# A Model of Systemic Risk

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# The Financial System

We have $N$ private banks and the central bank.

- Central bank chooses the discount rate: interest rate of lending to private banks
- There can be cash flows between each pair of private banks
- Private banks borrow from central bank and pay back interest
- Private banks invest in risky assets

# Private Banks

The $i$th private bank has capital $X_i(t)$, with $Y_i(t) = \log X_i(t)$

It borrows the amount $Z_i(t) \geq 0$ from the central bank, under discount interest rate $r(t)$

Therefore, it pays interest $r(t)Z_i(t)\,\mathrm{d}t$ during $[t, t + \mathrm{d}t]$

The case $Z_i(t) \leq 0$ is when the bank does not borrow anything but sets aside money $-Z_i(t)$ in cash, earning zero interest.

In both cases, the bank invests $X_i(t) + Z_i(t)$ in a risky asset $S_i(t)$, and pays interest $r(t)(Z_i(t))_+\,\mathrm{d}t$

# Risky Assets

The $i$th risky asset is given by

$$S_i(t) = \exp\left(M_i(t) - \langle M_i \rangle_t\right)$$

where $M = (M_1, \ldots, M_N)$ is a Brownian motion with drift vector $\mu = (\mu_1, \ldots, \mu_N)$ and covariance matrix $A = (a_{ij})$.

In particular, the component $M_i$ is a Brownian motion with drift $\mu_i$ and diffusion $a_{ii} = \sigma_i^2$:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mathrm{d}M_i(t) = \mu_i\,\mathrm{d}t + \sigma_i\,\mathrm{d}W_i(t)$$

# Equation Without Interbank Flows

Combining investment and borrowing for the $i$th bank:

$$\mathrm{d}X_i(t) = (X_i(t) + Z_i(t))\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - r(t)(Z_i(t))_+\,\mathrm{d}t$$

By Itô's formula, for $Y_i(t) = \log X_i(t)$ and $\alpha_i(t) := Z_i(t)/X_i(t)$

$$\mathrm{d}Y_i(t) = (1 + \alpha_i(t))\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - \left[\frac{\sigma_i^2}{2}(1 + \alpha_i(t))^2 + r(t)(\alpha_i(t))_+\right]\mathrm{d}t$$

# Equation with Added Interbank Flows

$$\mathrm{d}Y_i(t) = \frac{1}{N}\sum_{j=1}^{N} c_{ij}(t)(Y_j(t) - Y_i(t))\,\mathrm{d}t + (1+\alpha_i(t))\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - \left[\frac{\sigma_i^2}{2}(1+\alpha_i(t))^2 + r(t)(\alpha_i(t))_+\right]\mathrm{d}t$$

Here, $c_{ij}(t) = c_{ji}(t) \ge 0$ are controlled by the central bank and are used to keep banks close to one another, to minimize the possibility of bankruptcy

This model is taken from (Carmona, Fouque, Sun, 2013), where they had $c_{ij}(t) \equiv c > 0$

# Objective of Each Private Bank

The $i$th bank chooses the amount $Z_i$ of borrowing (or, equivalently, $\alpha_i = Z_i/X_i$) to maximize its expected terminal logarithmic utility:

$$\mathbf{E} \log X_i(T) = \mathbf{E}\, Y_i(T)$$

The $i$th bank takes as given $X_j(t)$, $Z_j(t)$ for $j \neq i$ (of other banks) and $r(t)$, $c_{ij}(t)$ (instruments of the central bank)

# Objective of Central Bank

Central bank chooses the discount interest rate $r$ to control (as we see below) the overall size of the system:

$$\overline{Y}(t) = \frac{1}{N}\sum_{i=1}^{N} Y_i(t),$$

and the rates $c_{ij}$ to make $Y_i$ closer to this average $\overline{Y}$ by directing flow of cash to this bank from other banks (or vice versa).

Objective: to prevent $Y_i(t)$ from becoming too small (which corresponds to bankruptcy)

# Opitmal Control for Private Bank

$$\Phi_i(t, y) := \sup_{\alpha_i} \mathbf{E}(Y_i(T) \mid Y(t) = y)$$

satisfies $\Phi_i(T, y) = y_i$, and HJB equation:

$$0 = \frac{\partial \Phi_i}{\partial t} + \sup_{\alpha_i} \Bigg[ \sum_{j=1}^{N} h_j(\alpha_j) \frac{\partial \Phi_i}{\partial y_j} + \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{\partial^2 \Phi_i}{\partial y_j \partial y_k} a_{jk} (1 + \alpha_j)(1 + \alpha_k) \Bigg]$$

$$h_j(\alpha_j) := (1 + \alpha_j)\mu_j - \frac{\sigma_j^2}{2}(1 + \alpha_j)^2 - r(\alpha_j)_+$$

# Optimal Control for Private Bank

$$\text{Try anzats} \quad \Phi_i(t,y) = g_{i0}(t) + \sum_{j=1}^{N} g_{ij}(t) y_j$$

Because the objective and dynamics are linear in $Y_j$, we can solve this problem explicitly. Just find $\alpha_i$ which maximizes

$$h_i(\alpha_i) := (1+\alpha_i)\mu_i - \frac{\sigma_i^2}{2}(1+\alpha_i)^2 - r(\alpha_i)_+$$

$$\text{This is} \quad \alpha_i^* := \begin{cases} \left(\frac{\mu_i - r(t)}{\sigma_i^2} - 1\right)_+, & \mu_i \ge \sigma_i^2; \\ \mu_i - \frac{\sigma_i^2}{2}, & \mu_i \le \sigma_i^2 \end{cases}$$

# Liquidity Trap

If $\mu_i \leq \sigma_i^2$ for all $i$, then banks do not borrow from the central bank; rather, they set aside some cash

Even setting $r = 0$ cannot induce banks to borrow

Below, we assume that $\mu_i \geq \sigma_i^2$ for all $i$

# Dynamics Under Optimal Control

Under optimal choice $\alpha_i = \alpha_i^*$, $i = 1, \ldots, N$, we have:

$$\mathrm{d}Y_i(t) = \mathrm{d}M_i^*(t) + \frac{1}{N}\sum_{j=1}^{N} c_{ij}(t)(Y_j(t) - Y_i(t))\,\mathrm{d}t$$

$$\mathrm{d}M_i^*(t) = h_i(\alpha_i^*(t))\,\mathrm{d}t + \sigma_i(1 + \alpha_i^*(t))\,\mathrm{d}W_i(t)$$

If $r$ is constant, then $\alpha_i^*$ are too, and $M^* = (M_1^*, \ldots, M_N^*)$ is an $N$-dimensional Brownian motion

# Dynamics of Total Size of System

$$\overline{Y}(t) = g(r(t))\,\mathrm{d}t + \rho(r(t))\,\mathrm{d}W(t),$$

$$g(r) = \frac{1}{N}\sum_{i=1}^{N} g_i(r), \quad g_i(r) := \begin{cases} \frac{(\mu_i - r)^2}{2\sigma_i^2} + r, & r \le \mu_i - \sigma_i^2; \\ \mu_i - \frac{\sigma_i^2}{2}, & r \ge \mu_i - \sigma_i^2. \end{cases}$$

$$\rho^2(r) := \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\rho_i(r)\rho_j(r), \quad \rho_i(r) := \begin{cases} \frac{\mu_i - r}{\sigma_i^2}, & r \le \mu_i - \sigma_i^2; \\ 1, & r \ge \mu_i - \sigma_i^2. \end{cases}$$

# Goals of Central Bank

Maximize $\mathbf{E} U_\lambda(\overline{Y}(T))$ for $U_\lambda(y) := -e^{-\lambda y}$, $\lambda > 0$.

Central bank is even more risk-averse than private banks.

Parameter of risk aversion: $\lambda$

The HJB equation becomes

$$g(r) - \frac{\lambda}{2}\rho^2(r) \to \sup_{r \geq 0}$$

# The Case of the Same Asset

$S_1 = \ldots = S_N$

Then $\mu_1 = \ldots = \mu_N = \mu$, $\sigma_1 = \ldots = \sigma_N = \sigma$

Let $\lambda_* := 1 - 2\left(\frac{\mu}{\sigma^2} + 1\right)^{-1}$. Then the maximum is attained at

$$r = \begin{cases} 0, \ \lambda < \lambda_*; \\ \mu - \sigma^2, \ \lambda > \lambda_* \end{cases}$$

$\lambda < \lambda_*$: a less risk-averse central bank, slashes the rate to zero

$\lambda > \lambda_*$: a more risk-averse central bank, increases the rate

# The Case of Independent Assets

Assume $\mu_1 = \ldots = \mu_N = \mu$ and $\sigma_1 = \ldots = \sigma_N = \sigma$

Then same holds true for $N\lambda_*$ instead of $\lambda_*$

Even a more risk-averse central bank (than in case of same asset) can slash rate $r$ to zero

Independence of assets creates diversification and reduces risk

# The Case of Correlated Assets

$$W_i(t) = \rho \tilde{W}_0(t) + \rho' \tilde{W}_i(t),$$

$\rho^2 + \rho'^2 = 1$, $\tilde{W}_i$ i.i.d. Brownian motions

If $\mu_1 = \ldots = \mu_N = \mu$ and $\sigma_1 = \ldots = \sigma_N = \sigma$, then same holds true for $N\lambda_*/((N-1)\rho + 1)$ instead of $\lambda_*$

Here, the room for risk is less than in case of independent assets, but more than in case of the same asset

# Stability of the System

Let $\tilde{Y}_i(t) = Y_i(t) - \overline{Y}(t)$. Then $\tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_N)$ is the solution of an SDE on the hyperplane $\Pi = \{y_1 + \ldots + y_N = 0\}$.

## Theorem

*Assume $c_{ij}(t) = c_{ij}$ do not depend on $t$, and the graph $G$ on $\{1, \ldots, N\}$ defined as $i \leftrightarrow j$ iff $c_{ij} > 0$ is connected. Then $\tilde{Y}$ has a unique stationary distribution $\pi$ on $\Pi$. For any bounded measurable $f : \Pi \to \mathbb{R}$, we have:*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tilde{Y}(t))\,\mathrm{d}t = \int_\Pi f(y)\pi(\mathrm{d}y).$$

# The Case of Similar Flows

Assume all $c_{ij}(t) = c > 0$.

Then $\tilde{Y}$ is an Orntsein-Uhlenbeck process on $\Pi$.

And $\pi$ is a multivariate normal distribution on $\Pi$ with $i$th marginal

$$\pi_i \sim \mathcal{N}\left(\frac{\tilde{g}_i}{c}, \frac{\tilde{\sigma}_i^2}{2c}\right),$$

where $\tilde{g}_i,\ \tilde{\sigma}_i$ can be explicitly found.

# Control Problem for Flow Rate

This allows to formulate control problem, assuming the process $\tilde{Y}$ is in the stationary distribution $\pi$:

$$\int_{\Pi} \|y\|^2 \pi(\mathrm{d}y) + k(c) = M_1 c^{-1} + M_2 c^{-2} + k(c) \to \min_{c>0}$$

$$M_1 := \frac{1}{2} \sum_{i=1}^{N} \tilde{\sigma}_i^2, \ M_2 := \sum_{i=1}^{N} \tilde{g}_i^2.$$

$k(c)$: cost of maintaining flow rate $c$

Thanks!