Wave Overreflection and Shear Instability

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ABSTRACT

It is shown that the necessary conditions for the instability of unstratified plane-parallel shear flow, rotating barotropic flows and rotating baroclinic flows are also sufficient conditions for the existence of propagating waves (essentially Rossby waves) and their overreflection (reflection coefficient exceeds 1 in magnitude) from critical levels (where flow speed and phase speed are equal). The identification of the unstable modes with overreflected waves is strongly suggested and allows greater insight into the meaning of various theorems such as Rayleigh's inflection point theorem.

The present results also suggest an important distinction between instabilities associated with mass redistribution such as Bénard convective instability and instabilities, such as those we are concerned with, associated with the self-excitation of waves.

1. Introduction

In recent years a close, though ill-defined, relation has been noted between the instability of stably stratified shear flows and the existence of overreflection for internal gravity waves incident on a critical level [i.e., the existence of a reflection coefficient exceeding unity for an internal gravity wave incident on a level where the wave's horizontal phase velocity and the mean flow velocity are equal (e.g., Jones, 1968, Lindzen, 1974)]. Both phenomena require that the Richardson number be less than $\frac{1}{4}$, and both involve an extraction of energy from the mean flow. Lindzen (1974) suggested that overreflected waves, if contained, would become unstable modes. This has been confirmed in a number of calculations (e.g., Lindzen and Rosenthal, 1976; Davis and Peltier, 1976) and, indeed, a close reading of Jones (1968) reveals the presence of such modes in his results. These calculations reveal that the destabilized gravity waves must not only be contained (by a physical boundary), but must also satisfy a quantization (wherein roughly an odd number of quarter-wavelengths must fit between the critical level and the "ground").

The above calculations all led to conventional Kelvin-Helmholtz instabilities (instabilities closely confined to the unstable shear zones and, to a large extent, independent of the existence of boundaries) in addition to the unstable gravity waves. In general, the two types of instability were considered to be physically distinct, though Lindzen and Rosenthal (1976) did find a tendency for them to become indistinguishable at long horizontal wavelengths. Lindzen and Rosenthal have now discovered that Kelvin-Helmholtz instabilities are destabilized vorticity waves—contained by internal turning points rather than physical boundaries. Such waves are, as we shall show in this paper, natural extensions of Rossby waves. This work is currently being prepared for publication. The important point, for our present purposes, is that we now have a total identification of the instability of stratified shear flows with wave overreflection.

The question we wish to address in this paper is whether a similar identification of wave overreflection and hydrodynamic instability may exist for other situations. We will show that for barotropic instability (of which the stability of unstratified shear flow is a special case) the relevant waves are essentially horizontally propagating Rossby waves and that the conditions for the existence and overreflection of such waves are intimately related to the well-known necessary conditions for barotropic instability given by Rayleigh's inflection point theorem and Fjørtoft's theorem (*viz*, Yih, 1969). Our results make clear the reason why existing theorems give only necessary conditions insofar as they usually guarantee only overreflection, but not the quantization of waves. Finally, we shall show that it is also possible to view the normal baroclinic instability problem in terms of overreflection where, however, the relevant waves are vertically propagating internal Rossby waves.

2. Barotropic Rossby waves and overreflection

In this paper we will confine ourselves to the conventional β-plane geometry [our equations are derived in Charney (1973)]. The equation for the streamfunction ψ of linearized barotropic perturbations of the form

$$e^{ik(x-ct)} \quad (1)$$

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on a basic zonal flow $U(y)$ is

$$\frac{d^2\psi}{dy^2}+\left(\frac{\beta-U_{yy}}{U-c}-k^2\right)\psi=0, \tag{2}$$

where

- x eastward coördinate
- y northward coordinate
- β df/dy, f being the Coriolis parameter
- p perturbation pressure $[=\rho_0 f\psi]$ where ρ_0 is the basic-state density
- u perturbation zonal velocity $[=-(\partial/\partial y)\psi]$
- v perturbation meridional velocity $[=(\partial/\partial x)\psi]$.

When the quantity in parentheses in Eq. (2) is positive, Eq. (2) describes Rossby waves which propagate in the y direction.

It is easy to show from Eq. (2) and the equations of motion from which it is derived that for c real and for $c\neq U$, that

$$\overline{uv}=(-k/2)\,\mathrm{Im}\{\psi^*\psi_y\}=\text{constant}, \tag{3}$$

where

$$(\overline{fg})=(1/2\pi)\int_0^{2\pi}(\mathrm{Re}f)\cdot(\mathrm{Re}g)d(kx),$$

the asterisk represents the complex conjugate, and

$$\overline{pv}=-(U-c)\rho_0\overline{uv}. \tag{4}$$

In general, $\overline{pv}$ is negative (positive) for a southward (northward) propagating wave. If $U\neq c$ everywhere and there is a rigid boundary at one end of the domain, then $\overline{pv}\equiv 0$. This simply indicates that a rigid boundary leads to perfect reflection and hence there is as much northward as southward energy flux leading to zero net flux.

We now consider a situation where for $y>y_1$, $U>c$, and the bracketed quantity in Eq. (2) is greater than zero. The latter requires $\beta-U_{yy}>0$ (if $U-c<0$, $\beta-U_{yy}<0$ is called for); even so, meridional propagation will only exist for sufficiently small k's. For the moment, we do not specify conditions for $y\leqslant y_1$, except to state that there is *no direct wave forcing* in this region. In general, $U(y)$ for $y\leqslant y_1$ and any boundary condition at some $y_b<y_1$ will serve to determine $\overline{pv}$ in the region $y>y_1$. For $y>y_1$ there will be two linearly independent solution to Eq. (2): one (ψ_1) may be chosen to be a southward propagating mode for which $\overline{p_1v_1}<0$, while the other (ψ_2) may be chosen to be northward propagating mode for which $\overline{p_2v_2}>0$. If ψ_1 and ψ_2 are identically normalized then we may write

$$\psi=\psi_1+R\psi_2, \tag{5}$$

where R is a complex reflection coefficient.

It can be shown that

$$(1-|R|^2)=+A^2\overline{uv}=-A^2[\overline{pv}/\rho_0(U-c)], \tag{6}$$

where A^2 is a positive constant. Now, the calculation of A^2 may require detailed consideration of conditions in the region $y\leqslant y_1$, but if we can determine the sign of $\overline{pv}$ we will know whether $|R|$ is less or greater than 1, i.e., whether or not we have over-reflection. If $U\neq c$ for $y\leqslant y_1$, then $\overline{pv}\leqslant 0$ (since there is no wave source in $y\leqslant y_1$) and $|R|\leqslant 1$; if, moreover, there is a rigid boundary in $y\leqslant y_1$, then we have already shown that $\overline{pv}=0$ and $|R|=1$.

The only possibility for $|R|$ to be greater than 1 is that there be some $y\leqslant y_1$, where $U(y)=c$; we shall refer to this point as $y=y_c$. Let us assume, for the moment, that there is only one such point. We know that $\overline{uv}$ must be independent of y away from y_c; thus, we let

$$\overline{uv}=\begin{Bmatrix}\overline{uv_+} & \text{for} & y>y_c\\ \overline{uv_-} & \text{for} & y<y_c\end{Bmatrix}. \tag{7}$$

It is readily shown along lines developed for the inviscid Orr-Sommerfeld problem (Wasow, 1950; Lin, 1955) that in the limit of vanishing damping, $\overline{uv}$ undergoes a jump at y_c such that

$$\overline{uv_+}-\overline{uv_-}=\pi[k\hat{\beta}_c/2U_y(y_c)]|B|^2, \tag{8}$$

where

$$\hat{\beta}_c=(\beta-U_{yy})|_{y_c}$$

and $|B|^2$ is a positive definite quantity depending on the flow. [The derivation of Eq. (8) is given in the Appendix.] In the case where there is a rigid boundary at some $y<y_c$, $\overline{uv_-}=0$, and Eq. (8) becomes

$$\overline{uv_+}=\pi[k\hat{\beta}_c/2U_y(y_c)]|B|^2. \tag{9}$$

The values of $\overline{uv_+}$ will be negative and from (4) and (6), $|R|$ will exceed one, if $(\hat{\beta}_c)/[U_y(y_c)]<0$. For definiteness we shall assume $U_y(y_c)$, the shear at the critical level, to be positive. Then overreflection requires $\beta-U_{yy}<0$ at the critical level for the case under consideration.[1] Thus, in order for a wave incident on y_1 from $y>y_1$ to be overreflected, the quantity $\beta-U_{yy}$ must change sign at some y between y_1 and y_c. We shall refer to this point as y_0. The situation called for by the above conditions is shown schematically in Fig. 1. It should be noted that $\overline{uv_-}=0$ is *not* essential for overreflection; all that is required is that

$$|\overline{uv_-}|<[-\pi k\hat{\beta}_c/2U_y(y_c)]|B|^2.$$

Such a condition can be met if there is significant partial reflection at some $y<y_c$; explicit determination

[1] Clearly, if $U-c<0$ for $y>y_1$, then waviness for $y<y_1$ requires $\beta-U_{yy}<0$ for $y>y_1$ and overreflection requires that $\beta-U_{yy}<0$ at $y-y_c$. Also, conditions may exist when $|B|^2=0$, but such conditions are likely to be special.

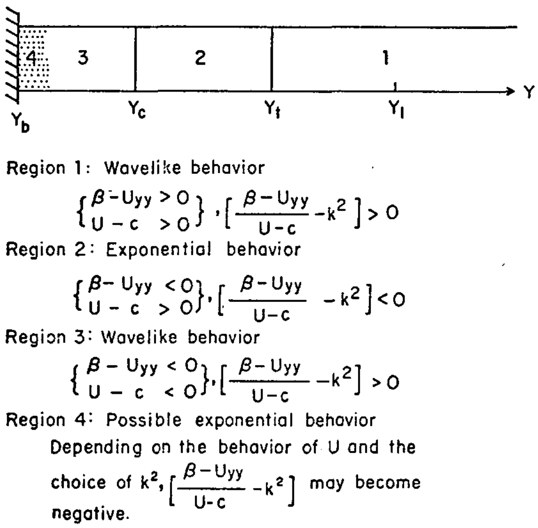

FIG. 1. The flow divided into regions according to wave characteristics.

of such a condition, however, requires a detailed calculation. For purposes of this paper we will continue to assume a wall at $y=y_b<y_c$.

Before proceeding we briefly summarize the conditions under which meridionally propagating Rossby waves will exist for $y>y_1$, and those conditions under which such waves will be overreflected from the south. For simplicity we will assume $U(y)$ is monotonic in y and increasing northward.

(i) Propagation for $y>y_1$ requires that

$$\frac{\beta-U_{yy}}{U-c}>0 \quad \text{for} \quad y>y_1.$$

Note that such wave propagation can exist even if $\beta=0$; all that is needed is a vorticity gradient. We shall refer to all waves whose restoring force arises from vorticity gradients as Rossby waves.

(ii) Overreflection requires a critical surface at some $y=y_c$ where $U(y_c)=c$.

(iii) Overreflection requires that $\beta-U_{yy}(y_c)$ have the opposite sign of $\beta-U_{yy}$ for $y>y_1$.

(iv) Overreflection requires a sufficient degree of wave reflection in the region $y<y_c$ so that

$$|\overline{uv}_-|<[-\pi k\hat{\beta}_c/2U_y(y_c)]|B|^2.$$

This is trivially satisfied when a rigid surface exists at some $y=y_b$, but clearly less restrictive conditions will also suffice.

The extension of the above analysis to non-monotonic $U(y)$'s is readily achieved, but the simple purposes of the present paper do not warrant the added complications.

3. Necessary conditions for barotropic instability and their relation to wave overreflection

There exist a number of well-known necessary conditions for barotropic instability (which, of course, also apply to the special case of the instability of nonrotating, unstratified shear flows). The three most important of these are the Rayleigh inflection point theorem, Fjørtoft's theorem and the semicircle theorem. Derivations of all of these may be found in a number of texts (Yih, 1969; Charney, 1973). We shall briefly review each of these conditions in order to show how they are related to the conditions for wave overreflection discussed in Section 2.

The oldest of the above conditions is Rayleigh's theorem [originally derived for unstratified shear flows and extended to barotropic instability by Kuo (1949)]. This theorem requires for instability that for a flow contained between boundaries at y_b and y_2 the quantity

$$\int_{y_b}^{y_2}(\beta-U_{yy})\left|\frac{\psi}{U-c}\right|^2 dy=0 \qquad (10)$$

which, in turn, requires that the quantity $\beta-U_{yy}$ must change sign someplace between y_b and y_2 (i.e., we must have an inflection point in the case $\beta\equiv0$; when $\beta\neq0$, we will refer to the point where $\beta-U_{yy}=0$ as an "inflection" point). As we see from condition (iii) the change of sign of $\beta-U_{yy}$ is also a necessary condition for the overreflection of meridionally propagating Rossby waves. Rayleigh's theorem does not, however, guarantee the existence of appropriate meridionally propagating Rossby waves; i.e., Rossby waves which propagate in the region north (south) of the "inflection" points if the critical level is south (north) of the "inflection" point. The existence of such waves, however, is assured (at least for simple $u(y)$'s with a single "inflection" point) by Fjørtoft's theorem.

Fjørtoft's theorem shows that the following inequality is necessary for instability:

$$\int_{y_b}^{y_2}(\beta-U_{yy})\{U-U(y_s)\}\left|\frac{\psi}{U-c}\right|^2 dy>0, \qquad (11)$$

where y_s is a zero of $\beta-U_{yy}$. For flows with only one such point, (11) simplifies to the requirement that

$$(\beta-U_{yy})[U-U(y_s)]>0, \qquad (12)$$

but this is precisely what is needed for the existence of appropriately propagating Rossby waves as described in condition (i). For more complicated flows Eq. (11) does require the existence of waves some place, but interpretation is more complicated.

Both of the above theorems also assume a wall at $y=y_b$, thus automatically satisfying condition (iv). For the simple situation, therefore, *Fjørtoft's and Rayleigh's theorems are not only necessary conditions for barotropic instability but also sufficient conditions for the existence* *of overreflected Rossby waves*. The discussion of Section 2 shows that the existence of a wall at $y=y_b$ is a more restrictive condition that is actually needed for overreflection, and explicit calculations of barotropic instability by Dickinson and Clare (1973) shows that the existence of such a wall is also unnecessary for the existence of instabilities.

Condition (ii) requires that waves which are overreflected must have a critical surface. This requirement is similar to the requirements imposed by various semicircle theorems. In the absence of β, Howard's (1961) semicircle theorem (see also Yih, 1969), in fact requires

$$U_{\min}<c_r<U_{\max}, \tag{13}$$

which does demand a critical surface. However, when β is included, Pedlosky's semicircle theorem (Pedlosky, 1964) requires

$$U_{\min}-\frac{\beta}{2k^2}<c_r<U_{\max}, \tag{14}$$

which still allows the *possibility* of an instability without a critical surface. We have, however, no assurance that (14) represents the tightest necessary constraint on c_r. All instabilities explicitly calculated (e.g., Dickinson and Clare, 1973) *do* have critical surfaces. Still, we have here considered a critical surface for the waves in the limit of vanishing c_i. When the effect of a finite c_i is included, the singularity associated with the critical surface is moved off the real axis, and the effect of the singularity is spread over a layer rather than a single surface. Under such circumstances, it is conceivable that an unstable mode associated with overreflection might still have a c_r slightly outside the range given by (13).

Emerging from the above discussion is the suggestion that barotropic instability can be understood in terms of the overreflection of meridionally propagating Rossby waves. What is meant by this is readily described. Consider a wavetrain in a region of wave propagation heading toward an overreflecting critical surface. The wave train will be amplified on overreflection. The amplified wavetrain will now move away from the critical surface. If it encounters a reflecting surface (which may consist of a physical wall or a turning point; even a rapid change in index of refraction producing partial reflection—provided that the partial reflection is greater than the inverse of the overreflection—will do), it will be returned toward the overreflecting surface where it will be further amplified. Clearly, overreflection combined with an energy-containing reflecting surface should lead to instability. However, for the instability to take the form of a normal mode, successive overreflections must occur in phase—a condition we will refer to as quantization. In practice, we find that quantization, when it exists, determines c_r. When quantization is obtained, growth rates can be estimated by means of a so-called laser formula:

$$e^{2kc_i\tau}=|R|\,|r|, \tag{15}$$

where $|R|$ is the magnitude of the overreflection, $|r|$ the magnitude of the partial reflection from the energy containing surface (in the cases where the energy-containing surface is either a physical wall or a turning point followed by a semi-infinite evanescent region, $|r|=1$), and τ is the time for a wave to travel the length of the region where propagation is possible at the wave group velocity. When the c_r which leads to quantization is close to the c_r of a calculated unstable mode, and when the c_i as determined by Eq. (15) is the same or greater than the c_i of the same unstable mode, then it appears reasonable to associate the instability with wave overreflection.[2] We believe that this type of instability corresponds to what is sometimes called critical level instability (Bretherton, 1966), though the identification of the latter with wave propagation has not been previously noted. In explicit calculations, which we shall report separately, we find that all known barotropic instabilities appear to be associated with wave overreflection.[3] If this is indeed the case, then we can readily understand why the above-described theorems provide necessary but not sufficient conditions for instability. As has already been noted in the case of stratified shear flows (Lindzen and Rosenthal, 1976), wave overreflection (even infinite overreflection) does not, by itself, lead to unstable modes. In addition, the overreflected wave energy must be contained and the overreflected wave must satisfy the required quantization. Confinement conditions are, in fact, generally assumed in deriving necessary conditions for instability.

The above applies to normal mode instabilities. It appears likely that the existence of overreflected waves which are contained but do not satisfy quantization conditions could lead to algebraic growth, but we have not yet demonstrated this.

4. Baroclinic instability

The purpose of this section is simply to show that the most common problem in baroclinic instability (wherein quasigeostrophy is assumed, and where the

[2] The reason why Eq. (15) may lead to an overestimate of c_i is implicitly found in McIntyre and Weissman (1978). They find that overreflection takes time to develop. If the requisite time exceeds 2τ then the full calculated overreflection will not be available to the instability.

[3] In response to a question from an anonymous referee, it is interesting to note that in the study of barotropic instability by Dickinson and Clare (1973), there were two modes of instability: one was characterized by the existence of a turning point bounding the region of wave propagation, the other by the absence of such a turning point, although the index of refraction [the quantity in parentheses in Eq. (2)] changed markedly. For the former case, $|r|=1$, and from Eq. (15), neutrality requires $|R|=1$. The corresponding neutral modes in Dickinson and Clare (1973) have critical levels at "inflection" points, where, according to Eq. (9), $\overline{uv}_+=0$ and hence $|R|$ does equal 1. For the latter modes $|r|<1$, so that neutrality requires $|R|>1$ and hence, the critical surface must occur away from the "inflection" point. This explains the differences between singular and nonsingular neutral modes. Both are consistent with wave overreflection concepts.

basic flow is purely zonal and dependent only on height) can be made homomorphic to the barotropic problem discussed in the preceding two sections. This will, in turn, permit us to identify the role of wave overreflection in this problem as well.

The problem of baroclinic instability is described in detail by Charney (1973); we shall, therefore, restrict ourselves to a minimum of detail here. For our purposes it will be adequate to restrict ourselves to a semi-infinite Boussinesq fluid. The governing instability is

$$\frac{d^2\psi}{dz^2}+\left(\frac{\epsilon^{-1}\beta-U_{zz}}{U-c}-\frac{k^2}{\epsilon}\right)\psi=0, \tag{16}$$

where, again, solutions of the form

$$f(z)e^{ik(x-ct)} \tag{17}$$

have been assumed. Here

- ψ perturbation streamfunction
- z height
- ϵ f^2/N^2
- β df/dy
- f Coriolis parameter
- N Brunt-Väisälä frequency
- y distance to the north
- $U(z)$ basic zonal velocity.

The following boundary conditions are usually adopted:

$$\psi\rightarrow 0 \quad \text{as} \quad z\rightarrow\infty, \tag{18}$$

$$\frac{d\psi}{dz}-\frac{U_z}{U-c}\psi=0 \quad \text{at} \quad z=0. \tag{19}$$

Eq. (16) is clearly mathematically identical to Eq. (2) for the barotropic problem. Moreover, in the absence of vertical shear (U_z) at $z=0$, the complete problem is equivalent to a conventional barotropic problem and may be approached precisely in terms of the preceding two sections. In particular, a necessary condition for instability would be that

$$\epsilon^{-1}\beta-U_{zz}\equiv q_y \tag{20}$$

(the y derivative of the pseudo-potential vorticity) change sign at some height. Normally, however, q_y does not change sign and, in any case, the most important baroclinic instabilities do not appear to be associated with such sign changes.

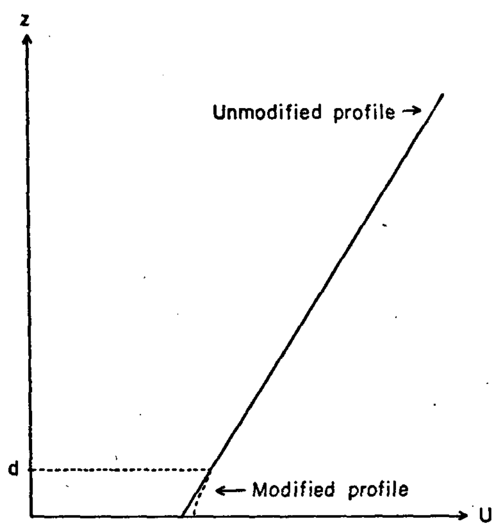

FIG. 2. $U(z)$ for the baroclinic instability problem. The modified profile differs from the unmodified profile only for $z<d$.

A more applicable, stereotypical profile is shown in Fig. 2; here we take U_z to be a positive constant. The unboundedness of U, while clearly unrealistic, leads to no special difficulties. The resulting problem has been considered at length in many studies (Charney, 1949; Burger, 1962; Geisler and Garcia, 1977). For such a choice of U, q_y is always positive. It turns out that all known unstable modes for this problem have critical levels where $U(z_c)=c_r$. For $z<z_c$, $U-c_r<0$. If we ignore, for the moment, the imaginary part of c (c_i), we see that vertical Rossby wave propagation is permitted only between z_c and $z_t>z_c$, where the quantity in parentheses in (16) changes sign [as it must for finite k^2 and constantly increasing $U(z)$]. If we consider the region $z>z_c$ detached from the region $z\leqslant z_c$, it is easily shown that a wave propagating upward from below z_t will be totally reflected at z_t—but *not* overreflected. Similarly, if we consider the region $z<z_t$ detached from the region $z\geqslant z_t$, we can show (using methods similar to those shown in the Appendix) that a wave propagating downward in the region $z>z_c$ will only be partially reflected (unless $z_c=0$ in which case one gets total reflection)—even with the boundary condition given by Eq. (19). Thus, the concept of overreflection does not, on the face of it, seem relevant to baroclinic instability. This turns out, however, not to be strictly so.

Charney and Stern (1962; see also Charney, 1973) have shown that the role of q_y in the barotropic problem [*viz*, Eq. (10)] is played in the present baroclinic problem by

$$\tilde{q}_y=q_y-\epsilon U_z(0)\delta(z). \tag{21}$$

This "effective" pseudo-potential vorticity gradient contains a delta function contribution due to shear at the ground (or equivalently, due to meridional temperature gradients at the ground). Clearly, if $U_z>0$, $\tilde{q}_y$ can be said to change sign in an infinitesimal neighborhood of the ground, thus satisfying Charney and Stern's extension of Rayleigh's inflection point theorem. More illuminatingly, we see that the present baroclinic problem is equivalent to a problem where the basic flow is identical to that shown in Fig. 2, but where $U_z=0$ at $z=0$. In this case

$$\tilde{q}_y=q_y,$$

but q_y now includes the delta function contribution due to the infinite curvature (U_{zz}) *required at* the ground [*viz*, Eq. (20)].[4] This feature has previously been noted by Bretherton (1966). The equivalent version of the problem is now (albeit in extreme form) again identical to the barotropic problem. The role of overreflection in this problem is most readily seen if we first consider a modified problem wherein the infinite curvature is spread over a finite neighborhood of the ground as in the modified profile shown in Fig. 2. We choose the neighborhood to be sufficiently small so that for positive U_z, the quantity U_{zz} is sufficiently positive so that

$$\left(\frac{\epsilon^{-1}\beta-U_{zz}}{U-c_r}-\frac{k^2}{\epsilon}\right)>0 \quad near \quad z=0 \tag{22}$$

[*viz*, Eq. (16)]. This neighborhood will now allow vertically propagating Rossby waves which, by the criteria developed in Section 2, will be overreflected from above.[5,6]

In the explicit calculations, which will be presented in a separate paper, we show [in a manner analogous to that presented in Lindzen and Rosenthal (1976)] that this overreflection does, in fact, lead to instabilities corresponding to all previously found baroclinically unstable modes—both Charney (1947) modes and Green (1960) modes.

In Lindzen and Rosenthal it was noted that for finite overreflection, growth rates increase as the distance between the boundary and the overreflecting surface decreases. One might therefore question whether, in the limit of the curvature going to infinity at the ground (i.e., d in Fig. 2, approaching zero), the above description remains appropriate. It is readily shown that in this limit, overreflection approaches 1 from above so that one obtains finite growth rates which correspond to the growth rates conventionally calculated for baroclinic instability. For purposes of this paper, however, we wish merely to note that the problems of barotropic and baroclinic instability are mathematically homomorphic, and that the necessary condition for baroclinic instability derived by Charney and Stern (1962) leads to wave overreflection as well.

[4] This interpretation suggests that quasigeostrophic theory does not, in fact, allow horizontal temperature gradients at rigid horizontal boundaries. In nature, of course, the atmosphere near the ground is turbulent and ageostrophic effects are important.

[5] Clearly, if U_z were negative, q_y would never change sign and there would be no overreflection.

[6] If one broadens this neighborhood sufficiently so that the transition from zero shear at ground to the interior shear is made by a profile for which

$$\epsilon^{-1}\beta-U_{zz}=0,$$

then the resulting profile will be baroclinically neutral. For realistic values of parameters, such a layer proves to be appreciably less than a scale height in depth—thus leaving much of the available potential energy unavailable. The profile described above is similar to one which develops in a numerical study of the finite-amplitude evolution of baroclinic instability by Simmons and Hoskins (1978).

5. Concluding remarks

In the preceding sections we have shown that for a number of shear instability problems the necessary conditions for instability are also the conditions for the propagation and overreflection of waves. To the extent that we can identify instabilities with overreflected waves this adds a measure of comprehensibility to the hitherto abstract conditions on such features as inflection points. Most significantly, we see that the inflection point per se is not the locus of wave-mean flow interaction; the critical surface is.

In addition, and more speculatively, the preceding results suggest that among hydrodynamic instabilities there may exist two distinct classes. For such problems as Bénard convective instability and Taylor inertial instability (e.g., Chandrasekhar, 1961) the instabilities are not associated with waves and the unstable modes clearly lead to mass redistribution. On the other hand, one can see, from this paper and others, in such problems as stably stratified shear instability, unstratified shear instability, barotropic instability and baroclinic instability, that instability appears to be associated with the self-excitation of waves rather than with the direct redistribution of mass. Such distinctions appear likely to be important for questions like the relationship between hydrodynamic instability and the onset of turbulence since turbulence is basically associated with the redistribution of mass. There exists some observational evidence (Klein *et al.*, 1967) which suggests that for instabilities of the second kind, the onset of turbulence occurs when these instabilities themselves become unstable to other disturbances. We would tentatively suggest that the other disturbances ought to be of the mass-redistributing type, and that wave-type instabilities, themselves, do not lead directly to turbulence.

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APPENDIX

Jump in Momentum Flux Across a Critical Level

Near the critical level $y=y_c$, where $U(y)-c=0$, one can expand

$$U(y)-c=U_y(y_c)(y-y_c)+\tfrac{1}{2}U_{yy}(Y_c)(y-y_c)^2+\ldots,$$
$$\beta-U_{yy}(y)=\hat{\beta}_c+[-U_{yyy}(y_c)](y-y_c)+\ldots.$$

Since there is a logarithmic singularity for Eq. (2) at $y=y_c$, the Frobenius solution takes the form

$$\psi(y)=Af(y-y_c)+Bg(y-y_c), \tag{A1}$$

where

$$f(x)=x+\sum_{n=2}^{\infty} a_n x^n,$$

$$g(x)=1+\sum_{n=2}^{\infty} b_n x^n-\frac{\hat{\beta}_c}{U_y(y_c)}\cdot \ln x\cdot f(x).$$

The recursion relations for a_n and b_n can be found by substituting (A1) into Eq. (2). The results of classical hydrodynamics (see Lin, 1945) show that when the correct branch is taken at the singularity $y-y_c=0$, there should be a $-\pi$ phase shift for the linear waves under consideration. Thus we have

$$\psi(y)=\begin{cases}\psi_+(y)=Af(|y-y_c|)+Bg(|y-y_c|), & y-y_c>0,\\ \psi_-(y)=\left[A-i(-\pi)\frac{\hat{\beta}_c}{U_y(y_c)}B\right]f(|y-y_c|)+Bg(|y-y_c|), & y-y_c<0.\end{cases}\quad \text{(A2)}$$

Using Eq. (3), the momentum fluxes are calculated to be

$$\begin{aligned}\overline{uv}_+&=-\tfrac{1}{2}k\,\mathrm{Im}[\psi_+^*\psi_{+y}]\\&=-\tfrac{1}{2}k\,\mathrm{Im}\left[AB^*\frac{d}{dy}fg+A^*Bf\frac{d}{dy}g\right]\\&=-\tfrac{1}{2}k\,\mathrm{Im}[AB^*],\end{aligned}\quad \text{(A3)}$$

where the identity

$$g\frac{df}{dy}-f\frac{ds}{dy}=1$$

has been used. Similarly,

$$\begin{aligned}\overline{uv}_-&=-\tfrac{1}{2}k\,\mathrm{Im}[\psi_-^*\psi_{-y}]\\&=-\tfrac{1}{2}k\,\mathrm{Im}\left\{\left[A+i\pi\frac{\hat{\beta}_c}{U_y(y_c)}B\right](B^*)\right\}\\&=-\tfrac{1}{2}k\,\mathrm{Im}[AB^*]-\tfrac{1}{2}k\pi\frac{\hat{\beta}_c}{U_y(y_c)}|B|^2.\end{aligned}\quad \text{(A4)}$$

Therefore the jump is found to be

$$\overline{uv}_+-\overline{uv}_-=\tfrac{1}{2}k\pi\frac{\hat{\beta}_c}{U_y(y_c)}|B|^2.\quad \text{(A5)}$$

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