Controlled McKean-Vlasov Equations and Related Master Equations

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Introduction

Consider n-player mean-field game:

$$dX_t^i = b(t, X^i, \mu^n, \alpha_t^i)dt + \sigma(t, X^i, \mu^n, \alpha_t^i)dB_t^i$$

with empirical distribution $\mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^j}$.

Question: What kind of information should the controls α^i use?

Strong formulation:

- α^i depend on the random noise B^i.
- But W is usually unobservable in practice.

Weak formulation:

- α^i depend on state process X^i, which is observable.
- Markov? non-Markov? Since players have freedom to use past information, non-Markov seems more reasonable in prictice.

Two different problems

As $n \to \infty$, we get SDE of McKean-Vlasov type:

$$dX_t = b(t, X_{\cdot\wedge t}, \mu_{[0,t]}, \alpha_t)dt + \sigma(t, X_{\cdot\wedge t}, \mu_{[0,t]}, \alpha_t)dB_t$$

with objective $J(\alpha, \mu) = \mathbb{E}[\int_0^T f(t, X_{\cdot\wedge t}, \mu_{[0,t]}, \alpha_t)dt + g(X, \mu)]$

Mean field game problem:

- Find $\alpha^\star$ so that $J(\alpha^\star, \mu^{\alpha^\star}) = \sup_\alpha J(\alpha, \mu^{\alpha^\star})$
- Fixed point problem: $\mu \to \alpha^\star \to \mu^{\alpha^\star}$
- Use fixed point to find approximate equilibrium of finite-player game (work done by Carmona, Lacker 2015)

Stochastic control problem of McKean-Vlasov type:

- Find $\alpha^\star$ so that $J(\alpha^\star, \mu^{\alpha^\star}) = \sup_\alpha J(\alpha, \mu^\alpha)$
- Non-standard control problem $\to$ our topic

Problem formulation

Let $\Omega := C([0,T],\mathbb{R})$ be endowed the uniform norm $\|\cdot\|_\infty$, $X_t(\omega) := \omega_t, \mathcal{F}_t := \mathcal{F}_t^X$ and

$$\mathcal{P}_2(\Omega) := \left\{\mathbb{P} \in \mathcal{P}(\Omega) \middle| \|X\|_\infty \text{ is square integrable under } \mathbb{P}\right\}$$

By DCT, it easy to see the following characterization:

$$\mu \in \mathcal{P}_2(\Omega) \Leftrightarrow \begin{array}{c} \text{marginal } \mu_t\text{'s are square integrable and} \\ \mathbb{E}^\mu \left\| X_\cdot - \left(\sum_{i=0}^{n-1} X_{t_i} 1_{[t_i,t_{i+1})}(\cdot) + X_T 1_{\{T\}}(\cdot) \right) \right\|_\infty^2 \to 0 \\ \text{as } |p| \to 0, \text{ for } p: 0 = t_0 < t_1 < \cdots < t_n = T \end{array}$$

The Wasserstein metric on $\mathcal{P}_2(\Omega)$ is

$$W_2(\mu,\nu) := \inf_{\overline{\mathbb{P}} \in \Gamma(\mu,\nu)} \left(\mathbb{E}^{\overline{\mathbb{P}}} \|X' - X''\|_\infty^2\right)^{\frac{1}{2}}$$

Let $\mathbb{P}_{[0,t]} := \mathbb{P} \circ (X_{\cdot\wedge t})^{-1}$, $\Lambda := [0,T] \times \mathcal{P}_2(\Omega)$. Define pseudometric on Λ as

$$W_2((t,\mu),(s,\nu)) := \left(|t-s| + W_2(\mu_{[0,t]},\nu_{[0,s]})^2\right)^{\frac{1}{2}}$$

We say function $F : \Lambda \to \mathbb{R}$ is adapted if

$$F(t,\mu) = F(t,\mu_{[0,t]})$$

Control problem

Consider the following simplified version of optimal control problem of McKean-Vlasov type (no dependence on state):

$$V(t,\mu) := \sup_{\alpha \in \mathcal{A}_t} g(\mathbb{P}^{t,\mu,\alpha})$$

where $\mathbb{P}^{t,\mu,\alpha}$ is the law of solution of McKean-Vlasov equation

$$X_s^{t,\mu,\alpha} = \xi_t + \int_t^s b(r, \mathcal{L}_{X_{\cdot\wedge r}^{t,\mu,\alpha}}, \alpha_r)dr + \int_t^s \sigma(r, \mathcal{L}_{X_{\cdot\wedge r}^{t,\mu,\alpha}}, \alpha_r)dB_r \quad (\star)$$

and $X_{\cdot\wedge t} = \xi_{\cdot\wedge t}$ with process $\xi \sim \mu$.

Above SDE is wellposed when the usual Lipschitz condition for b, σ holds and α is any fixed open loop control.

Strong formulation: a filtration issue

When α is open-loop, we are in the strong formulation. There are two choices for admissible controls:

$$\begin{aligned} \text{In } \mathcal{A}_t^1, \qquad & \alpha_s = \alpha(s, (B_r - B_s)_{t \le r \le s}) \\ \text{In } \mathcal{A}_t^2, \qquad & \alpha_s = \alpha(s, (B_r)_{0 \le r \le s}) \end{aligned}$$

On one hand, we cannot establish a weak solution for the master equation from any of $\mathcal{A}^1, \mathcal{A}^2$ alone. On the other hand, we don't know if

$$\sup_{\alpha \in \mathcal{A}_t^1} g(\mathbb{P}^{t,\mu,\alpha}) \stackrel{?}{=} \sup_{\alpha \in \mathcal{A}_t^2} g(\mathbb{P}^{t,\mu,\alpha})$$

Even the fact that the value function $V(t, \mu)$ is well defined under $\mathcal{A}^2$ is nontrivial!

Weak formulation

- When α is closed-loop, SDE $(\star)$ may not be well posed even in the weak sense unless we assume regularity in α.
- Study of wellposedness of weak solution of MKV SDE $(\star)$ is difficult even if α is of feedback form. Existing works always assume no volatility control. See Carmona, Lacker (2015); Li, Hui (2016).
- However, if $\alpha \in \mathbb{F}^X$ is assumed to be piecewise constant, then $(\star)$ is well posed in the strong sense.
- If $\sigma \neq 0$, then $B \in \mathbb{F}^X$. So $X, B, \mathbb{P}^{t,\mu,\alpha}$ can be constructed on $\Omega = C([0, T])$.

Motivating example

Warning: When set $\mathcal{A}_t$ is too small, continuity of V may fail.

For example, let

$$dX_t = (1 + \alpha_t^2 \wedge 1)dB_t, \quad g(\mathcal{L}_X) = \frac{1}{3}\mathbb{E}[X_1^4] - (\mathbb{E}[X_1^2])^2,$$

and $\mathcal{A}_t$ consists only of constant controls, then

$$\varliminf_{\varepsilon \to 0} V(0, \frac{1}{2}(\delta_\varepsilon + \delta_{-\varepsilon})) \geq \frac{9}{4} \neq 0 = V(0, \delta_0)$$

Main results

Assumption

b, σ, g are bounded, (uniformly) Lipschitz continuous, and $\sigma > 0$.

Theorem

Under the above Assumption and let

$$\mathcal{A}_t := \left\{ \alpha \middle| \alpha_s(X) = \sum_{i=0}^{n-1} h_i(X_{[0,t_i]}) 1_{[t_i,t_{i+1})}(s), h_i\text{'s are bdd. meas.} \right\},$$

then $V(t,\mu)$ is Lipschitz continuous in μ, uniformly in t, under W_2.

Note it is implicit that functions h_i could also depend on the (deterministic) law of $X_{[0,t_i]}$.

Technical lemma

Lemma

Fix $\mu, \nu \in \mathcal{P}_2, \pi \in \Gamma(\mu, \nu)$. For any $\varepsilon, \delta > 0$ and process $(\eta_s)_{0 \le s \le T}$ defined on a rich enough probability space with $\mathcal{L}_\eta = \nu$ and partition $0 \le t_1 < \cdots < t_m \le T$, there exist another process $(\xi_s)_{0 \le s \le T}$ and Brownian motion $(B_s)_{0 \le s \le \delta}$ such that: (i) $\mathcal{L}_\xi = \mu$, (ii) $\eta \perp B$, (iii) $\xi_{t_i} \in \sigma(\eta_{t_1, \cdots, t_m}, B_{[0,\delta]})$ and (iv)

$$W_2(\mathcal{L}_{\xi_{t_1,\cdots,t_m}}, \mathcal{L}_{\eta_{t_1,\cdots,t_m}}) \le \left(\int_{\Omega \times \Omega} \max_j |\omega'_{t_j} - \omega''_{t_j}|^2 d\pi(\omega', \omega'') \right)^{\frac{1}{2}} + \varepsilon.$$

- The key part of this result is (iii); otherwise it is trivial.
- This result relies on the fact that any random vector can be constructed from i.i.d. $U(0,1)$ random variables and its multivariate distribution function.

Change admissible controls

Proposition

All the following four cases define the same value function $V(t,\mu)$:

(i) $\alpha_s(X) = h_i(X_{[0,t_i]})$, *$h_i$'s are bounded measurable;*
(ii) $\alpha_s(X) = h_i(X_{[0,t_i]})$, *$h_i$'s are bounded continuous;*
(iii) $\alpha_s(X) = h_i(X_{s_1,\cdots,s_m}, X_{[t,t_i]})$, *$h_i$'s are bounded measurable;*
(iv) $\alpha_s(X) = h_i(X_{s_1,\cdots,s_m}, X_{[t,t_i]})$, *$h_i$'s are bounded continuous;*

Under weak formulation, DPP follows quite easily.

Theorem (Dynamic Programming Principle)

$$V(t,\mu) = \sup_{\alpha \in \mathcal{A}_t} V(s, \mathbb{P}^{t,\mu,\alpha}), \quad \forall s > t$$

By DPP, we immediately see that

Proposition

Value function $V : \Lambda \to \mathbb{R}$ is Lipschitz continuous under W_2.

Question: What kind of master equation is satisfied by this continuous value function on $[0, T] \times \mathcal{P}_2(\Omega)$?

Differentiation

Right now it is still not known to us how to define the proper derivatives in Λ, but here are several ideas from earlier works.

- In the Markovian case, V becomes a function on $[0, T] \times \mathcal{P}_2(\mathbb{R})$ and P.L. Lions studied how to define derivatives $\partial_\mu V$ through Frechét derivative of lifted function $\widetilde{V}$ on $L^2(\Omega; \mathbb{R})$. It turns out $D\widetilde{V}(t, \xi) = h(t, \mu, \xi), \xi \sim \mu$ for some deterministic function h defined on $[0, T] \times \mathcal{P}_2(\mathbb{R}) \times \mathbb{R}$. Generalized Itô's formula in this case was also proved by Carmona, Delarue (2014); Chassagneux, Crisan, Delarue (2014).

  Question: Does this form generalize to non-Markovian case?
- Since our value function V satisfies the adaptedness property, it's also possible for us to borrow the idea of Functional Itô Calculus.

Generalized Itô's formula: Markovian case

Fact: Most functions on $L^2(\Omega;\mathbb{R})$ are not twice Frechét differentiable, e.g. $f(X) = \mathbb{E}[\sin(X)]$, then $Df(X) = \cos(X)$ is not Frechét differentiable. But for Itô's formula, we only need directional derivatives (i.e. Gâteaux derivative) to exist.

Theorem (Itô's formula)

Suppose $dX_t = b_t dt + \sigma_t dB_t$ *such that* $E[\int_0^T |b_t|^2 + |\sigma_t|^4 dt] \leq \infty$ *and* $f \in C_b^2(\mathcal{P}_2(\mathbb{R}))$, *then*

$$f(\mathcal{L}_{X_t}) = f(\mathcal{L}_{X_0}) + \int_0^t \mathbb{E}\Big[\partial_\mu f(\mathcal{L}_{X_s}, X_s) b_s + \frac{1}{2}\partial_x \partial_\mu f(\mathcal{L}_{X_s}, X_s)\sigma_s^2\Big] ds$$

Note that derivative $\partial_\mu \partial_\mu f$ is not involved in above formula, so we only need "partial regularity" on f, see Chassagneux, Crisan, Delarue (2014).

A more general setting

Suppose function $f : L^2(\mathcal{F}_T^B) \to \mathbb{R}$, B is B.M.

- This generalizes the previous case, e.g. $f(\xi) = \mathbb{E}[\phi(\xi, B_\cdot)]$ for some functional ϕ.
- Another example is from BSDE: $f(\xi) = Y_0^\xi$, where Y^ξ is the solution of a BSDE with terminal value ξ.
- In fact, all functions $f(\xi)$ on $L^2(\mathcal{F}_T^B)$ are of the form $h(\mathcal{L}_{(\xi, B_\cdot)})$ for some function $h : \mathcal{P}_2(\mathbb{R} \times \Omega) \to \mathbb{R}$. This is because same information is provided by random variable ξ or measure $\mathcal{L}_{(\xi, B_\cdot)}$.

  (1) Given ξ, $\mathbb{P}(\xi \in A, B \in A') = \mathbb{P}(B \in A' \cap \xi^{-1}(A))$ $= \mathbb{P}_0(A' \cap \xi^{-1}(A))$

  (2) Given $\mathcal{L}_{(\xi, B_\cdot)}$, we can define the r.c.p.d. $\lambda : \Omega \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ for ξ given $\mathcal{F}_T^B$, and ξ is determined by relation $\delta_{\xi(\omega)} = \lambda(\omega, \cdot) \in \mathcal{P}(\mathbb{R})$

Differentiation: revisited

For $f : L^2(\mathcal{F}_T^B) \to \mathbb{R}$, we know that $Df(\xi) \in L^2(\mathcal{F}_T^B)$, so $\exists\, h$ such that $Df(\xi) = h(B_\cdot)$. However, this function h may depend on ξ itself. For example,

- $f(\xi) = \mathbb{E}[\xi^2] \;\Rightarrow\; Df(\xi) = 2\xi = 2\xi(B_\cdot)$
- $f(\xi) = (\mathbb{E}[\xi])^2 \;\Rightarrow\; Df(\xi) = 2\mathbb{E}[\xi]$, which is independent of B.

We see that h could depend on ξ in at least two ways (through distribution $\mathcal{L}_{(\xi,B_\cdot)}$ or composition $\xi(B_\cdot)$). In general, there always is a deterministic function $\phi : L^2(\mathcal{F}_T^B) \times \Omega \to \mathbb{R}$ which is independent of ξ, such that $Df(\xi) = \phi(\xi(\cdot), B_\cdot), \forall \xi$. So we can define $\partial_\xi f := \phi$.

- But this form of $\partial_\xi f$ is not satisfactory if we need to define higher order derivatives.

  e.g. $\partial_\xi f(\xi(\cdot), B_\cdot) = \xi(B_\cdot)$, how to define $\partial_\xi(\partial_\xi f)$?

Differentiation: revisited

So we prefer to define $\partial_\xi f$ as function

$$\begin{aligned} \phi : L^2(\mathcal{F}_T^B) \times \mathbb{R} \times \Omega &\to \mathbb{R} \\ (\xi(\cdot), x, \omega) &\mapsto \phi(\xi(\cdot), x, \omega) \end{aligned}$$

so that $Df(\xi) = \phi(\xi(\cdot), \xi, B.)$ and dependence of $Df(\xi)$ on $\xi(B.)$ is only from the second argument and no randomness comes from the first argument. We say f is differentiable when such a function ϕ exists and call it $\partial_\xi f$.

Question: How to prove that $\partial_\xi f$ is uniquely defined by above process?

Generalized Itô's formula: revisited

Suppose $X \in \mathbb{F}^B$ satisfies $dX_t = b_t dt + \sigma_t dB_t$ with b, σ bounded and $f : L^2(\mathcal{F}_T^B) \to \mathbb{R}$ is bounded and smooth enough, then

$$
\begin{aligned}
f(X_t(\cdot)) = f(X_0(\cdot)) + \int_0^t \mathbb{E}\Big[&\partial_\xi f(X_s(\cdot), X_s, B_\cdot) b_s \\
&+ \partial_\omega \partial_\xi f(X_s(\cdot), X_s, B_\cdot)\sigma_s + \frac{1}{2}\partial_x \partial_\xi f(X_s(\cdot), X_s, B_\cdot)\sigma_s^2\Big] ds
\end{aligned}
$$

- The derivatives above are not adapted, even though we could introduce an adapted version by using conditional expectation.
- The additional path derivative term vanishes in the Markovian case.
- These evidence suggest ∂_ω term should be expected when taking derivatives of function on $\Lambda = [0, T] \times \mathcal{P}_2(\Omega)$, which agrees with functional Itô's formula.

Suppose we are given the generalized Itô's formula for smooth functions on Λ, and the corresponding master equation $\mathbb{L}V(t,\mu) = 0$. Let

$$\begin{aligned}\mathcal{P}^L_t(\mu) &:= \{\mathbb{P} \in \mathcal{P} \,|\, \mathbb{P}_{[0,t]} = \mu_{[0,t]}, X \text{ is a } \mathbb{P}\text{-semimartingale} \\ &\qquad\qquad \text{on } [t,T] \text{ with drift and diffusion bounded by } L\} \\ \overline{\mathcal{A}}_L V(t,\mu) &:= \{\Phi \in C^{1,2}(\Lambda) \,|\, \exists \delta > 0, \text{ s.t. } [\Phi - V](t,\mu) = 0 \\ &\qquad\qquad = \sup_{t \le s \le t+\delta} \sup_{\mathbb{P} \in \mathcal{P}^L_t(\mu)} [\Phi - V](s,\mathbb{P})\}\end{aligned}$$

Definition

For $V \in C^0(\Lambda)$, we say it is a
(i) viscosity L-subsolution if $\mathbb{L}\Phi(t,\mu) \ge 0$ for any $(t,\mu) \in \Lambda$ and $\Phi \in \overline{\mathcal{A}}_L V(t,\mu)$;
(ii) viscosity solution if it is both a L-viscosity subsolution and L-viscosity supersolution for some $L > 0$.

Main results (Markovian case)

In the Markovian case (where Itô's formula is known), we can prove the following

Theorem

(i) The value function V defined earlier is a viscosity solution of the master equation.
(ii) Partial comparison: Let V_1 be a viscosity subsolution and V_2 a visocosity supersolution and $V_1(T,\cdot) \leq V_2(T,\cdot)$. If one of V_1, V_2 is in $C^{1,2}(\Lambda)$, then $V_1 \leq V_2$.

Future works

- Stability and comparison principle, and hence uniqueness
- General result under non-Markovian framework
- Classical solutions

Thank you!