Cover's universal portfolio and stochastic portfolio theory

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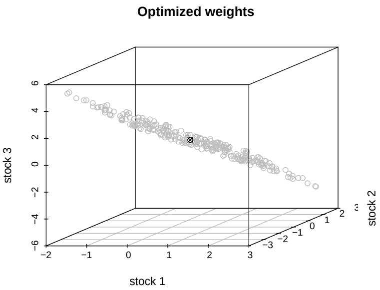
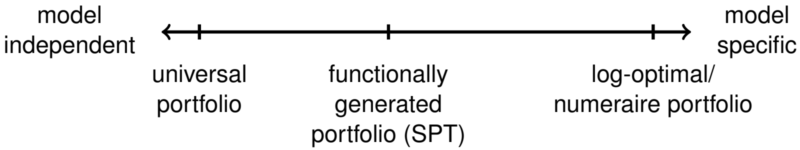
Robust portfolio selection

- Estimation error



- Changing parameters
- Model uncertainty

Approaches we study



- Universal portfolio: Cover (1991), Cover and Ordentlich (1996)
- Functionally generated portfolio: Fernholz (1998, 2002)
- Log-optimal portfolio: Kelly (1956), Breiman (1962), Long (1990)

This work

1. Theoretical results that connect the three kinds of portfolios.
2. To work in the SPT set up, we use the market portfolio as the numeraire:

$$\text{relative value of portfolio} = \frac{\text{portfolio value}}{\text{market portfolio value}}$$

3. Portfolio performance is measured in terms of *asymptotic* growth rate (relative to the market portfolio).

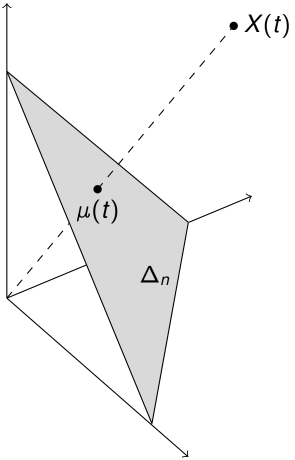
Market weight

We consider a stock market with $n \geq 2$ assets:

$$X_i(t) = \text{market cap of stock } i$$

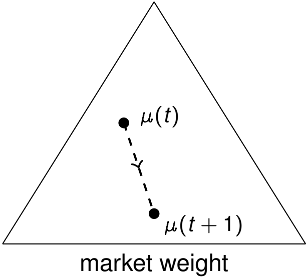
$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}$$

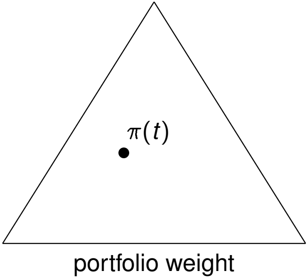
The market weight vector $\mu(t)$ takes values in the simplex Δ_n.



Portfolio relative value

- Having observed $\{\mu(s)\}_{0 \leq s \leq t}$, the investor picks $\pi(t) \in \overline{\Delta}_n$:





- The portfolio is all-long, fully invested and self-financed.
- In discrete time, portfolio relative value $V_\pi(t)$ satisfies

$$\frac{V_\pi(t+1)}{V_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)}.$$

Cover's universal portfolio

The portfolio is constant-weighted, or constant-rebalanced, if

$$\pi(t) \equiv b \quad \text{for some fixed } b \in \overline{\Delta}_n.$$

- Kelly (1956), Merton (1971), Cover (1991)
- Volatility harvesting: Fernholz and Shay (1982), Luenberger (1997), ...

Cover's (1991) problem:

- Find a portfolio strategy $\{\pi(t)\}$ such that

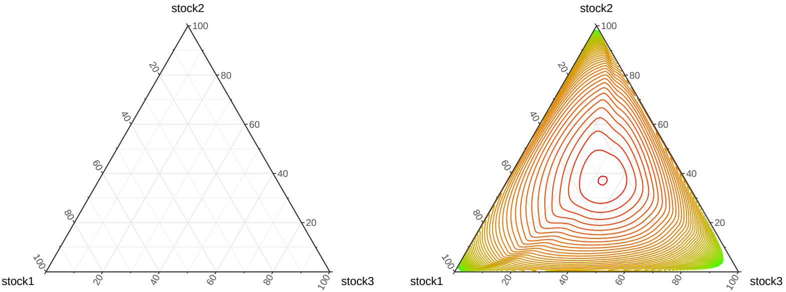
$$\frac{1}{t} \log \frac{V_\pi(t)}{\max_{b \in \overline{\Delta}_n} V_b(t)} \to 0$$

  for *all* sequences $\{\mu(t)\}_{t=0}^{\infty}$.
- $\pi(t) = F_t(\{\mu(s)\}_{0 \leq s \leq t})$ for some 'universal' functions F_t.

Cover's universal portfolio

- Take a 'prior' probability distribution ν_0 on Δ_n and consider the 'posterior' distribution

$$\nu_t(db) := \frac{V_b(t)}{\int V_\cdot(t) d\nu_0} d\nu_0(b).$$



- Cover's universal portfolio $\hat{\pi}(t)$ is the posterior mean:

$$\hat{\pi}(t) = \int b d\nu_t(b), \quad \hat{V}(t) = \int V_b(t) d\nu_t(b).$$

Functionally generated portfolio $\mathcal{FG}$

- Portfolio map:

$$\pi(t) = \boldsymbol{\pi}(\mu(t)), \quad \text{for some } \boldsymbol{\pi} : \Delta_n \to \overline{\Delta}_n$$

- Functionally generated portfolio: $\boldsymbol{\pi}(\cdot)$ is given in terms of the gradient of a generating function $\varphi : \Delta_n \to \mathbb{R}$.
  - Relative arbitrage (i.e., beating the market portfolio w.p.1) under conditions on market stability and volatility
  - Lyapunov function: Karatzas and Ruf (2016)
  - Optimal transport and information geometry: Pal and Wong (2015, 2016)
  - More recent papers: Wong (2015), Vervuurt and Karatzas (2016), Vervuurt and Samo (2016), Pal (2016)

- $\mathcal{FG}$ is convex and contains all constant-weighted portfolios.
  - Brod and Ichiba (2014): Cover's portfolio is 'functionally generated' (answering a question in Fernholz and Karatzas (2009))

UP, FGP and large deviation

Theorem (W. (2016))

Under suitable conditions on $\{\mu(t)\}_{t=0}^{\infty} \subset \Delta_n$ and ν_0 on $\mathcal{FG}$:

(i) *The sequence $\{\nu_t\}_{t=0}^{\infty}$ of wealth distributions on $\mathcal{FG}$ satisfies a pathwise large deviation principle as $t \to \infty$.*

(ii) *Cover's portfolio can be extended to $\mathcal{FG}$:*

$$\hat{\pi}(t) = \int_{\mathcal{FG}} \boldsymbol{\pi}(\mu(t)) d\nu_t(\boldsymbol{\pi})$$

and the following universality property holds:

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\hat{V}(t)}{\max_{\boldsymbol{\pi} \in \mathcal{FG}} V_{\boldsymbol{\pi}}(t)} = 0.$$

Log-optimal/numeraire portfolio

- A stochastic model for $\{\mu(t)\}$ is required.
- In the SPT set-up (relative to the market):

$$\pi_{\mathrm{num}}(t) := \operatorname*{arg\,max}_{b \in \overline{\Delta}_n} \mathbb{E}\left[\log\left(b \cdot \frac{\mu(t+1)}{\mu(t)}\right) \middle| \mathcal{F}_t\right]$$

  - Al-Aradia and Jaimungal (2017): explicit solutions using stochastic control techniques
- If $\{\mu(t)\}$ is a time homogeneous Markov chain, π_{num} can be realized by a portfolio map $\boldsymbol{\pi}_{\mathrm{num}} : \Delta_n \to \overline{\Delta}_n$.
  - Györfi, Lugosi and Udina (2006): universal portfolios assuming stock returns are stationary and ergodic over time

UP for Lipschitz portfolio maps

- For each $M > 0$, let $\mathcal{L}^M$ be the family of M-Lipschitz portfolio maps (with some boundary conditions).
- With topology of uniform convergence, $\mathcal{L}^M$ is compact. Let

$$V^{*,M}(t) := \max_{\boldsymbol{\pi} \in \mathcal{L}^M} V_{\boldsymbol{\pi}}(t).$$

- Consider Cover's portfolio $\hat{V}^M$ over $\mathcal{L}^M$.

Theorem (Cuchiero, Schachermayer and W. (2016))

Assume ν_0 has full support on $\mathcal{L}^M$. Then for every individual sequence $\{\mu(t)\}_{t=0}^{\infty}$ in Δ_n we have

$$\lim_{t \to \infty} \frac{1}{t} \left(\log V^{*,M}(t) - \log \hat{V}^M(t) \right) = 0.$$

Approximating $\boldsymbol{\pi}_{\text{num}}$ by Lipschitz portfolio maps

- Now let $\{\mu(t)\}_{t=0}^{\infty}$ be a time homogeneous ergodic Markov chain on Δ_n with a unique variant measure ρ, such that

$$L := \mathbb{E}_\rho \log \frac{V_{\boldsymbol{\pi}_{\text{num}}}(1)}{V_{\boldsymbol{\pi}_{\text{num}}}(0)} < \infty.$$

- We may construct Cover's portfolio $\hat{V}(t)$ on $\bigcup_{M=1}^{\infty} \mathcal{L}^M$ by splitting the prior over each $\mathcal{L}^M$.

Theorem (Cuchiero, Schachermayer and W. (2016))

It holds almost surely that

$$\lim_{M\to\infty} \lim_{t\to\infty} \frac{1}{t} \log V^{*,M}(t) = \lim_{t\to\infty} \frac{1}{t} \log \hat{V}(t) = \lim_{t\to\infty} \frac{1}{t} \log V_{\boldsymbol{\pi}_{num}}(t) = L.$$

A continuous time analogue

In continuous time, we let $\{\mu(t)\}_{t\geq 0}$ be a continuous semimartingale in Δ_n. For a portfolio process $\{\pi(t)\}_{t\geq 0}$,

$$\frac{dV_\pi(t)}{V_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{d\mu_i(t)}{\mu_i(t)}.$$

- The previous theorem cannot be generalized directly because of stochastic integrals.
- We will restrict to functionally generated portfolios where a pathwise decomposition for $V_{\boldsymbol{\pi}}(t)$ exists.
- To compare with π_{num}, $\{\mu(t)\}_{t\geq 0}$ needs to be a Markov diffusion process with a special structure.

Decomposition for functionally generated portfolio

- Assume $\boldsymbol{\pi}$ is generated by a positive C^2 concave function $\Phi = e^{\varphi}$ on Δ_n. In fact

$$\boldsymbol{\pi}_i(x) = x_i \left(D_i \varphi(x) + 1 - \sum_{j=1}^{n} x_j D_j \varphi(x) \right).$$

- Fernholz's pathwise decomposition:

$$V_{\boldsymbol{\pi}}(t) = V_{\boldsymbol{\pi}}(0) \frac{\Phi(\mu(t))}{\Phi(\mu(0))} e^{\Theta(t)},$$

  where $d\Theta(t) = -\frac{1}{2\Phi(\mu(t))} \sum_{i,j} D_{ij} \Phi(\mu(t)) d[\mu_i, \mu_j]_t$.
- The decomposition can be formulated using Föllmer's Itô calculus (Schied, Speiser and Voloshehenko (2016)).

Analytical considerations

- Analogous to $\mathcal{L}^M$, we define a *compact* Hölder space $\mathcal{FG}^{M,\alpha}$ for $M > 0$ and $0 \leq \alpha \leq 1$.
- Using the pathwise formulation in Schied et al (2016), we can define

$$V^{*,M,\alpha}(t) := \max_{\boldsymbol{\pi} \in \mathcal{FG}^{M,\alpha}} V_{\boldsymbol{\pi}}(t)$$

  and prove the existence of a maximizer, for any continuous path $\{\mu(t)\}_{t \geq 0} \subset \Delta_n$ whose quadratic variation exists.
- Cover's portfolio $\hat{V}^{M,\alpha}(t)$ can be generalized, in continuous time, to $\mathcal{FG}^{M,\alpha}$, and asymptotic universality holds under suitable conditions.

Conditions on the diffusion

We consider diffusions of the form

$$d\mu(t) = c(\mu(t))\lambda(\mu(t))dt + c^{1/2}(\mu(t))dW(t), \quad \text{where}$$

(i) $c(x)$ is positive definite,

(ii) $c(x)\mathbf{1} \equiv \mathbf{0}$,

(iii) $\sum_{i,j} c_{i,j}(x)\lambda_j(x)dx = 0$.

The log-optimal portfolio $\boldsymbol{\pi}_{\text{num}}$ maximizes the instantaneous drift of $\log V_{\boldsymbol{\pi}}(t)$.

Proposition

The log-optimal portfolio $\boldsymbol{\pi}_{num}$ is functionally generated if

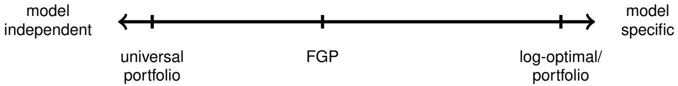
$$\lambda(x) = \nabla \log \Phi(x) \quad \textit{for some function } \Phi.$$

Continuous time result

Theorem (Cuchiero, Schachermayer and W. (2016))

Suppose $\{\mu(t)\}_{t\geq 0}$ *has the above form (with* $\lambda = \nabla \log \Phi$*), is ergodic with a unique invariant measure, and the coefficients satisfy some integrability conditions. Then the universal portfolio* $\hat{V}(t)$ *can be constructed on* $\bigcup_{M=1}^{\infty} \mathcal{FG}^{M,1/M}$*, such that almost surely*

$$\lim_{t\to\infty} \frac{1}{t} \log \hat{V}(t) = \lim_{M\to\infty} \lim_{t\to\infty} \frac{1}{t} \log V^{*,M,1/M}(t) = \lim_{t\to\infty} \frac{1}{t} \log V_{\boldsymbol{\pi}_{num}}(t).$$



Conclusion

Under suitable conditions, we proved asymptotic equivalence of the following portfolios:

- Best retrospectively chosen portfolio map/FGP
- Generalizations of Cover's universal portfolio
- Log-optimal portfolio

Further problems:

- Going beyond FGP in continuous time.
- Computation of these portfolios. Choice of the prior ν_0. Rank-based universal portfolio?
- Other performance measures, e.g. risk-adjusted return.
- Connections with machine learning approaches.

References

- T.-K. L. Wong, Universal portfolios in stochastic portfolio theory, arXiv:1510.02808
- C. Cuchiero, W. Schachermayer and T.-K. L. Wong, Cover's universal portfolio, stochastic portfolio theory and the numeraire portfolio, arXiv:1611.09631