Financial Contagion with Multiple Illiquid Assets

Zach Feinstein

Electrical and Systems Engineering,
Washington University in St. Louis

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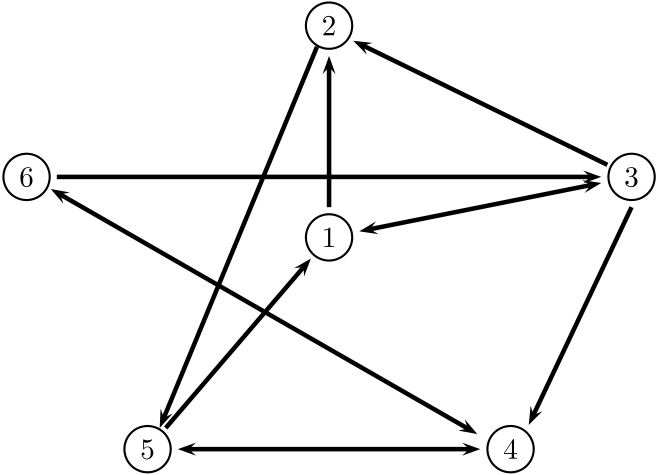
The Eisenberg & Noe Local Model

Network Model with Local Interactions Only:
Eisenberg & Noe (2001)

- n financial firms
- Nominal liability matrix: $(L_{ij})_{i,j=0,1,2,\ldots,n}$
- Total liabilities: $\bar{p}_i = \sum_{j=0}^{n} L_{ij}$
- Relative liabilities:

$$a_{ij} = \begin{cases} \frac{L_{ij}}{\bar{p}_i} & \text{if } \bar{p}_i > 0, \\ 0 & \text{if } \bar{p}_i = 0. \end{cases}$$


2
6
3
1
5
4

Network Model with Local Interactions Only:

- Liquid endowment: $x \in \mathbb{R}^n_+$
- Obligations fulfilled via transfers of the liquid asset.
- Equilibrium computed as fixed point: $p \in \mathbb{R}^n_+$:

$$p_i = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{n} a_{ji} p_j \right), \quad i = 1, 2, ..., n$$

- Existence: Tarski's fixed point theorem: maximal and minimal fixed points $p^- \leq p^+$.

Network Model with Local Interactions Only:

Uniqueness

- $S \subseteq \{1, 2, ..., n\}$ is a surplus set if $L_{ij} = 0$ and $\sum_{i \in S} x_i > 0$ for all $(i, j) \in S \times S^c$
- $o(i) = \{j \in \{1, 2, ..., n\} \mid \exists$ directed path from i to $j\}$
- If $o(i)$ is a surplus set for every bank i then there exists a unique payment vector $p := p^+ = p^-$ (Banach fixed point theorem)

Multilayered Financial Networks

Multilayered Network Model: Montagna & Kok (2013), Poledna, Molina-Borboa, Martinez-Jaramillo, Leij & Thurner (2015), Battiston, Caldarelli & D'Errico (2016), Feinstein (2017)

- Endowment: $x \in \mathbb{R}_+^{n \times m}$
- Nominal liabilities: $L \in \mathbb{R}_+^{n \times n \times m}$
- Obligations must be fulfilled via transfers of the physical assets.
- Assets may be transferred to cover obligations or maximize utility, but these are subject to price impact described by the inverse demand function.
- Inverse demand function: $F : \mathbb{R}^m \to \mathbb{R}_+^m$ maps units of illiquid assets being sold (positive input) or bought (negative input) into corresponding prices in some (possibly fictitious) numéraire.

Multilayered Network Model:

- Assume inverse demand function is continuous and nonincreasing with codomain $[\underline{q}, \bar{q}] \subseteq \mathbb{R}^m_{++}$.
- Assume the network model in asset k follows the Eisenberg & Noe (2001) model:
  - Total liabilities: $\bar{p}_i^k := \sum_{j=1}^n L_{ij}^k$
  - Relative liabilities: $a_{ij}^k := \begin{cases} \frac{L_{ij}^k}{\bar{p}_i^k} & \text{if } \bar{p}_i^k > 0 \\ 0 & \text{if } \bar{p}_i^k = 0 \end{cases}$
- Firm portfolio holdings: $y \in \mathbb{R}_+^{n \times m}$
- Initial portfolio wealth: for firm i in asset k is

$$x_i^k + \sum_{j=1}^n a_{ji}^k [\bar{p}_j^k \wedge y_j^k].$$

Multilayered Network Model:

- Payments must be made so that positive equity only accumulates after all obligations are paid

$$p_i \in P_i(y^*, q^*) \subseteq \operatorname*{Eff}_{p_i \in [0, \bar{p}_i]} \left\{ p_i \mid \sum_{k=1}^{m} q_k^* p_i^k \leq \sum_{k=1}^{m} q_k^* \left(x_i^k + \sum_{j=1}^{n} a_{ji}^k [\bar{p}_j^k \wedge y_j^{*k}] \right) \right\}.$$

- Holdings may involve futher transfers to maximize utility

$$y_i \in Y_i(y^*, q^*) = \operatorname*{arg\,max}_{e_i \in \mathbb{R}_+^m} \left\{ u_i(e_i; y_{-i}^*, q^*) \mid e_i \in H_i \right\}$$

$$H_i = \left\{ e_i \;\middle|\; \begin{array}{rcl} \bar{p}_i \wedge e_i & \in & P_i(y^*, q^*), \\ \sum_{k=1}^m q_k^* e_i & = & \sum_{k=1}^m q_k^* \left(x_i^k + \sum_{j=1}^n a_{ji}^k \left[\bar{p}_j^k \wedge y_j^{*k} \right] \right) \end{array} \right.$$

Multilayered Network Model:

- Prices update based on asset transfers

$$q(y^*, q^*) = F\left(\sum_{i=1}^n \left(x_i^k + \sum_{j=1}^n a_{ji}^k [\bar{p}_j^k \wedge y_j^{*k}] - y_i^{*k}\right)_{k=1,\dots,m}\right)$$
$$= F\left(\sum_{i=1}^n (x_i + [\bar{p}_i \wedge y_i^*] - y_i^*)\right)$$

- Equilibrium computed as fixed point: $(y, q) \in \mathbb{R}_+^{n \times m} \times [\underline{q}, \bar{q}]$

$$(y, q) \in \left(\prod_{i=1}^n Y_i(y, q)\right) \times \{q(y, q)\}$$

Multilayered Network Model: Existence

- Let P_i be given as the maximizer of a continuous regulatory function h_i which is strictly increasing and strictly quasi-concave in the first component

$$P_i(y^*, q^*) = \arg\max_{p_i \in [0, \bar{p}_i]} \left\{ h_i(p_i; y^*, q^*) \mid \sum_{k=1}^m q_k^* p_i^k \leq \sum_{k=1}^m q_k^* \left(x_i^k + \sum_{j=1}^n a_{ji}^k [\bar{p}_j^k \wedge y_j^{*k}] \right.\right.$$

- Let u_i be jointly continuous and quasi-concave in the first component
- There exists an equilibrium solution (via Berge maximum theorem and Kakutani fixed point theorem)

$$(y, q) \in \left(\prod_{i=1}^n Y_i(y, q) \right) \times \{q(y, q)\}$$

Multilayered Network Model: Existence

- Assume h_i and u_i satisfy a **dynamic programming principle**

$$h_i(p; y, x, \bar{p}) = h_i(p - P_i(y', q); y, x - P_i(y', q), \bar{p} - P_i(y', q))$$
$$u_i(e; y, x, \bar{p}) = u_i(e - Y_i(y', q); y, x - Y_i(y', q), (\bar{p} - Y_i(y', q))^+)$$

- For every q:
  - There exists a greatest and least equilibrium holdings $y^{\uparrow}(q) \geq y^{\downarrow}(q)$
  - Positive equity of all firms is equal for every fixed point $(y_i^{\uparrow}(q) - \bar{p}_i)^+ = (y_i^{\downarrow}(q) - \bar{p}_i)^+$
- If every bank owes positive amount to sink node (0) in every asset, then $y^{\uparrow}(q) = y^{\downarrow}(q)$ for every q.
- This unique equilibrium $y : [\underline{q}, \bar{q}] \to \mathbb{R}_+^{n \times m}$ is continuous.

Multilayered Network Model: Case Study A

- $m = 2$ assets; $F_1 \equiv 1$

$$F_2(z) = \begin{cases} f(z_2) & \text{if } z_2 \geq 0 \\ \frac{1}{f(\alpha^{-1}(-z_2))} & \text{if } z_2 < 0 \end{cases}$$

$$f(z) = \frac{3\tan^{-1}(-z) + 2\pi}{2\pi}; \quad \alpha(z) = z f(z)$$

- $n = 20$ firms and a society node
- 25% of connection of size 1 between firms in each asset independently
- All firms owe 1 in each asset to the society node
- Initial endowments uniformly chosen between 0 and 20 and split evenly between the two assets

Case Study A

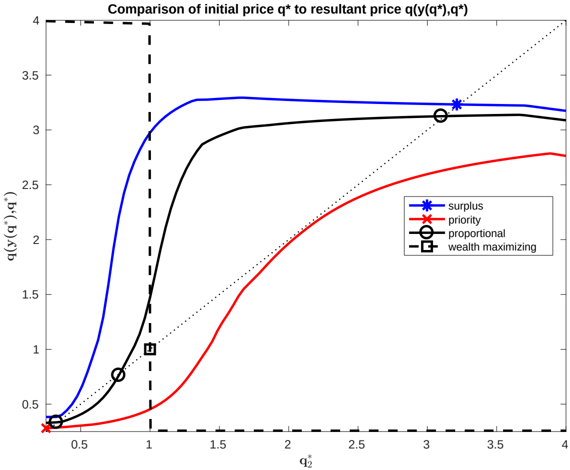


Figure: A comparison of different regulatory and utility schemes with two assets.

Multilayered Network Model: Case Study B

- $m = 3$ assets; $F_1 \equiv 1$

$$F_k(z) = \frac{\tan^{-1}(-1.5z_k) + \pi}{\tan^{-1}(-1.5z_1) + \pi}$$

- $n = 10$ firms and a society node
- 50% of connection of size 1 between firms in each asset independently
- All firms owe 1 in each asset to the society node
- Initial endowments of 5 split between the three assets (uniform between $\frac{10}{9}$ and $\frac{20}{9}$ in first asset, remainder split evenly in 2nd and 3rd)

Case Study B

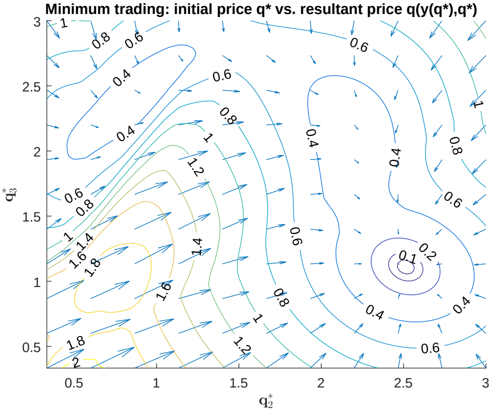


Figure: Proportional regulation and minimal trading utility with three assets.

Case Study B

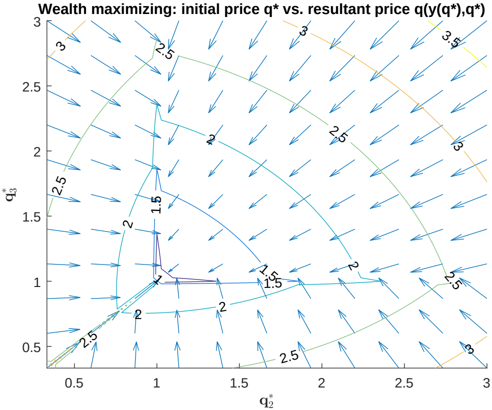


Figure: Proportional regulation and wealth maximizing utility with three assets.

Multilayered Network Model: Grexit Study

- $m = 2$ assets; $F_1 \equiv 1$

$$F_2(z) = \begin{cases} f(z_2) & \text{if } z_2 \geq 0 \\ \frac{1}{f(\alpha^{-1}(-z_2))} & \text{if } z_2 < 0 \end{cases}$$

$$f(z) = \frac{4\tan^{-1}(-10^{-4}z) + 3\pi}{3\pi}; \quad \alpha(z) = zf(z)$$

- $n = 87$ firms and a society node
- Calibrated to EBA data with Gandy & Veraart (2016)
- Under Eisenberg & Noe (2001): No failures

Multilayered Network Model: Grexit Study

- Initial endowments:

$$\begin{array}{lll} x_i^1 := x_i^{EN} - GE_i, & x_i^2 := GE_i & \forall i \in N \backslash G \\ x_i^1 := 0, & x_i^2 := x_i^{EN} & \forall i \in G \\ L_{ij}^1 := L_{ij}^{EN}, & L_{ij}^2 := 0 & \forall i \in N \backslash G \; \forall j \in N \cup \{0\} \\ L_{ij}^1 := L_{ij}^{EN}, & L_{ij}^2 := 0 & \forall i \in N \; \forall j \in N \backslash G \\ L_{ij}^1 := 0, & L_{ij}^2 := L_{ij}^{EN} & \forall i \in G \; \forall j \in G \cup \{0\}. \end{array}$$

Grexit Study

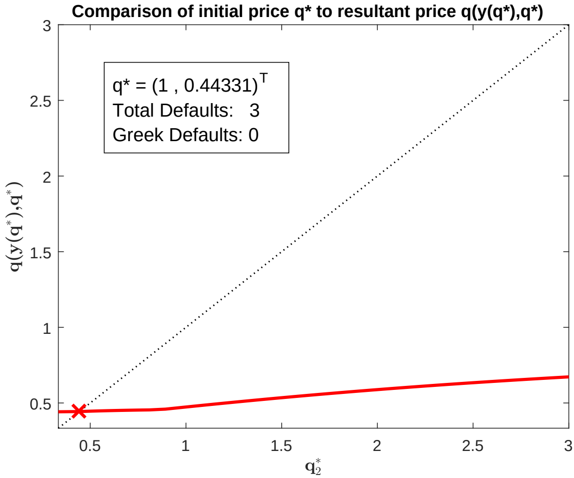


Figure: Proportional regulation and minimal trading utility with two assets.

Multilayered Network Model: Time Dynamics

Capponi & Chen (2015), Ferrara, Langfield, Liu & Ota (2016), Kusnetsov & Veraart (2016)

- Consider single asset with $T = m - 1$ time steps
- Assets can be traded through time with inverse demand function $F_1(z, q^*) = 1$ and

$$F_k(z, q^*) = F_{k-1}\left[1 \wedge f_k\left(z_k - \frac{1}{q_k^*}\left[\sum_{l=k+1}^{m} q_l^* z_l\right]^-\right)\right]$$

- Prioritize payments prior to times before default, proportional payments after default
- Maximize wealth at the final time period for solvent firms
- Default time $\mu(y^*, q^*) = \min\{k - 1 \mid y_i^{*k} < \bar{p}_i^k\}$

Multilayered Network Model: Time Dynamics: Existence

- If $\bar{\mu}$ is a continuous approximation of the default times μ then there exists an equilibrium solution:

$$(y^*, q^*, \mu^*) \in \left(\prod_{i=1}^n Y_i(y^*, q^*, \mu^*)\right) \times \left\{F\left(\sum_{i=1}^n (x_i + [\bar{p}_i \wedge y_i^*] - y_i^*), q^*\right)\right\} \times \{\bar{\mu}(y^*, q^*)\}$$

- For fixed (q^*, μ^*) then obtain greatest and least clearing holdings $y^\uparrow(q^*, \mu^*) \geq y^\downarrow(q^*, \mu^*)$ with unique positive equities
- If every bank owes positive amount to sink node (0) at every time, then $y^\uparrow(q^*, \mu^*) = y^\downarrow(q^*, \mu^*)$ for every (q^*, μ^*).
- This unique equilibrium $y : [\underline{q}, 1] \times [0, m] \to \mathbb{R}_+^{n \times m}$ is continuous.

Thank You!

- Eisenberg, Noe (2001): Systemic Risk in Financial Systems
- Feinstein (2017): Obligations with physical delivery in a multi-layered financial network