Optimal Portfolio under Fractional Stochastic Environment

Ruimeng Hu
Joint work with Jean-Pierre Fouque

Department of Statistics and Applied Probability
University of California, Santa Barbara

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Portfolio Optimization: Merton's Problem

An investor manages her portfolio by investing on a riskless asset B_t and one risky asset S_t (single asset for simplicity)

$$\begin{cases} \mathrm{d}B_t = rB_t\,\mathrm{d}t \\ \mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t \end{cases}$$

- π_t – amount of wealth invested in the risky asset at time t
- X_t^π – the wealth process associated to π

$$\mathrm{d}X_t^\pi = (rX_t^\pi + \pi_t(\mu - r))\,\mathrm{d}t + \pi_t\sigma\,\mathrm{d}W_t, \quad X_0^\pi = x$$

Objective:

$$M(t,x;\lambda) := \sup_{\pi\in\mathcal{A}(x,t)} \mathbb{E}\left[U(X_T^\pi)|X_t^\pi = x\right]$$

where $\mathcal{A}(x)$ contains all admissible π and $U(x)$ is a utility function on $\mathbb{R}^+$

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Stochastic Volatility

- In Merton's work, μ and σ are constant, complete market
- Empirical studies reveals that σ exhibits "random" variation
- Implied volatility skew or smile
- Stochastic volatility model: $\mu(Y_t), \sigma(Y_t)$ $\rightarrow$ incomplete market
- Rough Fractional Stochastic volatility:
  - Gatheral, Jaisson and Rosenbaum '14
  - Jaisson, Rosenbaum '16
  - Omar, Masaaki, Rosenbaum '16

We work with the following slowly varying fractional stochastic factor[1]

$$Z_t^{\delta,H} := \delta^H \int_{-\infty}^t e^{-\delta a(t-s)} \, \mathrm{d}W_s^{(H)}, \quad H \in (0,1)$$

[1] Garnier Solna '15: for **linear** problem of option pricing

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Fractional BM and Fractional OU

A fractional Brownian motion $W_t^{(H)}$, $H \in (0,1)$

- a continuous Gaussian process
- zero mean
- $\mathbb{E}\left[W_t^{(H)} W_s^{(H)}\right] = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$
- $H < 1/2$: short-range correlation; $H > 1/2$: long-range correlation

Consider the Langevin equation driven by fractional Brownian motion

$$\mathrm{d}Z_t^H = -aZ_t^H \,\mathrm{d}t + \mathrm{d}W_t^{(H)}$$

- stationary solution $Z_t^H = \int_{-\infty}^t e^{-a(t-s)} \,\mathrm{d}W_s^{(H)} = \int_{-\infty}^t \mathcal{K}(t-s)\,\mathrm{d}W_s^Z$
- correlated with risky asset $\mathrm{d}\left\langle W, W^Z \right\rangle_t = \rho \,\mathrm{d}t$
- Gaussian process with zero mean and constant variance

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Our Study Gives....

Under the slowly varying fSV model and power utility

$$\begin{cases} \mathrm{d}S_t = S_t\left[\mu(Z_t^{\delta,H})\,\mathrm{d}t + \sigma(Z_t^{\delta,H})\,\mathrm{d}W_t\right], \\ Z_t^{\delta,H} = \int_{-\infty}^t \mathcal{K}^\delta(t-s)\,\mathrm{d}W_s^Z, \end{cases} \qquad \mathrm{d}\left\langle W, W^Z\right\rangle_t = \rho\,\mathrm{d}t.$$

- The value process $V_t^\delta := \sup_{\pi\in\mathcal{A}_t^\delta} \mathbb{E}\left[U(X_T^\pi)\middle|\,\mathcal{F}_t\right]$
- The corresponding optimal strategy π^*
- First order approximations to V_t^δ and π^*
- A practical strategy to generate this approximated value process

$Z_t^{\delta,H}$ is not Markovian nor a semi-martingale $\Rightarrow$ HJB PDE is not available

A General Non-Markovian Model

Dynamics of the risky asset S_t

$$\begin{cases} \mathrm{d}S_t = S_t \left[\mu(Y_t)\,\mathrm{d}t + \sigma(Y_t)\,\mathrm{d}W_t\right], \\ Y_t\text{: a general stochastic process, } \mathcal{G}_t := \sigma\left\{\left(W^Y\right)_{0\leq u\leq t}\right\}\text{-adapted,} \end{cases}$$

with

$$\mathrm{d}\left\langle W, W^Y\right\rangle_t = \rho\,\mathrm{d}t.$$

Dynamics of the wealth process X_t (assume $r = 0$ for simplicity):

$$\mathrm{d}X_t^\pi = \pi_t\mu(Y_t)\,\mathrm{d}t + \pi_t\sigma(Y_t)\,\mathrm{d}W_t$$

Define the value process V_t by

$$V_t := \sup_{\pi\in\mathcal{A}_t} \mathbb{E}\left[U(X_T^\pi)\middle|\,\mathcal{F}_t\right]$$

where $U(x)$ is of power type $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Proposition: Martingale Distortion Transformation[2]

- The value process V_t is given by

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma}\left[\widetilde{\mathbb{E}}\left(e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\middle|\,\mathcal{G}_t\right)\right]^q$$

under $\widetilde{\mathbb{P}}$, $\widetilde{W}_t^Y := W_t^Y + \int_0^t a_s\,\mathrm{d}s$ is a BM.

- The optimal strategy π^* is

$$\pi_t^* = \left[\frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} + \frac{\rho q \xi_t}{\gamma\sigma(Y_t)}\right] X_t$$

where ξ_t is given by the martingale representation $\mathrm{d}M_t = M_t\xi_t\,\mathrm{d}\widetilde{W}_t^Y$ and M_t is

$$M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Y_s)\,\mathrm{d}s}\middle|\,\mathcal{G}_t\right]$$

[2] Tehranchi '04: different utility function, proof and assumptions

Remarks

- only works for one factor model
- assumptions: integrability conditions of ξ_t, X_t^π and π_t
- $\gamma = 1 \to$ case of log utility, can be treated separately
- degenerate case $\lambda(y) = \lambda_0$, M_t is a constant martingale, $\xi_t = 0$

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda_0^2(T-t)}, \quad \pi_t^* = \frac{\lambda_0}{\gamma\sigma(Y_t)} X_t.$$

- uncorrelated case $\rho = 0$, the problem is "linear" since $q = 1$

$$V_t = \frac{X_t^{1-\gamma}}{1-\gamma} \mathbb{E}\left[e^{\frac{1-\gamma}{2\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s} \middle| \mathcal{G}_t \right], \quad \pi_t^* = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)} X_t.$$

Sketch of Proof (Verification)

- V_t is a supermartingale for any admissible control π
- V_t is a true martingale following π^*
- π^* is admissible

Define $\alpha_t = \pi_t / X_t$, then

$$\mathrm{d}V_t = V_t D_t(\alpha_t)\,\mathrm{d}t + \mathrm{d}\,\text{Martingale}$$

with the drift factor $D_t(\alpha_t)$

$$D_t(\alpha_t) := \alpha_t \mu - \frac{\gamma}{2}\alpha_t^2 \sigma^2 - \frac{\lambda^2}{2\gamma} + \frac{q}{1-\gamma} a_t \xi_t + \frac{q(q-1)}{2(1-\gamma)} \xi_t^2 + \rho q \alpha_t \sigma \xi_t.$$

$\Rightarrow \alpha_t^*$ and $D_t(\alpha_t^*) = 0$ with the right choice of a_t and q:

$$a_t = -\rho\left(\frac{1-\gamma}{\gamma}\right)\lambda(Y_t), \quad q = \frac{\gamma}{\gamma + (1-\gamma)\rho^2}.$$

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$\Rightarrow$ α_t^* and $D_t(\alpha_t^*) = 0$ with the right choice of a_t and q:

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Relation to the Distortion Transformation [3]

In the Markovian setup, Y_t is a diffusion process

$$\mathrm{d}Y_t = k(Y_t)\,\mathrm{d}t + h(Y_t)\,\mathrm{d}W_t^Y,$$

and distortion transformation is given by

$$V(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma}\Psi(t,y)^q.$$

It solves the linear PDE

$$\Psi_t + \left(\frac{1}{2}h^2(y)\partial_{yy} + k(y)\partial_y + \frac{1-\gamma}{\gamma}\lambda(y)\rho h(y)\partial_y\right)\Psi + \frac{1-\gamma}{2q\gamma}\lambda^2(y)\Psi = 0,$$

and has the probabilistic representation

$$\Psi(t,y) = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_t^T \lambda^2(Y_s)\,\mathrm{d}s}\,\middle|\,Y_t = y\right].$$

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Merton Problem under Slowly Varying Fractional SV

Consider a rescaled stationary fOU process $Z_t^{\delta,H}$

$$Z_t^{\delta,H} = \delta^H \int_{-\infty}^{t} e^{-\delta a(t-s)} \, \mathrm{d}W_s^{(H)} = \int_{-\infty}^{t} \mathcal{K}^\delta(t-s) \, \mathrm{d}W_s^Z$$

together with the risky asset

$$\mathrm{d}S_t = S_t \left[\mu(Z_t^{\delta,H}) \, \mathrm{d}t + \sigma(Z_t^{\delta,H}) \, \mathrm{d}W_t \right],$$

Apply the martingale distortion transformation with $Y_t = Z_t^{\delta,H}$ gives

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma} \left[\widetilde{\mathbb{E}} \left(e^{\frac{1-\gamma}{2q\gamma} \int_t^T \lambda^2(Z_s^{\delta,H}) \, \mathrm{d}s} \middle| \mathcal{G}_t \right) \right]^q .$$

Approximation to the Value Process

Theorem (Fouque-H. '17)

For fixed $t \in [0, T)$, $X_t = x$ and the observed value $Z_0^{\delta,H}$, the value process V_t^δ takes the form

$$V_t^\delta = \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} + \frac{X_t^{1-\gamma}}{\gamma}\lambda(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H})\phi_t^\delta$$
$$+ \delta^H \rho \frac{X_t^{1-\gamma}}{1-\gamma} e^{\frac{1-\gamma}{2\gamma}\lambda^2(Z_0^{\delta,H})(T-t)} \lambda^2(Z_0^{\delta,H})\lambda'(Z_0^{\delta,H}) \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{(T-t)^{H+\frac{3}{2}}}{\Gamma(H+\frac{5}{2})}$$
$$+ \mathcal{O}(\delta^{2H}),$$

where ϕ_t^δ is the random component of order δ^H

$$\phi_t^\delta = \mathbb{E}\left[\int_t^T \left(Z_s^{\delta,H} - Z_0^{\delta,H}\right) \mathrm{d}s \,\middle|\, \mathcal{G}_t\right].$$

Approximation to the Optimal Strategy

Recall that

$$\pi_t^* = \left[\frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})} + \frac{\rho q \xi_t}{\gamma\sigma(Z_t^{\delta,H})} \right] X_t$$

and ξ_t is from the martingale rep. of $M_t = \widetilde{\mathbb{E}}\left[e^{\frac{1-\gamma}{2q\gamma}\int_0^T \lambda^2(Z_s^{\delta,H})\,\mathrm{d}s} \middle| \mathcal{G}_t \right]$.

Theorem (Fouque-H., '17)

The optimal strategy π_t^* is approximated by

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$$+ \mathcal{O}(\delta^{2H})$$
$$:= \pi_t^{(0)} + \delta^H \pi_t^{(1)} + \mathcal{O}(\delta^{2H}).$$

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How Good is the Approximation?

Corollary

In the case of power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\pi^{(0)} = \frac{\lambda(Z_t^{\delta,H})}{\gamma\sigma(Z_t^{\delta,H})}$ generates the approximation of V_t^δ up to order δ^H (leading order + two correction terms of order δ^H), thus asymptotically optimal in $\mathcal{A}_t^\delta$.

- $H = \frac{1}{2}$, $Z_t^{\delta,H}$ becomes the Markovian OU process, both approximation coincides with results in [Fouque Sircar Zariphopoulou '13]. The corollary recovers [Fouque -H. '16]
- Sketch of proofs: Apply Taylor expansion to $\lambda(z)$ at the point $Z_0^{\delta,H}$, and then control the moments $Z_t^{\delta,H} - Z_0^{\delta,H}$.

Denote by $v^{(0)}(t,x,z)$ the value function at frozen Sharpe-ratio $\lambda(z)$, we define $\pi^{(0)}$ by

$$\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)}\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$$

and the associated value process $V^{\pi^{(0)},\delta}$

$$V_t^{\pi^{(0)},\delta} := \mathbb{E}\left[U(X_T^{\pi^{(0)}})|\mathcal{F}_t\right].$$

- A first order approximation to $V^{\pi^{(0)},\delta}$ obtained by epsilon-martingale decomposition[4][5]
- Optimality of $\pi^{(0)}$ in a smaller class of controls of feedback form

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Epsilon-Martingale Decomposition

To find $Q^{\pi^{(0)},\delta}$ such that $Q_T^{\pi^{(0)},\delta} = V_T^{\pi^{(0)},\delta} = U(X_T^{\pi^{(0)}})$, and that can be decomposed as

$$Q_t^{\pi^{(0)},\delta} = M_t^\delta + R_t^\delta,$$

where M_t^δ is a martingale and R_t^δ is of order δ^{2H}. Then

$$\begin{aligned} V_t^{\pi^{(0)},\delta} &= \mathbb{E}\left[Q_T^{\pi^{(0)},\delta}|\mathcal{F}_t\right] = M_t^\delta + \mathbb{E}\left[R_T^\delta|\mathcal{F}_t\right] \\ &= Q_t^{\pi^{(0)},\delta} + \mathbb{E}\left[R_T^\delta|\mathcal{F}_t\right] - R_t^\delta, \end{aligned}$$

and $Q_t^{\pi^{(0)},\delta}$ is the approximation to $V^{\pi^{(0)},\delta}$.

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First order approximation to $V^{\pi^{(0)},\delta}$

Proposition

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$$V_t^{\pi^{(0)},\delta} = Q_t^{\pi^{(0)},\delta}(X_t^{\pi^{(0)}}, Z_0^{\delta,H}) + \mathcal{O}(\delta^{2H}),$$

where $Q_t^{\pi^{(0)},\delta}(x,z)$ is given by:

$$\begin{aligned} Q_t^{\pi^{(0)},\delta}(x,z) =& v^{(0)}(t,x,z) + \lambda(z)\lambda'(z)D_1 v^{(0)}(t,x,z)\phi_t^{\delta} \\ &+ \delta^H \rho\lambda^2(z)\lambda'(z)D_1^2 v^{(0)}(t,x,z)\frac{(T-t)^{H+3/2}}{\Gamma(H+\frac{5}{2})}. \end{aligned}$$

- For power utility, $Q_t^{\pi,\delta}$ coincides with the approximation of V_t^{δ}
- For the Markovian case $H = \frac{1}{2}$, recovers the results in [Fouque-H. '16]

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Asymptotically Optimality of $\pi^{(0)}$

Theorem (Fouque-H. '17)

The trading strategy $\pi^{(0)}(t,x,z) = -\frac{\lambda(z)}{\sigma(z)}\frac{v_x^{(0)}(t,x,z)}{v_{xx}^{(0)}(t,x,z)}$ *is asymptotically optimal in the following class:*

$$\widetilde{\mathcal{A}}_t^{\delta}[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha] := \left\{\pi = \widetilde{\pi}^0 + \delta^{\alpha}\widetilde{\pi}^1 : \pi \in \mathcal{A}_t^{\delta}, \alpha > 0, 0 < \delta \leq 1\right\}.$$

Thank you !

Stochastic Volatility with Fast Factor Y_t

S_t is modeled by:

$$\begin{cases} \mathrm{d}S_t = \mu(Y_t)S_t\,\mathrm{d}t + \sigma(Y_t)S_t\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = \frac{1}{\epsilon}b(Y_t)\,\mathrm{d}t + \frac{1}{\sqrt{\epsilon}}a(Y_t)\,\mathrm{d}W_t^Y, \end{cases}$$

with correlation $\mathrm{d}W_t W_t^Y = \rho\,\mathrm{d}t$.

Theorem (Fouque-H., in prep.)

Under appropriate assumptions, for fixed (t,x,y) and any family of trading strategies $\mathcal{A}_0(t,x,y)\left[\widetilde{\pi}^0, \widetilde{\pi}^1, \alpha\right]$, the following limit exists and satisfies

$$\ell := \lim_{\epsilon \to 0} \frac{\widetilde{V}^\epsilon(t,x,y) - V^{\pi^{(0)},\epsilon}(t,x,y)}{\sqrt{\epsilon}} \le 0.$$

Theorem (Fouque-H., in prep.)

The residual function $E(t,x,y) := V^{\pi^{(0)},\epsilon}(t,x) - v^{(0)}(t,x) - \sqrt{\epsilon}v^{(1)}(t,x)$ *is of order* ϵ*, where in this case,* $v^{(0)}$ *solves*

$$v_t^{(0)} - \frac{1}{2}\overline{\lambda}^2 \frac{\left(v_x^{(0)}\right)^2}{v_{xx}^{(0)}} = 0,$$

and $v^{(1)} = -\frac{1}{2}(T-t)\rho_1 B D_1^2 v^{(0)}(t,x)$*.*