Mean Field Games with Singular Controls, and Applications

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Outline

1. ABCs of MFGs
2. MFGs with Singular Controls
   - MFGs for systemic risk
   - MFGs for partially irreversible problems
   - Main results
3. Conclusion

Mean Field Games (MFGs)

- Stochastic strategic decision games with very large population of small interacting individuals
- Originated from physics on weakly interacting particles
- Theoretical works pioneered by Lasry and Lions (2007) and Huang, Malhamé and Caines (2006)
- About small interacting individuals, with each player choosing optimal strategy in view of the macroscopic information (mean field)

Key idea of MFGs

- Take an N-player game
- When N is large, consider the "aggregated" version of the N-player game
- SLLN kicks in as $N \to \infty$, the aggregated version, MFG, becomes an "approximation" of the N-player game

N-player game

$$\inf_{\alpha^i \in \mathcal{A}} E\{\int_0^T f^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt\}$$

$$subject \quad to \quad dX_t^i = b^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

$$\text{and} \quad X_0^i = x^i$$

- X_t^i is the state of player i at time t
- α_t^i is the action/control of player i at time t, in an appropriate control set $\mathcal{A}$
- f^i is the running cost for player i
- g^i is the terminal cost for player i
- b^i is the drift term for player i
- σ is a volatility term for player i
- W_t^i are i.i.d. standard Brownian motions

From N-player game to MFG

Consider

Aggregation

$$\inf_{\alpha^i \in \mathcal{A}} E\{\int_0^T \frac{1}{N}\sum_{i=1}^N f^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt\}$$

$$s.t. \quad dX_t^i = \frac{1}{N}\sum_{i=1}^N b^i(t, X_t^1, \cdots, X_t^N, \alpha_t^i)dt + \sigma dW_t^i$$

$$\text{and} \quad X_0^i = x^i$$

As $N \to \infty$, consider the mean information μ_t as an unknown external signal, instead of $X_t^1, \cdots, X_t^N$

MFG

$$\inf_{\alpha \in \mathcal{A}} E[\int_0^T f(t, X_t^i, \mu_t, \alpha_t) dt]$$

$$\text{such that} \quad dX_t^i = b(t, X_t^i, \mu_t, \alpha_t)dt + \sigma dW_t^i \quad \text{and} \quad X_0^i = x^i$$

Assumptions

Players are indistinguisheable: they are rational, identical, and interchangeable

Main results for general MFGs

Under proper technical conditions,

Theorem

The MFG admits a unique optimal control.

Theorem

The value function of MFG is an ϵ-Nash equilibrium to the N-player game, with $\epsilon = O(\frac{1}{\sqrt{N}})$.

PDE/control approach of MFG

- (i) Fix a deterministic function $t \in [0, T] \rightarrow \mu_t \in \mathcal{P}(\mathbb{R}^d)$
- (ii) Solve the stochastic control problem

$$\inf_{\alpha \in \mathcal{A}} \int_0^T f(t, X_t, \mu_t, \alpha_t) dt$$
$$s.t. \quad dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma dW_t \quad \text{and} \quad X_0 = x$$

- (iii) Update the function $t \in [0, T] \rightarrow \mu'_t \in \mathcal{P}(\mathbb{R}^d)$ so that $\mathcal{P}_{X_t} = \mu'_t$
- (iv) Repeat (ii) and (iii). If there exists a fixed point solution μ_t and α_t, then it is a solution for this model.

Three main approaches

- PDE/control approach:
  backward HJB equation + forward Kolmogorov equation
  Lions and Lasry (2007), Huang, Malhame and Caines (2006), Lions, Lasry and Guant (2009)
- Probabilistic approach: FBSDEs
  Buckdahn, Li and Peng (2009), Carmona and Delarue (2013)
- Stochastic McKean-Vlasov and DPP
  Pham and Wei (2016)

Growing literatures on MFGs (partial list)

- MFGs with common noise
  Sun (2006), Carmona, Fouque, and Sun (2013), Garnier, Papanicolaou and Yang (2012), Carmona, Delarue and Lacker (2016), Nutz (2016),
- MFGs with partial observations
  Buckdahn, Li, Ma (2015), Buckdahn, Ma, Zhang (2016)
- MFG for HFT
  Jaimungal and Nourian (2015), Lachapelle, Lasry, Lehalle, and Lions (2016)
- MFG for queuing system
  Manjrekar, Ramaswamy, and Shakkottai (2014), Wiecek, Altman, and Ghosh (2015), Bayraktar, Budhiraja, and Cohen (2016)
- MFG for energy
  Chan and Sircar (2016)

Why singular controls

- Natural from modeling perspective, controls are not necessarily (absolutely) continuous
- Explicit solutions for MFGs are important justification for MFGs, especially for application purpose
- Singular controls have distinct bang-bang type characteristics, could go beyond the LQ framework for regular control
- Fully nonlinear PDEs with additional gradient constraints can be both challenging and useful

Problem setup

$$v^i(s,x^i) = \inf_{\xi_\cdot^{i+},\xi_\cdot^{i-}\in\mathcal{U}} E\left[\int_s^T \left(f(x_t^i,\mu_t)dt + g_1(x_t^i)d\xi_t^{i+} + g_2(x_t^i)d\xi_t^{i-}\right)\right],$$

subject to

$$dx_t^i = b(x_t^i,\mu_t)dt + d\xi_t^{i+} - d\xi_t^{i-} + \sigma dW_t^i, \qquad x_s^i = x^i$$

- (ξ_t^{i+},ξ_t^{i-}), non-decreasing càdlàg processes of finite variaiton
- f, g_1, g_2 satisfies appropriate technical conditions
- $\mathcal{U}$ appropriate admissible control set
- $\{\mu_t\}$ the mean information process

Model by Carmona, Fouque and Sun (2013)

Let x_t^i be the log-monetary reserve for bank i with $i = 1, 2, \ldots, N$

$$\begin{aligned} dx_t^i &= \frac{a}{N}\sum_{j=1}^{N}(x_t^j - x_t^i)dt + \xi_t^i dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i) \\ &= a(m_t - x_t^i)dt + \xi_t^i dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i), \quad x_s^i = x^i \end{aligned}$$

Model by Carmona, Fouque and Sun (2013)

The objective of each bank i is to solve

$$v^i(s, x^i, m) = \inf_{\xi^i_\cdot \in \mathcal{A}} E_{s,x^i,m}\Big[\int_s^T \Big(\frac{1}{2}(\xi^i_t)^2 - q\xi^i_t(m_t - x^i_t) + \frac{\epsilon}{2}(m_t - x^i_t)^2\Big)dt + \frac{c}{2}(m_T - x^i_T)^2\Big]$$

subject to the dynamics of x^i_t

Solution by Carmona, Fouque and Sun (2013)

This MFG is shown to have a unique optimal control ξ_t^{i*}, with its mean information process m_t^* and value function v^i given by

$$dm_t^* = \rho\sigma dW_t^0,$$
$$\xi_t^{i*}(x^i, m) = q(m - x^i) - \partial_x v^i,$$
$$v^i(t, x^i, m) = \frac{F_t^1}{2}(m - x^i)^2 + F_t^2,$$

for some deterministic functions F_t^1 and F_t^2.

Model with singular control formulation

Add lending and borrowing rate constraint $\xi_t \in [\theta, -\theta]$

$$
\begin{aligned}
dx_t^i &= a(m_t - x_t^i)dt + d\xi_t^i + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i), \\
&= \left[a(m_t - x_t^i) + \dot{\xi}_t^i\right] dt + \sigma(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i), \quad x_s^i \quad = x^i
\end{aligned}
$$

Model with singular control formulation

The MFG is to solve

$$v^i(s, x^i, m) = \inf_{\dot{\xi}^i} E_{s,x^i,m}[\int_s^T (f(\dot{\xi}_t^i) + \frac{\epsilon}{2}(m_t - x_t^i)^2)dt + \frac{c}{2}(m_T - x_T^i)^2],$$

subject to the dynamics of x_t^i, with

- ξ_t being $\mathcal{F}_t$-progressively measurable, of finite variation, $\dot{\xi}_t \in [-\theta, \theta]$, $\xi_0 = 0$
- $f(\cdot)$ symmetric and convex

- Step 1: assuming $\rho = 0$, then the associated HJB for the value function with a fixed control is

$$\partial_t v^i + \frac{\epsilon}{2}(m-x)^2 + a(m-x)\partial_x v^i + \frac{1}{2}\sigma^2 \partial_{xx} v^i \\ + \theta \min\{0, r + \partial_x v^i, r - \partial_x v^i\} = 0,$$

  with the terminal condition $v^i(T, x, m) = \frac{c}{2}(m-x)^2$.
- Step 2: Utilize the symmetry of the problem structure and derive the explicit solution for $\rho \neq 0$

Solution to singular control formulation

The optimal control is of bang-bang type

$$\dot{\xi}_t^{i*}(x,m) = \begin{cases} \theta, & \text{if} \quad x \le x_1 = m - h, \\ 0, & \text{if} \quad x_2 < x < x_1, \\ -\theta, & \text{if} \quad x_2 = m + h \le x \end{cases}$$

$$dm_t^* = \rho\sigma dW_t^0,$$

Comparison between regular and singular

- Solution structure are consistent although singular control is not Lipschitz continuous
- Explicit solutions are of similar structure under the singular control framework, as long as the cost functional is convex and symmetric
- Instead of dealing with SSDE, SPDE, using the PDE/control approach via conditioning on W_t^0

Irreversible problem with single player (*G. Pham (2005)*)

$$\sup_{L_t, M_t} E\left[\int_0^\infty e^{-rt}[\Pi(K_t)dt - pd\xi_t^+ + (1-\lambda)pd\xi_t^-]\right]$$

subject to

$$dK_t = K_t(\delta dt + \sigma dW_t) + d\xi_t^+ - d\xi_t^-, K_0 = k$$

Model setup

- K_t the capacity of the company
- ξ_t^+ and ξ_t^- nondecreasing of finite variation, $\mathcal{F}_t^k$ progressively measurable, cádlág processes, with $\xi_0^+ = \xi_0^- = 0$
- Π Lipschitz continuous, nondecreasing, bounded and concave over k and satisfies $\lim_{k \downarrow 0} \frac{\Pi}{k} = \infty, \sup_{k>0}[\Pi - kz] < \infty$. For instance, $\Pi = K_t^\alpha$
- $\delta, \sigma, r, p, \lambda \in (0, 1)$ nonnegative constants

Explicit solution

If $\Pi(K_t) = K_t^\alpha$ with $\alpha \in (0, 1)$, the optimal strategy is characterized by (k_b, k_s), so that

- neither increasing nor reducing capacity when it is in the region (k_b, k_s);
- increasing capital when it is below than k_b in order to reach the threshold k_b; and
- reducing capital when it is above k_s in order to attain the level k_s.

The region $(0, k_b)$ is called the expansion region, and (k_s, ∞) the contraction region.

MFGs

$$\sup_{\xi^{i+},\xi^{i-}} E\left[\int_s^T e^{-rt}[\Pi(K_t^i, \mu_t)dt - pd\xi_t^{i+} + (1-\lambda)pd\xi_t^{i-}]\right]$$

subject to

$$dK_t^i = bK_t^i dt + \sigma K_t^i dW_t^i + d\xi_t^{i+} - d\xi_t^{i-}, \quad K_{s-}^i = k$$

Model setup

- $\Pi(k, \mu) = \mu k^{\alpha}$
- ξ_t^{i+}, ξ_t^{i-} are $\mathcal{F}_t$- progressively measurable, cádlág, of bounded velocity
- $\mu_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} K_t^i$
- $p > 0, r > 0, \lambda \in (0, 1)$ are constants.

Variational inequality

Theorem

The value function v is nondecreasing, concave, and continuous on $(0, \infty)$.

$$
\begin{aligned}
& \mathcal{N}\mathcal{J} = (k_b, k_s) \\
On \quad & (k_b, k_s), \\
& rv - \Pi - \mathcal{L}v = 0 \quad \text{and} \quad (1-\lambda)p < v' < p \\
On \quad & [k_s, \infty), \\
& rv - \Pi - \mathcal{L}v + \theta(v' - p(1-\lambda)) = 0 \quad \text{and} \quad v' \leq (1-\lambda)p \\
On \quad & (0, k_b], \\
& rv - \Pi - \mathcal{L}v + \theta(p - v') = 0 \quad \text{and} \quad v' \geq p
\end{aligned}
$$

Explicit solutions

- Kolmogorov forward equation translates into solving a piece-wise linear weakly reflected diffusion
- Value function and optimal control, and the fixed point μ^* explicitly solved
- For the fixed point μ^* $k_b^* = {\mu^*}^{\frac{1}{1-\alpha}} * C_1$ where C_1 is independent of μ^*. Similarly, $k_s^* = {\mu^*}^{\frac{1}{1-\alpha}} * C_2$ where C_2 is independent of μ^*
- The region of $(0, k_b^*)$ is of expansion and the region of (k_s^*, ∞) is of contraction

Comparison with and without MFGs

- In the case of μ^* sufficiently large (Good game)
  - $k_b < k_b^*$: everyone works harder
  - $k_s^* - k_b^* > k_s - k_b$: everyone benefit from other people's hard work
- In the case of μ^* sufficiently small (Bad game)
  - $k_b > k_b^*$: everyone works less hard
  - $k_s^* - k_b^* < k_s - k_b$: everyone gets hurt in the game

Main results for MFGs

Under proper technical conditions,

Theorem

The MFG of singular control with bounded velocity admits a unique optimal control.

Theorem

The value function of MFG is an ϵ-Nash equilibrium to the N-player game, with $\epsilon = O(\frac{1}{\sqrt{N}})$.

- MFG with singular control allows more model flexibility
- MFG with singular control is mathematically interesting and promising
  Zhang (2012), Hu, Oksendal and Sulem (2014), Fu and Horst (2016), G. Lee (2016)

Thank You!