	1.8	4.3
		8	0.7	1.0
2	0	0	0.8	319
	1	1	1.2	246
		2	2.9	258
	2	3	1.9	80
		4	3.3	69
	3	5	1.8	21
		6	1.4	9.3
	4	7	0.8	3.6
		8	—	—
3	0	0	0.8	83
	1	1	1.5	112
		2	1.7	84
	2	3	1.3	39
		4	2.1	38
	3	5	1.2	13
		6	1.0	6.9
	4	7	0.4	2.0
		8	—	—
4	0	0	0.9	38
	1	1	1.3	46
		2	1.7	47
	2	3	1.8	33
		4	1.5	20
	3	5	1.1	9.6
		6	0.6	4.1
	4	7	—	—
		8	—	—

[6] In the simple model considered in Sections 2 and 3, the resonant waves are barotropic in height. Therefore, Newtonian cooling has no appreciable effect on these waves. However, in more realistic models where the vertical shear of the mean wind is taken into account, the resonant waves are internally trapped and have finite vertical wavelengths.

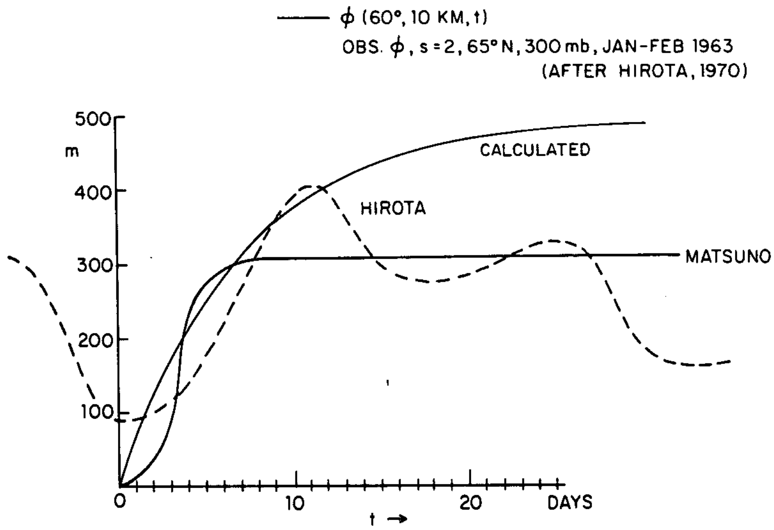

FIG. 2. Time behavior of the isobaric surface at 300 mb calculated in the present model with Eq. (48), together with the forcing function used in Matsuno's model and also the observed wave behavior.

due to horizontal diffusion does not become important until the wavenumbers are larger than 15.

8. Comparison with the lower boundary forcing function of Matsuno (1971)

As the lower boundary forcing at the tropopause (300 mb) for his numerical model of sudden warming, Matsuno (1971) used a forcing function given by

$$\Phi' = \{e^{is\lambda} \sin[\pi(\varphi - 30°)/60°]\}\phi(t),$$

where λ is the longitude and φ is the latitude. The latitudinal form was chosen based on the observations of Teweles (1958) and Hirota and Sato (1969), and happens to be exactly the same as the latitudinal structure for the symmetric mode $(s,n) = (2,1)$, which has been shown in our model to be a good candidate for resonance, both in terms of the magnitude of the forcing $\hbar_0$ and the attainability of the resonant speed U_r. The time behavior of $\phi(t)$ used by Matsuno is reproduced in Fig. 2, together with the observed wave behavior due to Hirota. The time behavior of the resonant wave calculated in our model is superimposed in Fig. 2. The agreement is seen to be quite good,[7] at least for $t < 2T_e$.[8]

As was pointed out by Matsuno, the initial time amplification of the wave in his boundary "forcing" is a very important aspect of his model. Without this initial "transient" behavior, the sudden warming cannot be initiated in his model. The origin of this amplifying wave in the troposphere was not investigated, but Matsuno did demonstrate that, given such a wave, his stratospheric model can produce a sudden warming. Our result here offers a possible explanation of the origin of the empirical forcing function used in Matsuno's numerical model of sudden warming.

9. Off-resonance wave response

Since the basic flow in the real atmosphere is not expected to remain in an exact resonance state for any length of time, it is important to study the sensitivity of the wave amplication mechanism presented in the previous sections for cases when the mean state is either off the resonance configuration or fluctuates slightly about it. It will be shown that, for wave amplification to occur, it is apparently not very critical that an exact resonance state be maintained.

The relevant equation to consider is Eq. (41), the vorticity equation in the presence of Ekman pumping. The exact solution of Eq. (41) at off-resonance will be given in Appendix C. Here for the purpose of presentation, a simpler case of barotropic waves is considered. Neglecting the vertical dependence of Φ', Eq. (41) reduces to [cf. Eq. (46)]

$$\frac{\partial}{\partial t}\Phi' + a_e\Phi' + ik(\bar{u} - U_r)\Phi' = \frac{f_0^2/H}{[(k^2 + l^2) + f_0^2/gH]} w_f, \quad (55)$$

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[7] In Fig. 2, the calculation is based on the topographic forcing alone, since we do not have the thermal-forcing amplitudes for January 1963 analyzed by Hirota.

[8] The oscillatory wave behavior for $t > 2T_e$ in Hirota's curve can be similated in the present model by letting $\bar{u}$ deviate from U_r after being at the resonant value for $2T_e$. However, such type of curve fitting is not done here in Fig. 2.

where a_e is the rate of damping due to Ekman pumping, given by (44). The balance of various terms clearly can be seen in the simple form of Eq. (55). Suppose the wave amplitude is initially small; then the forcing term on the right-hand side of Eq. (55) is balanced by the amplification term $\partial\Phi'/\partial t$, giving a growth in the wave amplitude which is initially linear in time. Such an amplification is slowed and eventually stopped when the damping term, the second term on the left-hand side of Eq. (55), becomes important. This case has been considered previously in Section 7. If the mean wind is not at the exact resonance state, so that the third term in Eq. (55) is nonzero, the atmosphere becomes capable of supporting Rossby wave oscillations, and a traveling Rossby wave with a frequency

$$\sigma = k(U_r - \bar{u}) \tag{56}$$

is generated and propagates away with some of the energy from the forcing. If damping does not take its toll, the traveling Rossby wave will alternately enhance and diminish the amplitude of the forced stationary wave as it comes into and out of phase with the latter.

When both the damping term and the free Rossby wave term are present, the relative importance of each depends on the magnitude of the parameter

$$\gamma \equiv \sigma/a_e \tag{57}$$

obtained by taking the ratio of the magnitudes of the second and third terms in Eq. (55). From the solution to Eq. (55)

$$\Phi'/\Phi'_{\max} = (1 - e^{-a_e t}e^{i\sigma t})/(1 - i\gamma), \tag{58}$$

the wave response as a function of γ is clearly exhibited:

$$|\Phi'/\Phi'_{\max}| = \frac{1}{(1+\gamma^2)^{1/2}} \times [1 + e^{-2(t/T_e)} - 2e^{-(t/T_e)}\cos\gamma(t/T_e)] \tag{59}$$

where

$$\Phi'_{\max} \equiv \frac{f_0^2/H}{(k^2+l^2) + f_0^2/gH} w_f T_e$$

is the maximum amplitude that can be achieved at *resonance*. The wave response as given by (59) is plotted for various values of γ in Fig. 3 as a function of time in units of T_e. Note that (59) is invariant with respect to a change of sign for γ, so Fig. 3, though plotted for positive values of γ only, is the same if it is changed to negative. [Observationally, however, a positive γ implies that the traveling wave is retrograde (i.e., $c < 0$), while a negative γ suggests progression.] The curve denoted by $\gamma = 0$ in Fig. 3 is the same as the one in Fig. 1, and depicts the resonant response of the stationary wave in the presence of damping with a damping time scale T_e. For nonzero values of γ, the response is diminished and wave oscillations become evident for $|\gamma| > 2$. For $|\gamma| < 1$, the response is not too different from that at exact resonance, and when $|\gamma| \lesssim 0.5$ the maximum amplitude achieved is within 10% of the resonance value. Thus one can say that the wave amplification is not too sensitive to being at exact resonance, provided that the "degree of off-resonance", as measured by the frequency of the free Rossby oscillation, is less than about half the damping rate. If we take an Ekman damping time scale of about one week, then the wave fre-

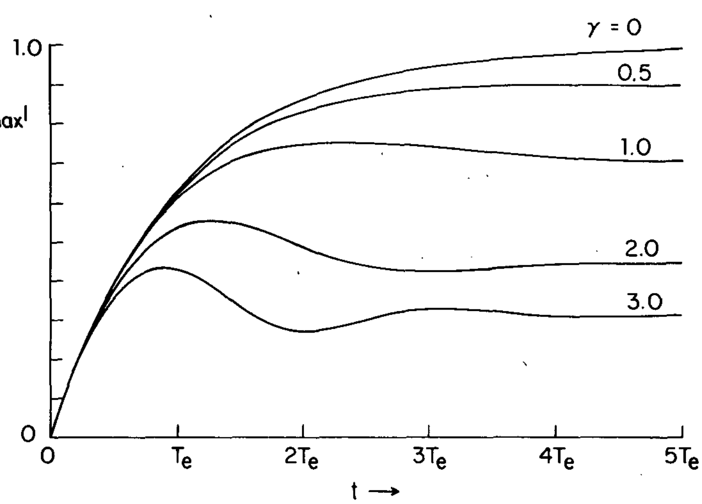

FIG. 3. Behavior of the magnitude of the wave height field as a function of time in units of the damping time scale T_e, for various values of $\gamma \equiv k(U_r - \bar{u})T_e$.

quency has to be less than 1/(2 weeks) in order to retain at least 90% of the resonance response.

We have also performed some calculations for the case of a mean wind slowly varying about the resonance value, utilizing the two-timing formalism developed in Section 3. The findings are to be expected from the results presented in Fig. 3 for the steady wind case: Resonant responses appear whenever $\bar{u} = U_r$ occurs, and such responses are maintained to within 10% over periods when the deviation of $\bar{u}$ from U_r are such that $|\gamma| \lesssim 0.5$.

10. The northward heat flux of the stationary waves

Traditionally, it has been a common practice for data analysts to attribute the observed heat fluxes in the atmosphere to baroclinic processes. It is our purpose here to point out that amplifying stationary long waves can also contribute significantly to the observed increases in heat flux in the atmosphere.

For a resonant wave of the form of Eq. (29), one has

$$\begin{aligned}\mathrm{Re}\, v' &= \frac{1}{f_0}\mathrm{Re}\frac{\partial}{\partial x}\Phi' \\ &= h_0 \frac{k}{f_0[1+(f_0^2/gH)(k^2+l^2)^{-1}]}\left[\frac{S}{H}z^* \sin kx - \frac{kU_0 f_0^2}{(k^2+l^2)H} t \cos kx\right] \sin l(y_p - y),\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\, T' &= \frac{1}{R}\mathrm{Re}\frac{\partial}{\partial z^*}\Phi' \\ &= -\frac{h_0 S}{HR[1+(f_0^2/gH)(k^2+l^2)^{-1}]} \times \cos kx \sin l(y_p - y),\end{aligned}$$

so that the northward heat flux is given by

$$\begin{aligned}\overline{v'T'} &\equiv \frac{1}{2\pi}\int_0^{2\pi} (\mathrm{Re}\, v')(\mathrm{Re}\, T') d(kx) \\ &= \frac{h_0^2 S f_0 U_r k^2/(k^2+l^2)}{2H^2 R[1+(f_0^2/gH)(k^2+l^2)^{-1}]^2} t \times \sin^2 l(y_p - y). \qquad (60)\end{aligned}$$

This implies a poleward flux at $t = T_e \approx 1$ week of 6.2°C m s^{-1} at the midlatitude for the mode $(s,n) = (2,1)$. This value should be applied to the lower atmosphere, since the present barotropic model greatly underestimates the fluxes at the upper levels due to the lack of the e^{z^*} growth with height.

As a result, the induced deceleration of the zonal flow in the present model does not have the desired property of increasing in magnitude with height. More important, through the thermal wind relation, the induced temperature change is zero. It will be shown, in Part II, that these undesirable qualities can be overcome in a more realistic model with vertical shears, where internally trapped Rossby waves propagate vertically below the stratospheric jet maximum with their amplitudes increasing in height roughly as $e^{z^*/2}$.

Nevertheless, the present model has the same resonance mechanism of the other more complicated models and the solution has the correct temporal behavior. Therefore, the present model is expected to describe the blocking, which is more or less a surface phenomenon, better than the sudden warming phenomenon aloft. As far as the latter is concerned, the present model provides an explanation for the origin of the lower-boundary forcing function used in the more realistic stratospheric models.

11. Conclusion

It is generally recognized that stratospheric sudden warming events are caused by stationary planetary waves generated in the troposphere. It has been observed that these waves amplify preceding the onset of warmings in the stratosphere; blockings caused by these amplifying waves also occur in the troposphere simultaneously with the warmings aloft. It is such transient time behavior of the waves that enables them to interact with the mean flow, as pointed out by Matsuno (1971), who was able to simulate a sudden warming event in his numerical model using amplifying waves.

In this paper a theory has been advanced to account for the unusual time behavior of the waves. We suggest that these amplifying waves are waves at resonance with the topographic forcing and land-sea differential heating. In order for a wave to be resonant, the following conditions have to be satisfied.

(i) The wave energy has to be contained both horizontally and vertically. The problem of horizontal confinement of the waves will be considered in detail in Part III, where it is shown that given the parameter values of viscosity and nonlinearity relevant to the real atmosphere, the zero wind line is likely to be a reflecting surface to waves propagated from the north. Stationary planetary waves are therefore confined between the zero wind line and the North Pole. In the simple, uniform wind model presented in this part, the requirement for vertical containment is that the wind speed is high enough so that the waves become evanescent in the atmosphere; the resonant waves are found to be barotropic. In the presence of realistic vertical shears, as will be discussed in Part II, the resonant waves are no longer necessarily barotropic; they can propagate in the lower atmosphere, but there must exist a turning point above which the waves are evanescent. The results indicate that the condition is easily satisfied by wavenumber 4, but is met by wavenumbers 1 and 2 only under abnormal conditions. This may explain why tropospheric blocking is common while stratospheric warmings are relatively rare.

(ii) To be resonant, the wind condition in the atmosphere has to be such that the phase speed of the free travelling wave is reduced to zero, so that the wave becomes stationary with respect to the surface of the Earth and hence, also to the topographic forcing and land-sea differential heating. This is called the "stationarity condition." In the present uniform wind model, this condition is satisfied for a wave mode with zonal wavenumber k and meridional wavenumber l when the mean wind reaches the Rossby-Haurwitz speed, $U_r = \beta/(k^2 + l^2)$. In models that incorporate the vertical and horizontal shears of the atmosphere (see Parts II and III), the stationarity condition becomes an integral condition on the wind values in the part of the atmosphere below the turning point mentioned in (i). This integral stationarity condition will be shown to be satisfied by realistic wind profiles.

(iii) Dampings due to various mechanisms cannot be too great; otherwise, little amplification of the wave will result. We find that a damping time scale of around 4–5 days is sufficiently long to allow the amount of amplification that is observed. In addition to the stationarity condition mentioned in (ii), damping provides another selective principle, which for most practical purposes eliminated waves with meridional wavenumber $n > 1$.

For the wave mode $(s,n) = (2,1)$, which is found by the present theory to be one of the most likely candidates for large-scale resonant waves in the troposphere and stratosphere, the calculated amplitude and resonant amplification behavior compare favorably with observations and the wave forcing function used in Matsuno's numerical model to generate a sudden warming in the stratosphere. Thus, it seems that the only diagnostic part in Matsuno's theory of sudden warming can be replaced by a more consistent predictive theory.

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APPENDIX A

List of Symbols

s	zonal wavenumber; number of waves in a zonal circle
f	Coriolis parameter [$=2\Omega \sin\varphi$, where φ is the latitude]
f_0	$2\Omega \sin\varphi_0$, $\varphi_0 = 45°$
β	$\frac{1}{a}\frac{d}{d\varphi}f$ at $\varphi = \varphi_0$, where a is the radius of the earth
z^*	$\ln(p_0/p)$ where p is pressure, $p_0 = 1000$ mb
$h(x,y)$	continental elevation plus equivalent elevation due to land-sea differential heating
$\hbar_0$	amplitude of a wave component of $h(x,y)$
R	gas constant
H	scale height (=7.5 km)
S	stability parameter [$=\kappa gH$, $\kappa = 2/7$]
U_r	resonant speed defined in Eq. (14)
k	dimensional zonal wavenumber [$=s/(a \times \cos\varphi)$]
l	dimensional meridional wavenumber
m	$l(a \cos\varphi)$
n	number of nodes in the meridional domain
n'	number of quarter-wavelengths in the meridional domain
y_p	β-plane's equivalent location of the pole [$=\pi a/2$]
y_c	location of the zero wind line
T_e	damping time scale due to Ekman pumping, defined in Eq. (50)
T_N	damping time scale due to Newtonian cooling, defined in Eq. (53)
T_ν	damping time scale due to horizontal diffusion

APPENDIX B

Solution of the Initial Value Problem

The governing equations are

$$\left\{\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{f_0^2}{S}\left(\frac{\partial^2}{\partial z^{*2}} - \frac{\partial}{\partial z^*}\right)\right] + \beta\frac{\partial}{\partial x}\right\}\Phi' = 0, \quad \text{(B1)}$$

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\Phi'_{z^*} - \kappa\frac{\partial}{\partial t}\Phi' = -w\frac{S}{H},$$

$$\kappa \equiv \frac{S}{gH}, \quad \text{(B2)}$$

subject to the lower boundary condition

$$w = \bar{u}\frac{\partial}{\partial x}\hbar(x,y), \quad \text{at} \quad z^* = 0, \tag{B3}$$

where

$$\hbar(x,y) = \hbar_0 e^{ikx}\sin[l(y_p - y)]. \tag{B4}$$

Due to the form (B4) of the forcing, the solutions can be expressed as

$$\Phi' = \psi(z^*,t)e^{ikx}\sin[l(y_p - y)], \tag{B5}$$

$$w = W(z^*,t)e^{ikx}\sin[l(y_p - y)]. \tag{B6}$$

For the switch-on problem where $\bar{u}(t)$ is given by

$$\bar{u}(t) = U\cdot H(t) \tag{B7}$$

the solutions for $t < 0$ are given by

$$W = 0 \tag{B8}$$

$$\psi = De^{-ikc_-t} \times \exp\left\{\left[\frac{1}{2} - \left(\frac{1}{4} + \frac{S}{f_0^2}\frac{\beta}{c_-} + k^2 + l^2\right)^{1/2}\right]z^*\right\}, \tag{B9}$$

where

$$c_- = -\frac{\beta}{[(k^2 + l^2) + (1 - \kappa)(f_0^2/gH)]}. \tag{B10}$$

These represent the free waves that exist in the absence of forcing, and since the equations are linear, they can be considered separately. Therefore, for the switch-on problem, we use the initial conditions:

$$\left.\begin{matrix} W \\ \psi \end{matrix}\right\} = 0 \quad \text{for} \quad t < 0. \tag{B11}$$

On defining the Laplace transformed variables by

$$\hat{W} = \int_0^\infty e^{-\sigma t}W(t)dt,$$

$$\hat{\psi} = \int_0^\infty e^{-\sigma t}\psi(t)dt, \tag{B12}$$

(B1) and (B2) become

$$\left(\frac{d^2}{dz^{*2}} - \frac{d}{dz^*}\right)\hat{\psi} + \lambda^2\hat{\psi} = 0, \tag{B13a}$$

$$\lambda^2 \equiv \frac{S}{f_0^2}\left[\frac{ik\beta}{ikU + \sigma} - (k^2 + l^2)\right], \tag{B13b}$$

$$(\sigma + ikU)\frac{d}{dz^*}\hat{\psi} - \kappa\sigma\hat{\psi} = -\frac{S}{H}\hat{W}, \tag{B14}$$

while (B3) becomes

$$\hat{W} = ikU\hbar_0/\sigma, \quad \text{at} \quad z^* = 0. \tag{B15}$$

The solutions are therefore

$$\hat{\psi}(z^*,\sigma) = \frac{-ikU\hbar_0(S/H)\exp\{[\tfrac{1}{2} - (\tfrac{1}{4} - \lambda^2)^{1/2}]z^*\}}{\sigma\{(ikU + \sigma)[\tfrac{1}{2} - (\tfrac{1}{4} - \lambda^2)^{1/2}] - \kappa\sigma\}} \tag{B16}$$

and

$$\hat{W}(z^*,\sigma) = \frac{ikU\hbar_0}{\sigma} \times \exp\{[\tfrac{1}{2} - (\tfrac{1}{4} - \lambda^2)^{1/2}]z^*\}. \tag{B17}$$

Eq. (B17) has the same form as the solution for $\hat{\omega}$ in Eq. (18) of Clark (1972) while (B16) is slightly different from Clark's. The difference came from the different boundary condition used by us which includes a non-Doppler term. This term changes only the phase speed of the free wave [see Eq. (12)] and does not affect the large time asymptotic behavior of the transient solution. Inverting (B16) and (B17), one finds

$$W = W_0 + W_{\text{transient}},$$

$$\psi = \psi_0 + \psi_{\text{transient}},$$

where W_0 and ψ_0 are the forced waves given in (9) and (11), while the transient waves can be shown to decay in time as

$$W_{\text{transient}} = O(t^{-5/6}),$$

$$\psi_{\text{transient}} = O(t^{-1/2}).$$

APPENDIX C

Solution at Off-Resonance

The governing equation is [Eq. (41)]

$$\left[\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \beta\frac{\partial}{\partial x} + \frac{f_0^2}{gH}\frac{\partial}{\partial t}\frac{\partial}{\partial z^*}\right](e^{-z^*}\Phi') = \frac{f_0^2}{H}\frac{\partial}{\partial z^*}(e^{-z^*}w), \tag{C1}$$

subject to the lower boundary condition of [cf. Eq. (43)]

$$w(z_1^*) = w_f + (H/f_0^2)a_e\nabla^2\Phi'(z_1^*), \tag{C2}$$

where z_1^* is the top of the Ekman layer. Integration of (C1) from $z^* = z_1^*$ to ∞ yields

$$\frac{\partial}{\partial t}\exp z_1^*\int_{z_1^*}^\infty (e^{-z^*}\Phi')dz^* + \frac{f_0^2/gH}{(k^2 + l^2)}\frac{\partial}{\partial t}\Phi'(z_1^*) + a_e\Phi'(z_1^*)$$

$$+ ik(U_r - \bar{u})\Phi'(z_1^*) = \frac{f_0^2/H}{(k^2 + l^2)} w_f. \quad \text{(C3)}$$

In (C3), the boundary conditions of $\Phi e^{-z^*} \to 0$ as $z^* \to \infty$ and (C2) have been used. The solution to (C3) is

$$\Phi' = \frac{(f_0^2/H)w_f}{(k^2 + l^2)[a_e + ik(U_r - \bar{u})]} \times \{[1 - e^{-\theta t}] \exp[(\tfrac{1}{2} - \mu)(z^* - z_1^*)]\}, \quad \text{(C4)}$$

where

$$\theta = [a_e + ik(U_r - \bar{u})]\left[\frac{1}{(\tfrac{1}{2} + \mu)} + \frac{f_0^2/gH}{(k^2 + l^2)}\right]^{-1},$$

$$\mu = \left\{\frac{1}{4} + \frac{S}{f_0^2}(k^2 + l^2)\left[\frac{\theta + ikU_r - \bar{u})}{\theta - ik\bar{u}}\right]\right\}^{1/2}.$$

As far as the time behavior of the solution and its dependence on the parameter $\gamma \equiv k(U_r - \bar{u})/a_e$ is concerned, Eq. (C4) is not too different from the approximate solution used in Section 9.

REFERENCES

Bates, J. R., 1977: Dynamics of stationary ultra-long waves in middle latitudes. *Quart. J. Roy. Meteor. Soc.*, **103**, 397–430.

Brezowsky, H., H. Flohn and P. Hess, 1951: Some remarks on the climatology of blocking action. *Tellus*, **3**, 191–194.

Charney, J. G., 1973: Planetary fluid dynamics. *Dynamic Meteorology*, P. Morel, Ed., D. Reidel 97–352.

——, and A. Eliassen, 1949: A numerical method for predicting the perturbations of middle latitude westerlies. *Tellus*, **1**, 38–54.

——, and P. G. Drazin, 1961: Propagation of planetary scale disturbances from the lower into the upper atmosphere. *J. Geophys. Res.*, **66**, 83–109.

——, and M. E. Stern, 1962: On the stability of internal baroclinic jets in a rotating atmosphere. *J. Atmos. Sci.* **19**, 159–172.

Clark, J. H. E., 1972: The vertical propagation of forced atmospheric planetary waves. *J. Atmos. Sci.* **29**, 1430–1451.

——, 1974: Atmospheric response to the quasi-resonant growth of forced planetary waves. *J. Meteor. Soc. Japan*, **52**, 143–195.

Craig, R., and W. Hering, 1959: The stratospheric warming of January–February 1957. *J. Meteor.*, **16**, 91–107.

Dickinson, R. E., 1968: Planetary Rossby waves propagating vertically through weak westerly wind wave guides. *J. Atmos. Sci.*, **25**, 984–1002.

——, 1969: Theory of planetary wave-zonal flow interaction. *J. Atmos. Sci.*, **25**, 73–81.

Döö, B. R., 1962: The influence of exchange of sensible heat with the Earth's surface on the planetary flow. *Tellus*, **14**, 133–147.

Eliasen, E., and B. Machenhauer, 1965: A study of the fluctuations of the atmospheric planetary flow patterns represented by spherical harmonics. *Tellus*, **17**, 220–238.

——, and ——, 1969: On the observed large-scale atmospheric wave motions. *Tellus*, **21**, 149–166.

Finger, F., and S. Teweles, 1964: The mid-winter 1963 stratospheric warming and circulation change. *J. Appl. Meteor.*, **3**, 1–15.

Geisler, J. E., 1974: A numerical model of the sudden stratospheric warming mechanism. *J. Geophys. Res.*, **79**, 4989–4999.

Hirota, I., and Y. Sato, 1969: Periodic variation of the winter stratospheric circulation and intermittent vertical propagation of planetary waves. *J. Meteor. Soc. Japan*, **47**, 390–402.

Johnson, K., 1969: A preliminary study of the stratospheric warming of December 1967–January 1968. *Mon. Wea. Rev.*, **97**, 553–564.

Julian, P. R., and K. B. Labitzke, 1965: A study of atmospheric energetics during the January–February 1963 stratospheric warming. *J. Atmos. Sci.*, **22**, 597–610.

Labitzke, K., 1965: On the mutual relation between stratosphere and troposphere during periods of stratospheric warmings in winter. *J. Appl. Meteor.*, **4**, 91–99.

Matsuno, T., 1971: A dynamic model of the stratospheric sudden warming. *J. Atmos. Sci.*, **28**, 1479–1494.

McIntyre, M. E., 1972: Baroclinic instability of an idealized model of the polar night jet. *Quart. J. Roy. Meteor. Soc.*, **98**, 165–174.

Muench, H. S., 1965: On the dynamics of the wintertime stratospheric circulation. *J. Atmos. Sci.*, **22**, 262–283.

Murakami, T., 1965: Energy cycle of the stratospheric warming in early 1958. *J. Meteor. Soc. Japan*, **43**, 262–283.

Namias, J., 1964: Seasonal persistence and recurrence of European blocking during 1958–60. *Tellus*, **16**, 394–407.

O'Connor, J. F., 1963: The weather and circulation of January 1963: One of the most severe months on record in the United States and Europe. *Mon. Wea. Rev.*, **92**, 209–218.

Quiroz, R. S., 1975: The stratospheric evolution of sudden warmings in 1969–74 determined from measured infrared radiation fields. *J. Atmos. Sci.*, **32**, 211–224.

——, 1977: The tropospheric-stratospheric polar vortex breakdown of January 1977. *Geophys. Res. Lett.*, **4**, 151–154.

——, A. J. Miller and R. M. Nagatani, 1975: A comparison of observed and simulated properties of sudden stratospheric warmings. *J. Atmos. Sci.*, **32**, 1723–1736.

Reed, R. J., J. L. Wolfe and H. Nishimoto, 1963: A spectral analysis of the energetics of the stratospheric sudden warming of early 1957. *J. Atmos. Sci.*, **20**, 265–275.

Rex, D., 1951: The effect of Atlantic blocking action upon European climate. *Tellus*, **3**, 100–111.

Sankar-Rao, M., 1965: Continental elevation influence on the stationary harmonics of the atmospheric motion. *Pure Appl. Geophys.*, **60**, 141–159.

——, and B. Saltzman, 1969: On a steady theory of global monsoons. *Tellus*, **21**, 308–330.

Sumner, E. J., 1954: A study of blocking in the Atlantic-European sector of the Northern Hemisphere. *Quart. J. Roy. Meteor. Soc.*, **80**, 402–416.

Teweles, S., 1958: Anomalous warming of the stratosphere over North America in early 1957. *Mon. Wea. Rev.*, **86**, 377–396.

Trenberth, K. E., 1973: Dynamic coupling of the stratosphere with the troposphere and sudden stratospheric warming. *Mon. Wea. Rev.*, **101**, 306–322.