Portfolio optimization in short time horizon

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Objective

- Portfolio optimization in a time horizon $[t, T]$ where $\tau := T - t \to 0$.
- We want to choose an investment strategy that will maximize the expected utility of terminal wealth.
- We assume a general strictly increasing, concave terminal utility function of wealth $U_T(x)$.
- Obtain closed-form approximating formulas as $\tau \to 0$ of the maximal expected utility and optimal portfolio.

Literature review

Some recent work where closed-form formulas are obtained in the incomplete market case.

- [LS16] “Portfolio Optimization under Local-Stochastic Volatility: Coefficient Taylor Series Approximations & Implied Sharpe Ratio” by Lorig and Sircar in 2016
- [FSZ13] “Portfolio optimization & stochastic volatility asymptotics” by Fouque, Sircar and Zariphopoulou.

We work under the assumptions on the market model and utility function in the 2013 paper "An approximation scheme for solution to the optimal investment problem in incomplete markets" by Zariphopoulou and Nadtochiy [NZ13].

Stochastic volatility model for risky asset price

The market consists of one risky asset and one riskless bond. The risky asset price, S_t satisfies

$$dS_t = \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW^{(1)}1_t$$

$$dY_t = b(Y_t)dt + a(Y_t)(\rho dW_t^{(1)} + \sqrt{1-\rho^2}dW_t^{(2)}).$$

where $W^{(1)}$ and $W^{(2)}$ are independent standard brownian motions, $-1 < \rho < 1$. Define $\lambda(y) := \frac{\mu(y)-r}{\sigma(y)}$ where r is the risk-free interest rate.

Assumptions on stochastic volatility model

Bounded continuous coefficients, volatilities bounded away from zero and $|a'|, |a''|, |b'|, |\lambda|, |\lambda'|, |\lambda''|$ are bounded.

Let π_s and π_s^0 denote the discounted amount of wealth invested in the risky asset and risk-free asset at time s. If x denotes the initial wealth at time t, then the wealth at time s is $X_s^{t,x,\pi} := \pi_s + \pi_s^0$, which evolves as

$$dX_s^{t,x,\pi} = \sigma(Y_s)\pi_s(\lambda(Y_s)ds + dW_s^1), \quad X_t^{t,x,\pi} = x,$$

assuming self-financing strategies (π_s, π_s^0).

Optimization problem

We wish to maximize the expected terminal utility at time T where the terminal utility function is given by U_T. We define the **value function** $J(t, x)$ as the optimal expected terminal utility:

$$J(t, x, Y_t) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} E[U_T(X_T^{t,x,\pi})|\mathcal{F}_t],$$

where $\mathcal{A}$ is the set of admissible trading strategies.

$\mathcal{A} = \{\mathcal{F}_t$-adapted processes π such that $E[\int_t^T \pi_s^2 \sigma^2(Y_s) ds] < \infty$, $(X_s^{t,x,\pi})_{s \in [t,T]}$ is strictly positive and $E[\int_t^T (X_s^{t,x,\pi})^{-p}(1 + \pi_s^2) ds] < \infty$, for every $p \geq 0\}$.

Utility function

- **Assumption 1:** The terminal utility function $U_T(x)$ is a strictly increasing, concave function of wealth, x, and $U_T \in C^5(\mathbb{R})$.
- **Assumption 2:** $U_T(x)$ behaves asymptotically as $x \to 0$ and $x \to \infty$ as an affine transformation of some power function $x^{1-\gamma}$, where $\gamma \neq 1$.
  $\left(\frac{U_T'(x)}{x^{-\gamma}} = O(1), \frac{U_T''(x)}{x^{-\gamma-1}} = O(1), \ldots \frac{U_T^{(5)}(x)}{x^{-\gamma-4}} = O(1)\right.$, as $\left.x \to 0, \infty.\right)$

HJB equation

The associated HJB equation for this optimization problem is:

$$\begin{aligned} &U_t + \max_{\pi}\Big(\frac{1}{2}\sigma^2(y)\pi^2 U_{xx} + \pi(\sigma(y)\lambda(y)U_x \\ &\qquad + \rho\sigma(y)a(y)U_{xy})\Big) + \frac{1}{2}a^2(y)U_{yy} + b(y)U_y = 0; \\ &U(T,x,y) = U_T(x). \end{aligned}$$

The maximum is achieved at

$$\pi(t,x,y) = \frac{-\lambda(y)U_x - \rho a(y)U_{xy}}{\sigma(y)U_{xx}}.$$

marginal HJB equation

If $U(t,x,y)$ is a solution to the HJB equation, then $V = U_x$ satisfies the following **marginal HJB** equation

$$V_t + H(y, V, V_x, V_y, V_{xx}, V_{xy}, V_{yy}) = 0, \tag{3}$$

where

$$H := \frac{1}{2}\left(\frac{\lambda(y)V + \rho a(y)V_y}{V_x}\right)^2 V_{xx} - \frac{\lambda(y)V + \rho a(y)V_y}{V_x}\rho a(y)V_{xy} + \frac{1}{2}a^2(y)V_{yy} - \lambda^2(y)V + (b(y) - \lambda(y)\rho a(y))V_y$$

Key results from [NZ13]

THEOREM (Nadtochiy&Zariphopoulou [NZ13])

The marginal HJB equation has a unique continuous viscosity solution V in the class $\mathcal{D}$
(Where, informally, $\mathcal{D}$ can be described as the class of continuous functions $f(t,x,y)$ such that $0 < \frac{1}{c}x^{-\gamma} \leq f(t,x,y) \leq cx^{-\gamma}$.)

THEOREM (Nadtochiy&Zariphopoulou [NZ13])

Let V be the unique viscosity solution of the marginal HJB equation in previous theorem. Define

$$U(t,x,y) := \begin{cases} U_T(0^+) + \int_0^x V(t,z,y)dz, & \text{if } \gamma \in (0,1) \\ U_T(\infty) - \int_x^\infty V(t,z,y)dz, & \text{if } \gamma > 1. \end{cases}$$

Then the value function

$$J(t,x,y) = U(t,x,y)$$

Approximating V- solution to the marginal HJB

Approximating V:

- **STEP 1:** We construct classical sub- and super-solutions to the marginal HJB equation.
  - Plug in the following formal expansion into the marginal HJB equations:

$$V(t,x,y) = V_0(x,y) + (T-t)V_1(x,y) + (T-t)^2 V_2(x,y) + \cdots .$$

  - Comparing coefficients of powers of $(T-t)$, we obtain the following expressions for V_0 and V_1:

$$V_0(x,y) = u(x) := U_T'(x), \quad V_1(x,y) = K(x,y);$$

  where $K(x,y) := \lambda^2(y)R(x)$ and

$$R(x) := \frac{1}{2}\frac{u^2(x)u''(x)}{(u'(x))^2} - u(x).$$

- Define $\underline{V}(t,x,y) = u(x) + (T-t)K(x,y) - Cx^{-\gamma}(T-t)^2$ and $\overline{V}(t,x,y) = u(x) + (T-t)K(x,y) + Cx^{-\gamma}(T-t)^2$, which for an appropriate choice of $C > 0$ are sub- and super-solutions, respectively, of the marginal HJB equation, i.e.

$$\partial_t \underline{V} + H(\underline{V}) \geq 0 \quad \text{and} \quad \partial_t \overline{V} + H(\overline{V}) \leq 0.$$

- **STEP 2:** Prove that $\underline{V} \leq V \leq \overline{V}$
  - Define $W(t,z,y) := \log(V^{\epsilon}(t,e^z,y)) + \gamma z$, $\underline{W}(t,z,y) := \log(\underline{V}^{\epsilon}(t,e^z,y)) + \gamma z$, and $\overline{W}(t,z,y) := \log(\overline{V}^{\epsilon}(t,e^z,y)) + \gamma z$
  - If V is a solution (or sub- or super-solution) of the marginal HJB equation, then W is the solution (or sub- or super-solution) of

$$W_t + \epsilon\Big[-\frac{1}{2}\frac{(\lambda + a\rho W_y)^2}{W_z - \gamma}\left(\frac{W_{zz}}{W_z - \gamma} - 1\right) - \frac{1}{2}a^2 W_{yy} + a\rho\frac{\lambda + a\rho W_y}{W_z - \gamma}W_{zy} + \frac{1}{2}\lambda^2 + (\lambda a\rho - b)W_y + \frac{1}{2}a^2(\rho^2 - 1)(W_y)^2\Big] = 0. \tag{4}$$

  - Apply Comparison principle for (4) to get $\underline{W} \leq W \leq \overline{W}$.

Thus V is sandwiched between $\underline{V}$ and $\overline{V}$ i.e.

$$U'_T(x) + (T-t)K(x,y) - Cx^{-\gamma}(T-t)^2 \leq V(t,x,y)$$
$$\leq U'_T(x) + (T-t)K(x,y) - Cx^{-\gamma}(T-t)^2.$$

This gives us

$$|V(t,x,y) - (U'_T(x) + (T-t)K(x,y))| \leq Cx^{-\gamma}(T-t)^2.$$

Approximating formulas

Approximating formula for the value function is given by

THEOREM

Define $u(x) := U'_T(x)$,

$$R(x) := \frac{1}{2}\frac{u^2(x)u''(x)}{(u'(x))^2} - u(x)$$

and $K(x,y) := \lambda^2(y)R(x)$.
There exists a constant $C > 0$ *such that*

$$\begin{cases} |J(t,x,y) - (U_T(x) + (T-t)\int_0^x K(r,y)dr| \leq C(T-t)^2 x^{1-\gamma} & \text{if } \gamma \in (0,1). \\ |J(t,x,y) - (U_T(x) - (T-t)\int_x^\infty K(r,y)dr| \leq C(T-t)^2 x^{1-\gamma} & \text{if } \gamma > 1, \end{cases}$$

as $T - t \to 0$.

Approximating formulas

The approximate closed-from formula for the optimal portfolio as $T - t \to 0$ is given by

$$\hat{\pi}(t,x,y) = -\frac{\lambda}{\sigma}\frac{\hat{U}_x}{\hat{U}_{xx}} - \frac{\rho a}{\sigma}\frac{\hat{U}_{xy}}{\hat{U}_{xx}}, \tag{5}$$

where

$$\hat{U}(t,x,y) = \begin{cases} U_T(x) + (T-t)\int_0^x K(r,Y_t)dr & \text{if } 0 < \gamma < 1 \\ U_T(x) - (T-t)\int_x^\infty K(r,Y_t)dr & \text{if } \gamma > 1. \end{cases}$$

Proof

Let

$$\tilde{\pi}(t,x,y) := -\frac{\lambda}{\sigma}\frac{\underline{U}_x}{\underline{U}_{xx}} - \frac{\rho a}{\sigma}\frac{\underline{U}_{xy}}{\underline{U}_{xx}} \tag{6}$$

where

$$\underline{U}(t,x,y) := \begin{cases} U_T(0^+) + \int_0^x \underline{V}(t,r,y)dr & \text{if } \gamma \in (0,1) \\ U_T(\infty) - \int_x^\infty \underline{V}(t,r,y)dr & \text{if } \gamma > 1, \end{cases}$$

and let $X_s^{t,x,\tilde{\pi}}$ be the discounted wealth process associated with the portfolio $\tilde{\pi}$.

Then, by construction, $\underline{U}$ is a subsolution of the HJB equation:

$$\partial_t \underline{U} + \mathcal{L}^{\tilde{\pi}} \underline{U} \geq 0,$$

where $\mathcal{L}^{\tilde{\pi}}$ is the generator of $(X^{t,x,\tilde{\pi}}, Y)$ with the control $\tilde{\pi}$.

Then, by construction, $\underline{U}$ is a subsolution of the HJB equation:

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where $\mathcal{L}^{\tilde{\pi}}$ is the generator of $(X^{t,x,\tilde{\pi}}, Y)$ with the control $\tilde{\pi}$. Applying Itô's formula to $\underline{U}(s, X_s^{t,x,\tilde{\pi}}, Y_s)$ and taking conditional expectations:

$$\underbrace{E[\underline{U}(T, X_T^{t,x,\tilde{\pi}}, Y_T)|\mathcal{F}_t]}_{E[U_T(X_T^{t,x,\tilde{\pi}})|\mathcal{F}_t]} - \underline{U}(t,x,Y_t)$$

$$= E\left[\int_t^T (\partial_s \underline{U}(s, X_s^{t,x,\tilde{\pi}}, Y_s) + \mathcal{L}^{\tilde{\pi}} \underline{U}(s, X_s^{t,x,\tilde{\pi}}, Y_s)) ds \Big| \mathcal{F}_t\right] \geq 0.$$

Thus,

$$\underline{U}(t, x, Y_t) \le E[U_T(X_T^{t,x,\tilde{\pi}})|\mathcal{F}_t] \le J(t, x, Y_t).$$

Thus,

$$\underline{U}(t,x,Y_t) \leq E[U_T(X_T^{t,x,\tilde{\pi}})|\mathcal{F}_t] \leq J(t,x,Y_t).$$

We know that

$$|J(t,x,y) - \underline{U}(t,x,y)| = O((T-t)^2)O(x^{1-\gamma}).$$

Lemma

For $X_s^{t,x,\tilde{\pi}}$ the wealth process under portfolio $\tilde{\pi}$, there exists a $c > 0$ such that

$$|J(t, x, Y_t) - E[U_T(X_T^{t,x,\tilde{\pi}})|\mathcal{F}_t] \leq c(T-t)^2 x^{1-\gamma}, \text{ as } T-t \to 0.$$

Lemma

For $\hat{\pi}$ and $\tilde{\pi}$ defined in (5) and (6), respectively, we have

$$|\tilde{\pi} - \hat{\pi}| = O((T-t)^2)O(1+x), \ \text{as } T-t \to 0. \tag{7}$$

Example

We consider the example where the stochastic volatility model is

$$dS_s = \mu S_s ds + \frac{1}{\sqrt{Y_s}} S_s dW_s^{(1)}$$
$$dY_s = (m - Y_s)ds + \beta\sqrt{Y_s}(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}),$$

where $2m \geq \beta^2$ and utility function $U_T(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Explicit formulas for the value function and optimal portfolio exist in this case.

In the following graphs we have taken: $\mu = 0.0811$, $m = 27.9345$, $\beta = 1.12$, $\rho = 0.5241$, and $\gamma = 3$. Terminal time $T = 2$ and y is fixed at 27.9345.

Value function approximation when $T - t = .5$

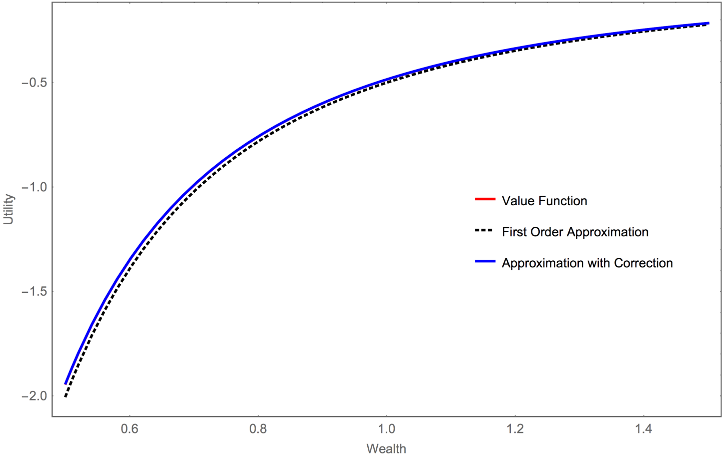

Figure: $t = 1.5$, $T = 2$; The value function is plotted against the first order approximation $U_T(x)$ and the first order approximation with the additional

Value function approximation when $T - t = .1$

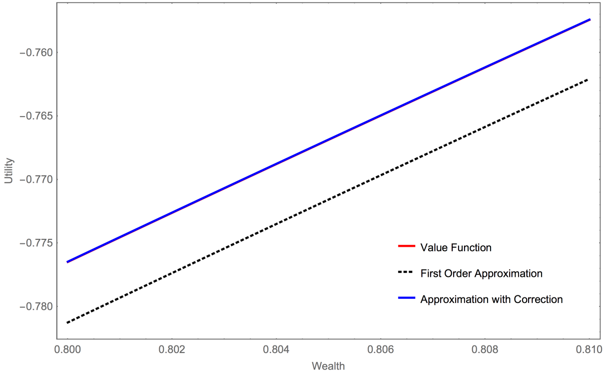

Figure: $t = 1.9,\ T = 2$; When the time interval is shortened from a length of 0.5 to a length of 0.1, the approximation with correction is much closer to the value function.

Portfolio approxaimtion when $T - t = .5$

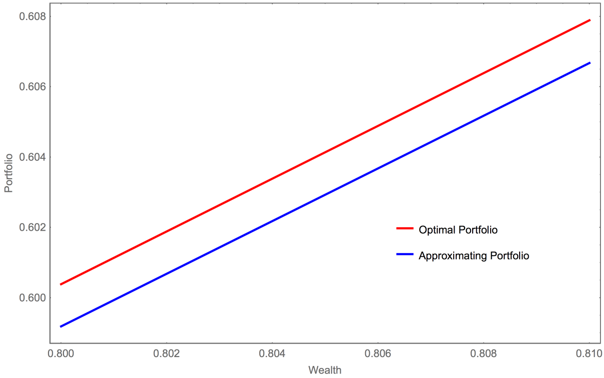

Figure: $t = 1.5$, $T = 2$; The portfolios generated by the value function and the approximation with correction are shown in this figure to be close.

Portfolio approximation when $T - t = .1$

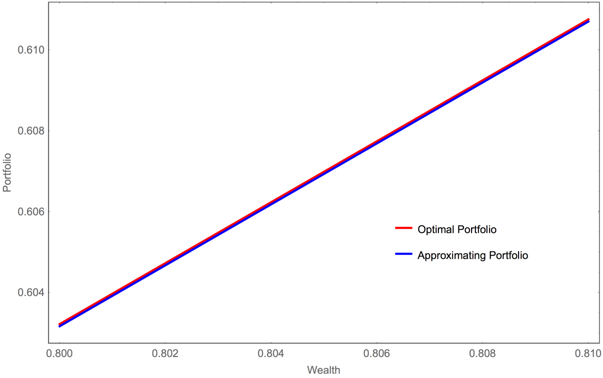

Figure: $t = 1.9$, $T = 2$; When the time interval is shortened from a length of 0.5 to a length of 0.1, the approximating portfolio is much closer to the value portfolio.

References

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Nadtochiy, Sergey and Zariphopoulou, Thaleia, An approximation scheme for solution to the optimal investment problem in incomplete markets, SIAM J. Financial Math., 2013, Vol 4, Number 1, p.494-538, http://dx.doi.org/10.1137/120869080

Thank You!

Lemma

For $\hat{\pi}$ and $\tilde{\pi}$ defined in (5) and (6), respectively, we have

$$|\tilde{\pi} - \hat{\pi}| = O((T-t)^2)O(1+x), \text{ as } T - t \to 0. \tag{9}$$

Proof.

For some constant $\tilde{C} > 0$, we have

$$
\begin{aligned}
|\tilde{\pi} - \hat{\pi}| &= \left| \frac{-\lambda \underline{U}_x - \rho a \underline{U}_{xy}}{\sigma \underline{U}_{xx}} - \frac{-\lambda \hat{U}_x - \rho a \hat{U}_{xy}}{\sigma \hat{U}_{xx}} \right| \\
&= \left| \frac{-\lambda(\underline{U}_x \hat{U}_{xx} - \hat{U}_x \underline{U}_{xx}) - \rho a(\underline{U}_{xy} \hat{U}_{xx} - \hat{U}_{xy} \underline{U}_{xx})}{\sigma \underline{U}_{xx} \hat{U}_{xx}} \right| \\
&= \left| \frac{-\lambda \frac{(T-t)^2}{2} \tilde{C}[u'(x)x^{-\gamma} + (T-t)K_x x^{-\gamma} + -\gamma u(x) x^{-\gamma-1}] - \rho a \gamma C}{\sigma[(u'(x))^2 + 2(T-t)K_x u'(x) - \gamma \tilde{C} \frac{(T-t)^2}{2} x^{-\gamma-1} u'(x) + (T-t)^2 K_x^2} \right. \\
&= \begin{cases} O((T-t)^2) O(1) & \text{as } x \to 0 \\ O((T-t)^2) O(x) & \text{as } x \to \infty \end{cases} \\
&= O((T-t)^2) O(1+x)
\end{aligned}
$$

where the second to last equality is by Assumption 2 and the definition of $K(x,y)$. $\square$