Approximate pricing of European and Barrier claims in a local-stochastic volatility setting

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Problem statement

We are interested in computing the price of a barrier-style claim

$$V = (V_t)_{0 \le t \le T} \qquad \text{(option)}$$

written on an asset

$$S = (S_t)_{0 \le t \le T} \qquad \text{(asset)}$$

whose payoff at the maturity date T is given by

$$\mathbb{1}_{\{\tau > T\}} \widehat{\varphi}(S_T), \qquad \tau = \inf\{t \ge 0 : S_t \notin \widehat{I}\}. \qquad \text{(payoff)}$$

where $\widehat{I}$ is an interval in $\mathbb{R}$.

- The option becomes worthless if S leaves $\widehat{I}$ at any time $t \le T$.
- These types of options are known as *knock-out* options.

Problem statement

Examples:

- $\widehat{I} = (L, U)$ - double-barrier knock-out
- $\widehat{I} = (L, \infty)$ - single-barrier option with lower barrier
- $\widehat{I} = (-\infty, U)$ - single-barrier option with upper barrier
- $\widehat{I} = (-\infty, \infty)$ - European option

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We can price knock-in options by pricing European and knock-out options using knock-in knock-out parity

$$V_{\widehat{I}}^{(\text{knock-in})} + V_{\widehat{I}}^{(\text{knock-out})} = V^{(\text{European})},$$

where the payoff of a knock-in option is given by

$$\mathbb{1}_{\{\tau \leq T\}} \widehat{\varphi}(S_T).$$

Asset model

For an asset S, we consider models of in a general local-stochastic volatility setting

$$\begin{aligned} S_t &= \mathrm{e}^{X_t}, \\ \mathrm{d}X_t &= \mu(X_t, Y_t)\mathrm{d}t + \sigma(X_t, Y_t)\mathrm{d}W_t, \\ \mathrm{d}Y_t &= c(X_t, Y_t)\mathrm{d}t + g(X_t, Y_t)\mathrm{d}B_t, \\ \mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t, \end{aligned}$$

where W and B are correlated Brownian motions under the pricing probability measure $\mathbb{P}$.

Risk-neutral price

Let

- $r = 0$,
- $I = \log \widehat{I}$,
- $\varphi(x) = \widehat{\varphi}\left(\mathrm{e}^{x}\right) = \widehat{\varphi}(s)$.

To avoid arbitrage, all traded assets must be martingales under the pricing measure $\mathbb{P}$. The value V_t of the claim with the payoff

$$\mathbb{1}_{\{\tau > T\}}\varphi(X_T), \qquad \tau = \inf\{t \geq 0 : X_t \notin I\} \qquad \text{(payoff)}$$

at time $t \leq T$ is given by

$$V_t = \mathbb{1}_{\{\tau > t\}} u(t, X_t, Y_t),$$

where

$$u(t, x, y) := \mathbb{E}\left(\mathbb{1}_{\{\tau > T\}}\varphi(X_T) | X_t = x, Y_t = y, \tau > t\right).$$

Possible Approaches

How might one solve the pricing problem?

- Simulation
  - Ex: Monte Carlo
  - Limitation: Simulation gives you the price for one (X_0, Y_0) and parameter choice.
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  - Ex: Solve PDE using finite difference or finite element
  - Limitation: Numerical solvers suffer from the "curse of dimensionality."
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  - Limitation: Discretized solution
- Analytical techniques on the PDE
  - Ex: perturbation theory
  - Advantage: Fast evaluation at higher dimension
  - Advantage: Ease of implementation

Pricing PDE

The function u

$$u(t,x,y) = \mathbb{E}\left(\mathbb{1}_{\{\tau > T\}}\varphi(X_T) | X_t = x, Y_t = y, \tau > t\right),$$

is the unique classical solution of the Kolmogorov Backward equation

$$0 = (\partial_t + \mathcal{A})u, \qquad u(T,\cdot) = \varphi,$$

where $\mathcal{A}$, the generator of (X,Y), is given explicitly by

$$\mathcal{A} = \frac{1}{2}\sigma^2(x,y)\partial_x^2 + \rho\sigma(x,y)g(x,y)\partial_x\partial_y + \frac{1}{2}g^2(x,y)\partial_y^2 + \mu(x,y)\partial_x + c(x,y)\partial_y,$$

and the domain of $\mathcal{A}$ is given by

$$\text{dom}(\mathcal{A}) := \{g \in C^2 : \lim_{x \to \partial I} g(x,y) = 0\}.$$

Our approach

The full pricing PDE

$$\begin{aligned}
0 &= \partial_t u + \frac{1}{2}\sigma^2(x,y)\partial_x^2 u + \rho\sigma(x,y)g(x,y)\partial_x\partial_y u \\
&\quad + \frac{1}{2}g^2(x,y)\partial_y^2 u + \mu(x,y)\partial_x u + c(x,y)\partial_y u, \\
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\end{aligned}
$$

is not generally solvable in closed form.

If σ, g, μ, c were constant and $\rho = 0$, the pricing PDE would be

$$\partial_t u + \frac{1}{2}\sigma^2\partial_x^2 u + \frac{1}{2}g^2\partial_y^2 u + \mu\partial_x u + c\partial_y u = 0,$$

which is solvable. This suggests a perturbation expansion...

Perturbation framework

Let $f \in \{\frac{1}{2}\sigma^2, \sigma g, \frac{1}{2}g^2, \mu, c\}$ and $(\bar{x}, \bar{y}) \in I \times \mathbb{R}$. We introduce $\varepsilon \in [0,1]$ and define

$$f^{\varepsilon}(x,y) := f(\bar{x} + \varepsilon(x - \bar{x}), \bar{y} + \varepsilon(y - \bar{y})).$$

Note that $f^{\varepsilon}(x,y)|_{\varepsilon=1} = f(x,y)$ and $f^{\varepsilon}(x,y)|_{\varepsilon=0} = f(\bar{x},\bar{y})$.

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Taylor expanding f^ε about the point $\varepsilon = 0$ yields

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots ,$$

where

$$f_n(x, y) = \sum_{i=0}^{n} \frac{\partial_x^{n-i} \partial_y^i f(\bar{x}, \bar{y})}{i!(n-i)!} (x - \bar{x})^{n-i} (y - \bar{y})^i.$$

Perturbation framework

Recall

$$\mathcal{A} = \frac{1}{2}\sigma^2(x,y)\partial_x^2 + \rho\sigma(x,y)g(x,y)\partial_x\partial_y + \frac{1}{2}g^2(x,y)\partial_y^2 + \mu(x,y)\partial_x + c(x,y)\partial_y,$$

Replacing $f \in \{\frac{1}{2}\sigma^2, \sigma g, \frac{1}{2}g^2, \mu, c\}$ with f^ε in $\mathcal{A}$ and expanding yields

$$\mathcal{A}^{\varepsilon,\rho} = \sum_{n=0}^{\infty} \varepsilon^n \left(\mathcal{A}_{n,0} + \rho\mathcal{A}_{n,1}\right),$$

where

$$\mathcal{A}_{n,0} := (\tfrac{1}{2}\sigma^2)_n\partial_x^2 + (\tfrac{1}{2}g^2)_n\partial_y^2 + \mu_n\partial_x + c_n\partial_y$$
$$\mathcal{A}_{n,1} := (\sigma g)_n\partial_x\partial_y.$$

Perturbation framework

We try to solve

$$(\partial_t + \mathcal{A}^{\varepsilon,\rho})u^{\varepsilon,\rho} = 0, \qquad u^{\varepsilon,\rho}(T,\cdot,\cdot) = \varphi$$

by expanding $u^{\varepsilon,\rho}$ in powers of ε and ρ as follows

$$u^{\varepsilon,\rho} = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \varepsilon^{n-i}\rho^{i} u_{n-i,i}.$$

An approximation to the solution of the original pricing PDE

$$(\partial_t + \mathcal{A})\, u = 0, \qquad u(T,\cdot,\cdot) = \varphi$$

will be obtained by setting $\varepsilon = 1$ in $u^{\varepsilon,\rho}$.

Perturbation framework

We now have the parameterized set of PDEs

$$(\partial_t + \mathcal{A}^{\varepsilon,\rho})\, u^{\varepsilon,\rho} = 0, \qquad u^{\varepsilon,\rho}(T,\cdot,\cdot) = \varphi.$$

Inserting $\mathcal{A}^{\varepsilon,\rho}$ and $u^{\varepsilon,\rho}$ and collecting powers of ε and ρ gives

$$\begin{aligned} O(\varepsilon^0\rho^0):& \quad (\partial_t + \mathcal{A}_{0,0})\, u_{0,0} = 0, \quad u_{0,0}(T,\cdot,\cdot) = \varphi, \\ O(\varepsilon^n\rho^k):& \quad (\partial_t + \mathcal{A}_{0,0})\, u_{n,k} + F_{n,k} = 0, \quad u_{n,k}(T,\cdot,\cdot) = 0, \end{aligned}$$

where

$$F_{n,k} = \sum_{i=0}^{n}\sum_{j=0}^{k}(1-\delta_{i+j,0})\mathcal{A}_{i,j}u_{n-i,k-j}.$$

- $O(\varepsilon^0\rho^0)$ is a constant coefficient heat equation.
- $O(\varepsilon^n\rho^k)$ is a constant coefficient heat equation with a forcing term.

Nth order approximation

Definition

Let u be the unique classical solution of PDE problem (1).

$$(\partial_t + \mathcal{A})\, u = 0, \qquad u(T, \cdot, \cdot) = \varphi, \tag{1}$$

We define $\bar{u}_N^{\rho}$, the *Nth order approximation of u*, as

$$\bar{u}_N^{\rho}(t,x,y) := \sum_{i=0}^{N} \sum_{j=0}^{i} \varepsilon^j \rho^{i-j} u_{j,i-j}(t,x,y) \Big|_{(\bar{x},\bar{y},\varepsilon)=(x,y,1)},$$

where $u_{0,0}$ satisfies (2) and $u_{n,k}$ satisfies (3) for $(n,k) \neq (0,0)$.

$$(\partial_t + \mathcal{A}_{0,0})\, u_{0,0} = 0, \qquad u_{0,0}(T,\cdot,\cdot) = \varphi, \tag{2}$$

$$(\partial_t + \mathcal{A}_{0,0})\, u_{n,k} + F_{n,k} = 0, \qquad u_{n,k}(T,\cdot,\cdot) = 0. \tag{3}$$

Duhamel's principal

Duhamel's principle states that the the unique classical solution to

$$(\partial_t + \mathcal{A}_{0,0})u + F = 0, \qquad\qquad u(T,\cdot,\cdot) = h,$$

is given by

$$u(t,x,y) = \mathcal{P}_{0,0}(t,T)h(x,y) + \int_t^T \mathrm{d}s\, \mathcal{P}_{0,0}(t,s)F(s,x,y),$$

where we have introduced $\mathcal{P}_{0,0}$ the *semigroup* generated by $\mathcal{A}_{0,0}$, which is defined as follows

$$\mathcal{P}_{0,0}(t,s)h(x,y) = \int_I \mathrm{d}\xi \int_{\mathbb{R}} \mathrm{d}\eta\, \Gamma_{0,0}(t,x,y;s,\xi,\eta)h(\xi,\eta),$$

where $0 \le t \le s \le T$, and $\Gamma_{0,0}$ is the solution of

$$0 = (\partial_t + \mathcal{A}_{0,0})\Gamma_{0,0}(\cdot,\cdot,\cdot;T,\xi,\eta), \quad \Gamma_{0,0}(T,\cdot,\cdot;T,\xi,\eta) = \delta_{\xi,\eta}.$$

Formula for $u_{n,k}$

Proposition

The function $u_{0,0}$ is given by

$$u_{0,0}(t) = \mathcal{P}_{0,0}(t,T)\varphi,$$

and for $(n,k) \neq (0,0)$, we have

$$u_{n,k}(t) = \sum_{j=1}^{n+k} \sum_{I_{n,k,j}} \int_t^T \mathrm{d}s_1 \int_{s_1}^T \mathrm{d}s_2 \cdots \int_{s_{j-1}}^T \mathrm{d}s_j$$
$$\mathcal{P}_{0,0}(t,s_1)\mathcal{A}_{n_1,k_1} \cdots \mathcal{P}_{0,0}(s_{j-1},s_j)\mathcal{A}_{n_j,k_j}\mathcal{P}_{0,0}(s_j,T)\varphi,$$

with $I_{n,k,j}$ given by

$$I_{n,k,j} = \left\{ \begin{pmatrix} n_1, \cdots, n_j \\ k_1, \cdots, k_j \end{pmatrix} \in \mathbb{Z}_+^{2\times j} \;\middle|\; \begin{array}{l} n_1 + \cdots + n_j = n, \\ k_1 + \cdots + k_j = k, \\ 1 \leq n_i + k_i, \text{ for all } 1 \leq i \leq j \end{array} \right\}.$$

Asymptotic accuracy for European claims

Let $I = \mathbb{R}$ (European option), and let $h - 1$ be the number of Lipschitz continuous derivatives of φ. Then under certain regularity assumptions on the coefficients (μ, σ, g, c), the approximate solution satisfies the following:

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$$|(u - \bar{u}_0^{\rho})(t,x,y)| \leq C\,(T-t)^{\frac{h+1}{2}}, \quad 0 \leq t < T, \quad x \in I, y \in \mathbb{R}.$$

For $N \geq 1$, we have

$$|(u - \bar{u}_N^{\rho})(t,x,y)| \leq C\,((T-t)^{\frac{1}{2}} + |\rho|) \sum_{i=0}^{N} |\rho|^i (T-t)^{\frac{N-i+h}{2}}$$

$$0 \leq t < T, x \in I, y \in \mathbb{R}.$$

The positive constants C in depend only on N, φ (and σ, g, μ, c).

Numerical example: CEV model

Suppose that $S = \mathrm{e}^X$ has Constant Elasticity of Variance (Cox (1975)) dynamics i.e.

$$\begin{aligned} \mathrm{d}S_t &= \sigma S_t^\gamma \mathrm{d}W_t, \\ \mathrm{d}X_t &= -\frac{1}{2}\sigma^2 \mathrm{e}^{2X_t(\gamma-1)}\,\mathrm{d}t + \sigma \mathrm{e}^{X_t(\gamma-1)}\,\mathrm{d}W_t. \end{aligned}$$

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We consider double-barrier knock-out calls and puts with the following parameters fixed

X_0	K	T	σ	γ
0.62	0.62	0.083	0.32	0.019

CEV double-barrier call

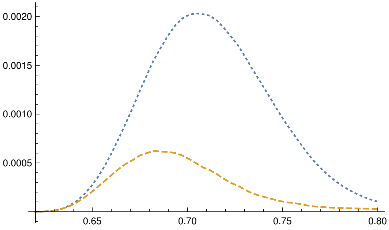

Figure 1: For the CEV with $L = 0$, we plot $u - \bar{u}_0$ (blue dotted) and $u - \bar{u}_2$ (orange dashed) as a function of the upper barrier U for a call option.

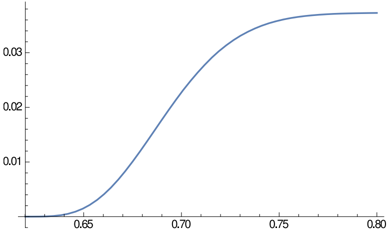

Figure 2: For the CEV model with $L = 0$, we plot u as a function of the upper barrier U for a call option.

CEV double-barrier put

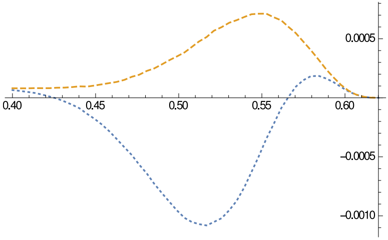

Figure 3: For the CEV model with $U = 1$, we plot $u - \bar{u}_0$ (blue dotted) and $u - \bar{u}_2$ (orange dashed) as a function of the lower barrier L for a put option.

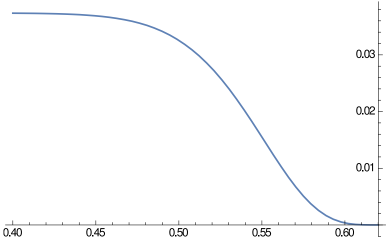

Figure 4: For the CEV model with $U = 1$, we plot u as a function of the lower barrier L for a put option.

Numerical example: Heston model

Suppose that $S = \mathrm{e}^X$ has Heston (Heston (1993)) dynamics i.e.

$$\begin{aligned}
\mathrm{d}S_t &= \sqrt{Y_t} S_t \mathrm{d}W_t, \\
\mathrm{d}X_t &= -\frac{1}{2} Y_t \,\mathrm{d}t + \sqrt{Y_t}\,\mathrm{d}W_t, \\
\mathrm{d}Y_t &= \kappa(\theta - Y_t)\,\mathrm{d}t + \delta\sqrt{Y_t}\,\mathrm{d}B_t, \\
\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t
\end{aligned}$$

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\mathrm{d}\langle W, B\rangle_t &= \rho\,\mathrm{d}t
\end{aligned}$$

We specify a model

X_0	Y_0	K	T	ρ	κ	θ	δ
0.62	0.04	.62	0.083	-0.4	1.15	0.04	0.2

Heston double-barrier call

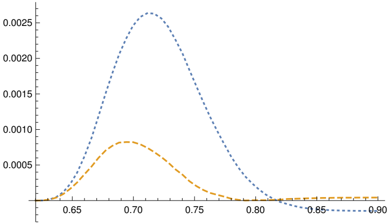

Figure 5: For the Heston model, we plot $u - \bar{u}_0^\rho$ (blue dotted) and $u - \bar{u}_2^\rho$ (orange dotted-dashed) as a function of the upper barrier U for a call option.

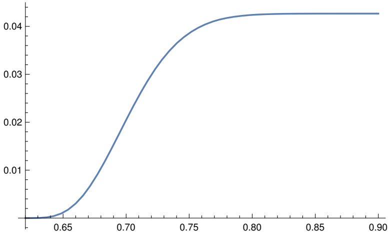

Figure 6: For the Heston model, we plot u as a function of the upper barrier U for a call option.

Heston double-barrier put

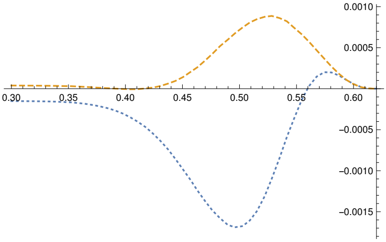

Figure 7: For the Heston model, we plot $u - \bar{u}_0^\rho$ (blue dotted) and $u - \bar{u}_2^\rho$ (orange dotted-dashed) as a function of the lower barrier L for a put option.

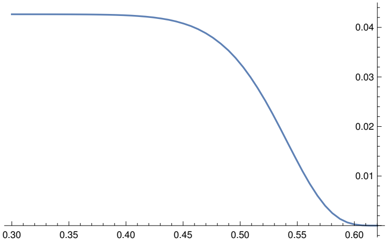

Figure 8: For the Heston model, we plot u as a function of the lower barrier L for a put option.

Conclusion

- Limitations of numerical methods and simulations
- Pricing options exactly under general dynamics is impossible, so we turn to asymptotics
- Constant coefficient PDE theory is used to solve the asymptotic problem
- Rigorous accuracy results for European options
- Numerical accuracy demonstrations for barrier options

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