Mean-field and n-agent games for optimal investment under relative performance criteria

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Portfolio management under competition and asset specialization

References

- Mean-field and n-agent games for optimal investment under relative performance criteria (with Dan Lacker)

- Relative forward performance criteria: passive and competitive cases, under asset specialization and diversification (with Tianran Geng)

Competition among fund manangers in mutual and hedge funds

Chevalier and Ellison (1997)
Sirri and Tufano (1998)
Agarwal, Daniel and Naik (2004)
Ding, Getmansky, Liang and Wermers (2007)
Goriaev et al. (2003)
Li and Tiwari (2006)
Gallaher, Kaniel and Starks(2006)
Brown, Goetzmann and Park(2001)
Kempf and Ruenzi (2008)
Basak and Makarov (2013, 2016)
Espinoza and Touzi (2013), ...

Career advancement motives, seeking higher money inflows from their clients, preferential compensation contracts,...
Only two managers, mainly discrete-time models, criteria involving risk neutrality, relative performance with respect to an absolute benchmark or a critical threshhold, constraints on the managers' risk aversion parameters, ...

Asset specialization for fund managers

Brennan (1975)
Merton (1987)
Coval and Moskowitz (1999)
Karperczyk, Sialm and Zheng (2005)
van Nieuwerburgh and Veldkamp (2009, 2010)
Uppal and Wang (2003)
Boyle, Garlappi, Uppal and Wang (2012)
Liu (2012)
Mitton and Vorkink (2007), ...

Familiarity, learning cost reduction, ambiguity aversion, solvency requirements, trading costs and constraints, liquidation risks, informational frictions, ...

The n-player game

The competition setting

n fund managers

- common investment horizon $[0, T]$
- common riskless asset (bond)
- asset specialization
- individual stock S^i, $i = 1, ..., n$

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \nu_i dW_t^i + \sigma_i dB_t,$$

$\mu_i > 0$, $\sigma_i \geq 0$, and $\nu_i \geq 0$, $\sigma_i + \nu_i > 0$
$(\widetilde{W}_t^n)_{t \in [0,T]} := (B_t, W_t^1, \ldots, W_t^n)_{t \in [0,T]}$ is an $(n+1)$-dim. BM

- B common noise and W^i an indiosyncratic noise

Special case

Single stock

- Coefficients $(\mu_i, \sigma_i) = (\mu, \sigma)$, and $\nu_i = 0$, $i = 1, ..., n$
- All stocks are identical
- Managers invest in identical markets
- Managers differ only in their risk preferences

  and personal competition concerns

Policies and wealth processes

i^{th} fund manager, $i = 1, ..., n$

- Uses self-financing portfolios π^i (other usual admissibility conds)
- Trades in $[0, T]$
- Has wealth process X^i

$$dX_t^i = \pi_t^i(\mu_i dt + \nu_i dW_t^i + \sigma_i dB_t)$$

- W^i : indiosyncratic noise
- B : common noise

Utility under competition

- Utility function $U_i : \mathbb{R}^2 \to \mathbb{R}$ depends on both her individual wealth x, and the average wealth of all investors , m,

$$U_i(x, m) := -\exp\left(-\frac{1}{\delta_i}\left(x - \theta_i m\right)\right)$$

- $\delta_i > 0$ is the personal risk tolerance
- $\theta_i \in [0, 1]$ as the personal social comparison parameter
- $\theta_i = 0$ means no relative concerns
- Both δ_i, θ_i are unitless quantities

Expected utility under competition

- Fund managers choose admissible strategies $\pi_t^1, \dots, \pi_t^n$, $t \in [0, T]$
- The payoff for investor i is given by

$$J_i(\pi^1, \dots, \pi^n) := \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(X_T^i - \theta_i \bar{X}_T\right)\right)\right]$$

- Average wealth of the managers' population

$$\bar{X}_T = \frac{1}{n}\sum_{k=1}^{n} X_T^k$$

- Alternatively,

$$J_i(\pi^1, \dots, \pi^n) = \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left((1-\theta_i)X_T^i + \theta_i(X_T^i - \bar{X}_T)\right)\right)\right]$$

- X_T^i: personal, absolute wealth
- $X_T^i - \bar{X}_T$: personal, relative to the population, wealth

Nash equilibrium

Nash equilibrium

- A vector $(\pi^{1,*}, \ldots, \pi^{n,*})$ of admissible strategies is a Nash equilibrium if, for all admissible $\pi^i \in \mathcal{A}$ and $i = 1, \ldots, n$,

$$J_i(\pi^{1,*}, \ldots, \pi^{i,*}, \ldots, \pi^{n,*}) \geq J_i(\pi^{1,*}, \ldots, \pi^{i-1,*}, \pi^i, \pi^{i+1,*}, \ldots, \pi^{n,*})$$

- A constant Nash equilibrium is one in which, for each i, $\pi^{i,*}$ is constant in time, i.e.,

$$\pi_t^{i,*} = \pi_0^{i,*}, \quad \text{for all} \quad t \in [0, T]$$

- A constant Nash equilibrium is thus a vector

$$\pi^* = (\pi^{1,*}, \ldots, \pi^{n,*}) \in \mathbb{R}^n$$

Construction of Nash equilibria

Main result

- $\delta_i > 0, \theta_i \in [0, 1]$
- $\mu_i > 0, \sigma_i \geq 0, \nu_i \geq 0$, and $\sigma_i + \nu_i > 0$
- Define the constants

$$\varphi_n := \frac{1}{n} \sum_{i=1}^{n} \delta_i \frac{\mu_i \sigma_i}{\sigma_i^2 + \nu_i^2 (1 - \theta_i/n)} \quad \text{and} \quad \psi_n := \frac{1}{n} \sum_{i=1}^{n} \theta_i \frac{\sigma_i^2}{\sigma_i^2 + \nu_i^2 (1 - \theta_i/n)}$$

Nash equilibria

- If $\psi_n < 1$, there exists a unique constant equilibrium, given by

$$\pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2 (1 - \theta_i/n)} + \theta_i \frac{\sigma_i}{\sigma_i^2 + \nu_i^2 (1 - \theta_i/n)} \frac{\varphi_n}{1 - \psi_n}$$

- If $\psi_n = 1$, there is no constant equilibrium

Main steps in the proof

- Fix i and assume that all other k^{th} agents, $k \neq i$, follow constant investment strategies, $\alpha_k \in \mathbb{R}$
- Competitor's wealth X_t^k,

$$X_t^k = x_0^k + \alpha_k \left(\mu_k t + \nu_k W_t^k + \sigma_k B_t \right)$$

- Competitors' aggregate wealth

$$Y_t := \frac{1}{n} \sum_{k \neq i} X_t^k$$

- The i^{th} fund manager solves the optimization problem

$$\sup_{\pi \in \mathcal{A}} \mathbb{E} \left[-\exp\left(-\frac{1}{\delta_i} \left(\left(1 - \frac{\theta_i}{n} \right) X_T^i - \theta_i Y_T \right) \right) \Big| X_0 = x_0^i, Y_0 = \frac{1}{n} \sum_{k \neq i} x_0^k \right]$$

with

$$dX_t = \pi_t (\mu_i dt + \nu_i dW_t^i + \sigma_i dB_t),$$
$$dY_t = \widehat{\mu\alpha} dt + \widehat{\nu\alpha} dW_t^k + \widehat{\sigma\alpha} dB_t$$

$$\widehat{\mu\alpha} := \frac{1}{n} \sum_{k \neq i} \mu_k \alpha_k, \ \ \widehat{\nu\alpha} := \frac{1}{n} \sum_{k \neq i} \nu_k \alpha_k \ \ \text{and} \ \ \widehat{\sigma\alpha} := \frac{1}{n} \sum_{k \neq i} \sigma_k \alpha_k$$

Connection with indifference valuation

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_i}\left(\left(1-\frac{\theta_i}{n}\right)X_T^i - \theta_i Y_T\right)\right) \middle| X_0 = x_0^i, Y_0 = \frac{1}{n}\sum_{k\neq i} x_0^k \right]$$

The i^{th} fund manager $\rightarrow$ writer of liability $G(Y_T) := \frac{\theta_i}{1-\theta_i/n} Y_T$,

Risk aversion $\gamma_i := \frac{1}{\delta_i}\left(1-\frac{\theta_i}{n}\right)$

Thus, the above supremum is equal to $v(X_0, Y_0, 0)$, with $v(x,y,t)$ solving the HJB eqn

$$v_t + \max_{\pi\in\mathbb{R}}\left(\frac{1}{2}(\sigma_i^2+\nu_i^2)\pi^2 v_{xx} + \pi\left(\mu_i v_x + \sigma_i \widehat{\sigma\alpha} v_{xy}\right)\right)$$
$$+\frac{1}{2}\left(\widehat{\sigma\alpha}^2 + \frac{1}{n}\widehat{(\nu\alpha)^2}\right) v_{yy} + \widehat{\mu\alpha} v_y = 0,$$

for $(x,y,t)\in\mathbb{R}\times\mathbb{R}\times[0,T]$, and $\widehat{(\nu\alpha)^2} := \frac{1}{n}\sum_{k\neq i}\nu_k^2\alpha_k^2$,

$$v(x,y,T) = -e^{-\gamma^i(x-G(y))} = -\exp\left(-\frac{1}{\delta_i}\left(\left(1-\frac{\theta_i}{n}\right)x - \theta_i y\right)\right)$$

Candidate Nash equilibria

- The i^{th} agent's optimal feedback control

$$\pi^{i,*}(x,y,t) := -\frac{\mu_i v_x(x,y,t)}{(\sigma_i^2+\nu_i^2)v_{xx}(x,y,t)} - \frac{\sigma_i \widehat{\sigma\alpha} v_{xy}(x,y,t)}{(\sigma_i^2+\nu_i^2)v_{xx}(x,y,t)}$$

- The HJB equation admits separable solutions

$$v\left(x,y,t\right) = -e^{-\gamma_i x} F(y,t)$$

- It then turns out that the optimal policy is of the form

$$\pi^{i,*} = \frac{\delta_i \mu_i}{(\sigma_i^2+\nu_i^2)(1-\theta_i/n)} + \frac{\theta_i \sigma_i}{(\sigma_i^2+\nu_i^2)(1-\theta_i/n)}\widehat{\sigma\alpha}$$

Construction of Nash equilibria

- For a candidate portfolio vector $(\alpha_1, \ldots, \alpha_n)$ to be a Nash equilibrium, we need $\pi^{i,*} = \alpha_i$, $i = 1, ..., n$

$$a_i = \frac{\delta_i \mu_i}{(\sigma_i^2 + \nu_i^2)(1 - \theta_i/n)} + \frac{\theta_i \sigma_i}{(\sigma_i^2 + \nu_i^2)(1 - \theta_i/n)} \widehat{\sigma\alpha}$$

- Set

$$\overline{\sigma\alpha} := \frac{1}{n} \sum_{k=1}^{n} \sigma_k \alpha_k = \widehat{\sigma\alpha} + \frac{1}{n} \sigma_i \alpha_i$$

- Then, we must have

$$\alpha_i = \pi^{i,*} = \frac{\delta_i \mu_i + \sigma_i \theta_i \overline{\sigma\alpha}}{(\sigma_i^2 + \nu_i^2)(1 - \theta_i/n)} - \frac{\theta_i \sigma_i^2}{n(\sigma_i^2 + \nu_i^2)(1 - \theta_i/n)} \alpha_i,$$

and

$$a_i = \frac{\delta_i \mu_i}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)} + \frac{\sigma_i \theta_i}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)} \overline{\sigma\alpha}$$

Construction of Nash equilibria (cont.)

$$a_i = \frac{\delta_i \mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} + \frac{\sigma_i \theta_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \overline{\sigma\alpha}$$

$$\overline{\sigma\alpha} := \frac{1}{n} \sum_{k=1}^{n} \sigma_k \alpha_k = \widehat{\sigma\alpha} + \frac{1}{n} \sigma_i \alpha_i$$

Multiplying both sides by σ_i and then averaging over $i = 1, \ldots, n$, gives

$$\overline{\sigma\alpha} = \varphi_n + \psi_n \overline{\sigma\alpha}$$

$$\varphi_n := \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \delta_i \frac{\sigma_i \mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \quad \text{and} \quad \psi_n := \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \theta_i \frac{\sigma_i^2}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)}$$

- Existence
- Uniqueness

Existence of Nash equilibria

$$a_i = \frac{\delta_i \mu_i}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)} + \frac{\sigma_i \theta_i}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)} \overline{\sigma\alpha}$$

$$\overline{\sigma\alpha} = \varphi_n + \psi_n \overline{\sigma\alpha}$$

$$\varphi_n := \frac{1}{n} \sum_{i=1}^n \delta_i \frac{\sigma_i \mu_i}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)} \quad \text{and} \quad \psi_n := \frac{1}{n} \sum_{i=1}^n \theta_i \frac{\sigma_i^2}{\sigma_i^2 + \nu_i^2(1 - \theta_i/n)}$$

- If $\psi_n < 1$, then $\overline{\sigma\alpha} = \varphi_n / (1 - \psi_n)$, and Nash equilibrium exists
- If $\psi_n = 1$ and $\varphi_n > 0$, eqn has no solution; no constant equilibria exist
- If $\psi_n = 1$ and $\varphi_n = 0$, eqn has infinitely many solutions, but this case not feasible

Uniqueness of smooth solutions to the HJB equation

Recall that the candidate Nash equilibria were constructed from the smooth solutions of the HJB eqn

$$v_t + \max_{\pi \in \mathbb{R}} \left(\frac{1}{2}(\sigma_i^2 + \nu_i^2)\pi^2 v_{xx} + \pi \left(\mu_i v_x + \sigma_i \widehat{\sigma\alpha} v_{xy} \right) \right)$$

$$+ \frac{1}{2}\left(\widehat{\sigma\alpha}^2 + \frac{1}{n}\widehat{(\nu\alpha)^2} \right) v_{yy} + \widehat{\mu\alpha} v_y = 0,$$

$$v(x,y,T) = -e^{-\gamma^i (x - G(y))} = -\exp\left(-\frac{1}{\delta_i}\left(\left(1 - \frac{\theta_i}{n} \right) x - \theta_i y \right) \right)$$

This equation has a unique smooth solution that is strictly concave and strictly increasing in x (Duffie et al. (1996), Musiela and Z. (2002))

Discussion on Nash equilibrium

$$\pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} + \theta_i \frac{\sigma_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \frac{\varphi_n}{1-\psi_n}$$

$\varphi_n := \frac{1}{n}\sum_{i=1}^n \delta_i \frac{\sigma_i \mu_i}{\sigma_i^2+\nu_i^2(1-\theta_i/n)}$ and $\psi_n := \frac{1}{n}\sum_{i=1}^n \theta_i \frac{\sigma_i^2}{\sigma_i^2+\nu_i^2(1-\theta_i/n)}$

Then, it turns out that

$$\pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} + \theta_i \frac{\sigma_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \frac{1}{n}\sum_{k=1}^n \sigma_k \pi^{k,*}$$

Thus, there is a "myopic" Merton-type component
and an average of weighted by the common-noise volatilities aggregate
Nash allocations

Discussion on Nash equilibrium (cont.)

$$\pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} + \theta_i \frac{\sigma_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \frac{1}{n} \sum_{k=1}^{n} \sigma_k \pi^{k,*}$$

- The myopic portfolio component dominates the no-competition one, which is $\delta_i \frac{\mu_i}{\sigma_i^2+\nu_i^2}$
- Competition always results in higher stock allocation
- No competition, $\theta_i = 0$: $\pi^{i,*} \to \delta_i \frac{\mu_i}{\sigma_i^2+\nu_i^2}$
- No common noise, $\sigma_i = 0$: $\pi^{i,*} = \tilde{\delta}_i \frac{\mu_i}{\nu_i^2(1-\theta_i/n)}, \;\; \tilde{\delta}_i := \frac{\delta_i}{(1-\theta_i/n)}$
- Nash policy $\pi^{i,*}$ is strictly increasing in δ_i and θ_i

Single common stock

- For all $i = 1, \ldots, n$, $\mu_i = \mu > 0$, $\sigma_i = \sigma > 0$, and $\nu_i = 0$
- Define the "representative" risk tolerance and social comparison parameters

$$\bar{\delta} := \frac{1}{n}\sum_{i=1}^{n}\delta_i \qquad \text{and} \qquad \bar{\theta} := \frac{1}{n}\sum_{i=1}^{n}\theta_i$$

- If $\bar{\theta} < 1$, there exists a unique constant equilibrium, given by

$$\pi^{i,*} = \left(\delta_i + \theta_i \frac{\bar{\delta}}{1-\bar{\theta}}\right)\frac{\mu}{\sigma^2} = \delta^{ef}\frac{\mu}{\sigma^2}$$

- If $\bar{\theta} = 1$, there is no constant equilibrium

All managers use myopic Merton portfolio with effective risk tolerance

$$\delta^{ef} := \delta_i + \theta_i \frac{\bar{\delta}}{1-\bar{\theta}} = \frac{\delta_i}{1-\bar{\theta}} + \frac{\theta_i\bar{\delta} - \delta_i\bar{\theta}}{1-\bar{\theta}}$$

Passing to the limit as $n \uparrow \infty$

The mean field game under CARA risk preferences

Passing to the limit as $n \uparrow \infty$

- each manager has her own type vector $\zeta_i := (x_0^i, \delta_i, \theta_i, \mu_i, \nu_i, \sigma_i)$, $i = 1, ..., n$
- these vectors induce an empirical measure: type distribution m_n
- type space : $\mathcal{Z}^e := \mathbb{R} \times (0, \infty) \times [0, 1] \times (0, \infty) \times [0, \infty) \times [0, \infty)$
- type measure

$$m_n(A) = \frac{1}{n} \sum_{i=1}^{n} 1_A(\zeta_i), \quad \text{for Borel sets} \quad A \subset \mathcal{Z}^e$$

- Recall each agent's Nash equilibrium strategy $\pi^{i,*}$,

$$\pi^{i,*} = \delta_i \frac{\mu_i}{{\sigma_i}^2 + {\nu_i}^2(1 - \theta_i/n)} + \theta_i \frac{\sigma_i}{{\sigma_i}^2 + {\nu_i}^2(1 - \theta_i/n)} \frac{\varphi_n}{1 - \psi_n},$$

$$\varphi_n := \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \delta_i \frac{\sigma_i \mu_i}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)} \quad \text{and} \quad \psi_n := \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \theta_i \frac{\sigma_i^2}{\sigma_i^2 + \nu_i^2(1-\theta_i/n)}$$

- Thus, $\pi^{i,*}$ depends only on ζ_i and φ_n, ψ_n, which essentially "aggregate" over all managers' type vectors

Defining the mean field game

- Assume that as $n \uparrow \infty$,

the empirical measure m_n has a weak limit m

- Let $\zeta = (\xi, \delta, \theta, \mu, \nu, \sigma)$ be a random variable with this limiting distribution m

- Then, the Nash strategy $\pi^{i,*}$ "should" converge to

$$\lim_{n\to\infty} \pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2} + \theta_i \frac{\sigma_i}{\sigma_i^2 + \nu_i^2} \frac{\varphi}{1 - \psi}$$

where

$\varphi := \lim_{n\to\infty} \varphi_n = \mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2+\nu^2}\right]$ and $\psi := \lim_{n\to\infty} \psi_n = \mathbb{E}\left[\theta \frac{\sigma^2}{\sigma^2+\nu^2}\right]$

Formulating the mean field game

Continum of managers $\longleftrightarrow$ Representative manager

- A game with a continuum of agents with type distribution m
- A single representative agent , randomly selected from the population
- This representative agent's type is a random variable with law m
- Heuristically, each manager in the continuum trades in a single stock driven by two Brownian Motions , one of which is unique to this agent while the other is common to all agents

Formulating the mean field game (cont.)

- The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supports (B, W) , independent BMs
- It also supports the type vector, a random variable in $\mathcal{Z}^e$,

$$\zeta = (\xi, \delta, \theta, \mu, \nu, \sigma)$$

- Its distribution is the type distribution
- Independence of ζ from (B, W)
- $\mathbb{F}^{\mathrm{MF}} = (\mathcal{F}_t^{\mathrm{MF}})_{t \in [0,T]}$ smallest filtration such that, ζ is $\mathcal{F}_0^{\mathrm{MF}}$-mble, and (B,W) adapted
- $\mathbb{F}^B = (\mathcal{F}_t^B)_{t \in [0,T]}$, natural filtration generated by the common noise B

The investment problem of the representative manager

The representative agent's wealth process

$$dX_t = \pi_t(\mu dt + \nu dW_t + \sigma dB_t), \qquad X_0 = \xi$$

$\pi \in \mathcal{A}_{\mathrm{MF}}$, self-financing, $\mathbb{F}^{\mathrm{MF}}$-prog. mble, $\mathbb{E}\int_0^T |\pi_t|^2 dt < \infty$

The type vector $\zeta = (\xi, \delta, \theta, \mu, \nu, \sigma)$ provides the random variables ξ (initial wealth), (μ, ν, σ) (market parameters) and (δ, θ) (personal risk preference and competition parameters)

Special case - a single stock

The vector (μ, ν, σ) is nonrandom , with $\nu = 0$, $\mu > 0$, and $\sigma > 0$
The continum of managers trades in the same market environment, randomness comes only from the distinct personal characteristics (δ, θ)

Defining the MFG

- Recall that in the n-player game, we first solved the investment problem faced by each single manager i, taking the strategies of the other agents $k \neq i$ as fixed.
- The ith agent faced a "liability" $Y_T \leftrightarrow \frac{1}{n}\sum_{k \neq i} X_T^k$, effectively the only source of agents' interaction
- We could had kept this average Y_T as constant instead
- Now take $\overline{X}$ a given random variable , representing the average wealth of the continuum of agents
- The representative agent has no influence on $\overline{X}$, as but one agent amid a continuum
- Then, this objective becomes to maximize the expected payoff

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta}\left(X_T - \theta\overline{X}\right)\right)\right]$$

Definition of the MFG

- For any $\pi^* \in \mathcal{A}_{\mathrm{MF}}$, consider the $\mathcal{F}_T^B$-mble random variable

$$\overline{X} := \mathbb{E}[X_T^* \,|\, \mathcal{F}_T^B],$$

  where $(X_t^*)_{t\in[0,T]}$ is the wealth process corresponding to this investment strategy π^*

- Then, π^* is a mean field equilibrium (MFE) if π^* is optimal for the optimization problem

$$\sup_{\pi\in\mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta}\left(X_T - \theta\overline{X}\right)\right)\right]$$

- A constant MFE is a MFE π^* which is constant in time, i.e., $\pi_t^* = \pi_0^*$ for all $t \in [0, T]$

- Essentially, a constant MFE π^* is the $\mathcal{F}_0^{\mathrm{MF}}$-mble random variable π_0^*

Solving the mean field game

- A MFE is computed as a fixed point
- Start with a generic $\mathcal{F}_T^B$-mble random variable $\overline{X}$, solve

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta}\left(X_T - \theta \overline{X}\right)\right)\right],$$

  find an optimal π^*, and then compute $\mathbb{E}[X_T^* \,|\, \mathcal{F}_T^B]$
- If the consistency condition , $\mathbb{E}[X_T^* \,|\, \mathcal{F}_T^B] = \overline{X}$, holds, then π^* is a MFE
- Intuitively, every agent in the continuum faces an independent noise W , an independent type vector ζ , and the same common noise B

  Therefore, conditionally on B, all agents face i.i.d. copies of the same optimization problem
- Heuristically, the law of large numbers suggests that the average terminal wealth of the whole population should be $\mathbb{E}[X_T^* \,|\, \mathcal{F}_T^B]$
- For example, if $\sigma \equiv 0$ a.s., (no common noise term), then $\overline{X} = \mathbb{E}[X_T^*]$
- Carmona-Delarue-Lacker, Lacker, Cardaliaguet, Sun, etc.

An alternative formulation of the mean field game

- Recall that the sources of randomness are (ζ, B, W), with $B \longleftrightarrow$ common noise
- For a fixed $\mathcal{F}^B$-mble rv $\overline{X}$,

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-e^{-\frac{1}{\delta}\left(X_T^\pi - \theta \overline{X}\right)}\right] = \mathbb{E}\left[u(\zeta)\right],$$

where $u(\cdot)$ is the value function for (deterministic) elements $\zeta_0 = (x_0, \delta_0, \theta_0, \mu_0, \nu_0, \sigma_0)$ of the type space $\mathcal{Z}^e$,

$$u(\zeta_0) := \sup_{\pi} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_0}\left(\widetilde{X}_T^{\zeta_0,\pi} - \theta_0 \overline{X}\right)\right)\right],$$

with $d\widetilde{X}_t^{\zeta_0,\pi} = \pi_t\left(\mu_0 dt + \nu_0 dW_t + \sigma_0 dB_t\right), \ \widetilde{X}_0^{\zeta_0,\pi} = x_0$

- For a deterministic ζ_0, $u(\zeta_0)$ is the value of an agent of type ζ_0
- On the other hand, the original optimization problem (lhs) gives the optimal expected value faced by an agent before the random assignment of types at time $t = 0$

An alternative formulation of the mean field game (cont.)

- Define

$$v_{\zeta_0}(x_0,0) := u(\zeta_0) := \sup_{\pi} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta_0}\left(\widetilde{X}_T^{\zeta_0,\pi} - \theta_0 \overline{X}\right)\right)\right],$$

as the time-zero value of the solution $\{v_{\zeta_0}(x,t) : t \in [0,T],\ x \in \mathbb{R}\}$ of an "indifference type" HJB eqn ,
with the writer's wealth process given by

$$d\widetilde{X}_t^{\zeta_0,\pi} = \pi_t\left(\mu_0 dt + \nu_0 dW_t + \sigma_0 dB_t\right),\ \ \widetilde{X}_0^{\zeta_0,\pi} = x_0$$

- Then the original problem reduces to

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta}\left(X_T - \theta \overline{X}\right)\right)\right] = \mathbb{E}[v_\zeta(\xi,0)]$$

Solution of the mean field game

- Assume that, a.s., $\delta > 0$, $\theta \in [0,1]$, $\mu > 0$, $\sigma \geq 0$, $\nu \geq 0$, and $\sigma + \nu > 0$
  Define the constants

$$\varphi := \mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2 + \nu^2}\right] \qquad \text{and} \qquad \psi := \mathbb{E}\left[\theta \frac{\sigma^2}{\sigma^2 + \nu^2}\right]$$

- There are two cases:

  If $\psi < 1$, there exists a unique constant MFE, given by

$$\pi^* = \delta \frac{\mu}{\sigma^2 + \nu^2} + \theta \frac{\sigma}{\sigma^2 + \nu^2} \frac{\varphi}{1 - \psi}$$

  If $\psi = 1$, there is no constant MFE

This MFG solution indeed provides a natural interpretation of the Nash equilibrium one as the number of agents $n \to \infty$

Key steps - formulating a MF indifference-type problem

- Solve the representative agent's stochastic optimization problem

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\Big[- \exp\Big(- \frac{1}{\delta}(X_T - \theta \overline{X})\Big)\Big]$$

- Enough to consider $\overline{X}$ of the form $\overline{X} = \mathbb{E}[X_T^{\alpha} | \mathcal{F}_T^B]$, $\alpha \in \mathcal{A}_{\mathrm{MF}}$,

$$dX_t = \alpha(\mu dt + \nu dW_t + \sigma dB_t), \qquad X_0 = \xi$$

- For constant equilibria, $\alpha \in \mathcal{F}_0^{\mathrm{MF}}$ -mble rv with $\mathbb{E}[\alpha^2] < \infty$

- Define, for $t \in [0, T]$, $\overline{X}_t := \mathbb{E}[X_t^{\alpha} | \mathcal{F}_T^B]$; then $\overline{X}_T = \overline{X}$

- Find $(\overline{X}_t)_{t \in [0,T]}$ and incorporate it into the state process of "indifference type" problem

- But, because $(\xi, \mu, \sigma, \nu, \alpha)$, B, and W are independent , we must have

$$\overline{X}_t = \overline{\xi} + \overline{\mu\alpha} t + \overline{\sigma\alpha} B_t \qquad (\overline{M} = \mathbb{E}[M])$$

Key steps - obtaining a random HJB

- For $\pi \in \mathcal{A}_{\mathrm{MF}}$, define for $t \in [0,T]$, the "centered" controlled state process
$$Z_t^\pi := X_t^\pi - \theta \overline{X}_t$$
Then, at $t=0$, $Z_0^\pi = \xi - \theta\bar{\xi} = \xi - \theta\mathbb{E}[\bar{\xi}]$ and
$$dZ_t^\pi = (\mu\pi_t - \theta\overline{\mu\alpha})dt + \nu\pi_t dW_t + (\sigma\pi_t - \theta\overline{\sigma\alpha})dB_t$$

- The new problem is now a Merton one,
$$\sup_{\pi\in\mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[-\exp\left(-\frac{1}{\delta}Z_T^\pi\right)\right]$$

- Then, the above supremum equals $\mathbb{E}[v(\xi,0)]$, where $v(x,t)$ solves
$$v_t + \max_\pi\left(\frac{1}{2}\left(\nu^2\pi^2 + (\sigma\pi - \theta\overline{\sigma\alpha})^2\right)v_{xx} + (\mu\pi - \theta\overline{\mu\alpha})v_x\right) = 0,$$
with $v(x,T) = -e^{-x/\delta}$
- This HJB eqn is random , as it depends on the $\mathcal{F}_0^{\mathrm{MF}}$-mble type parameters $(\delta,\theta,\mu,\nu,\sigma)$

Key steps - solving the random HJB

- The random HJB simplifies to

$$v_t - \frac{1}{2}\frac{\left(\mu v_x - \theta\sigma\overline{\sigma\alpha}v_{xx}\right)^2}{(\sigma^2+\nu^2)v_{xx}} - \theta\overline{\mu\alpha}v_x = 0$$

- Then,

$$v(x,t) = -e^{-x/\delta}e^{-\rho(T-t)}$$

with $\rho \in \mathcal{F}_0^{\mathrm{MF}}$ given by

$$\rho := -\frac{1}{\delta}\theta\overline{\mu\alpha} + \frac{\left(\mu + \frac{1}{\delta}\theta\sigma\overline{\sigma\alpha}\right)^2}{2(\sigma^2+\nu^2)}$$

- The optimal feedback $\pi^*(x,t)$, which is actually $\mathcal{F}_0^{\mathrm{MF}}$-mble , turns out to be

$$\pi^*(x,t) = -\frac{\mu v_x(x,t) - \theta\sigma\overline{\sigma\alpha}v_{xx}(x,t)}{(\sigma^2+\nu^2)v_{xx}(x,t)} = \mu\frac{\delta}{\sigma^2+\nu^2} + \theta\frac{\sigma\overline{\sigma\alpha}}{\sigma^2+\nu^2}$$

Key steps - solving for the fixed point

- Observe that a strategy α is an MFE if and only if

$$\mathbb{E}[X_T^\alpha|\mathcal{F}_T^B] = \mathbb{E}[X_T^{\pi^*}|\mathcal{F}_T^B], \text{ a.s.}$$

or, equivalently,

$$\bar{\xi} + \overline{\mu\alpha}T + \overline{\sigma\alpha}B_T = \bar{\xi} + \overline{\mu\pi^*}T + \overline{\sigma\pi^*}B_T, \text{ a.s.}$$

- Taking expectations, α is a constant MFE if and only if

$$\overline{\mu\alpha} = \overline{\mu\pi^*} \quad \text{and} \quad \overline{\sigma\alpha} = \overline{\sigma\pi^*}$$

- Using the form of π^*, $\overline{\sigma\alpha} = \overline{\sigma\pi^*}$ if and only if

$$\overline{\sigma\alpha} = \mathbb{E}\left[\delta\frac{\mu\sigma}{\sigma^2+\nu^2}\right] + \mathbb{E}\left[\theta\frac{\sigma^2}{\sigma^2+\nu^2}\right]\overline{\sigma\alpha} = \varphi + \psi\overline{\sigma\alpha}$$

If $\psi < 1$, then a unique solution $\overline{\sigma\alpha} = \varphi/(1-\psi)$
If $\psi = 1$ and $\varphi \neq 0$, then no solutions, thus no constant MFE
If $\psi = 1$ and $\varphi = 0$, this cannot happen

The value function of the representative fund manager

- The controlled process $(Z_t^\pi)_{t\in[0,T]}$ starts from $Z_0^\pi = \xi - \theta\bar{\xi}$
- The time-zero value to the representative agent

$$v(\xi - \theta\bar{\xi}, 0) = -\exp\left(\frac{1}{\delta}(\xi - \theta\bar{\xi}) - \rho T\right)$$

- Therefore,

$$\rho := -\frac{1}{\delta}\theta\overline{\mu\alpha} + \frac{\left(\mu + \frac{1}{\delta}\theta\sigma\overline{\sigma\alpha}\right)^2}{2(\sigma^2+\nu^2)} = \dots$$

$$= \frac{1}{2(\sigma^2+\nu^2)}\left(\mu + \sigma\frac{\theta}{\delta}\frac{\varphi}{1-\psi}\right)^2 - \frac{\theta}{\delta}\left(\tilde{\psi} + \tilde{\varphi}\frac{\varphi}{1-\psi}\right)$$

with

$$\tilde{\psi} = \mathbb{E}\Big[\delta\frac{\mu^2}{\sigma^2+\nu^2}\Big] \qquad \text{and} \qquad \tilde{\varphi} = \mathbb{E}\Big[\theta\frac{\mu\sigma}{\sigma^2+\nu^2}\Big]$$

Discussion of the equilibrium

$$\pi^* = \delta \frac{\mu}{\sigma^2 + \nu^2} + \theta \frac{\sigma}{\sigma^2 + \nu^2} \frac{\varphi}{1 - \psi}$$

- It turns out that

$$\frac{\varphi}{1 - \psi} = \frac{\mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2+\nu^2}\right]}{1 - \mathbb{E}\left[\theta \frac{\sigma^2}{\sigma^2+\nu^2}\right]} = \ldots = \mathbb{E}\left(\sigma\pi^*\right)$$

Thus,

$$\pi^* = \delta \frac{\mu}{\sigma^2 + \nu^2} + \theta \frac{\sigma}{\sigma^2 + \nu^2} \mathbb{E}\left(\sigma\pi^*\right)$$

- Competition always increases stock allocation
- Myopic component $\delta \frac{\mu}{\sigma^2+\nu^2}$
- Portfolio increasing in risk tolerance and competition weight

Single common stock

- The stock parameters (μ, σ) are deterministic , with $\nu \equiv 0$ and $\mu, \sigma > 0$

- Define the average representative parameters

$$\overline{\delta} := \mathbb{E}[\delta] \qquad \text{and} \qquad \bar{\theta} := \mathbb{E}[\theta]$$

- There are two cases:

  If $\overline{\theta} < 1$, there exists a unique constant MFE, given by the myopic portfolio

$$\pi^* = \delta^{ef} \frac{\mu}{\sigma^2} \qquad \text{with} \qquad \delta^{ef} := \delta + \theta \frac{\overline{\delta}}{1 - \overline{\theta}}$$

- If $\overline{\theta} = 1$, there is no constant MFE

The CRRA case

The n-agent game

- Same setting as in the exponential case
- Individual utilities, $U_i : \mathbb{R}^2_+ \to \mathbb{R}$, of CRRA type depending on the manager's individual wealth , x, and the geometric average wealth of all fund managers, m

$$U_i(x, m) := U(xm^{-\theta_i}; \delta_i),$$

where $U(x;\delta), x > 0$, $\delta > 0$ defined as

$$U(x;\delta) := \begin{cases} \left(1 - \frac{1}{\delta}\right)^{-1} x^{1-\frac{1}{\delta}}, & \text{for } \delta \neq 1, \\ \\ \log x, & \text{for } \delta = 1 \end{cases}$$

- The parameters $\delta_i > 0$, $\theta_i \in [0, 1]$ are the personal relative risk tolerance and social comparison parameters

Modeling competition

- The i^{th} fund manager's wealth process X_t^i solves

$$dX_t^i = \pi_t^i X_t^i (\mu_i dt + \nu_i dW_t^i + \sigma_i dB_t)$$

- If the competitors, $k = 1, ..., n$, $k \neq i$, use policies $(\pi^1, \ldots \pi^{i-1}, \pi^{i+1}, ..., \pi^n)$, his payoff is

$$J_i(\pi^1, \ldots, \pi^n) = \mathbb{E}\left[U\left(X_T^i (\bar{X}_T)^{-\theta_i}; \delta_i\right)\right]$$

- The aggregate wealth $\overline{X}_T$ is given by the geometric mean

$$\overline{X}_T = \left(\prod_{k=1}^{n} X_T^k\right)^{1/n}$$

- Alternatively,
$J_i(\pi^1, \ldots, \pi^n) = \mathbb{E}\left[U\left((X_T^i)^{1-\theta_i}(R_T^i)^{\theta_i}; \delta_i\right)\right]$, with $R_T^i := X_T^i / \overline{X}_T$

Basak-Makarov, Geng-Z.

Nash equilibrium

- Assume, for all $i = 1, \ldots, n$, that $x_0^i > 0$, $\delta_i > 0$, $\theta_i \in [0,1]$ and $\mu_i > 0$, $\sigma_i \geq 0$, $\nu_i \geq 0$, and $\sigma_i + \nu_i > 0$

- Let

$$\varphi_n := \frac{1}{n} \sum_{i=1}^n \delta_i \frac{\mu_i \sigma_i}{\sigma_i^2 + \nu_i^2 (1 + (\delta_i - 1)\theta_i / n)}$$

  and

$$\psi_n := \frac{1}{n} \sum_{i=1}^n \theta_i (\delta_i - 1) \frac{\sigma_i^2}{\sigma_i^2 + \nu_i^2 (1 + (\delta_i - 1)\theta_i / n)}$$

- There always exists a unique constant equilibrium, given by

$$\pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2 + \nu_i^2 (1 + (\delta_i - 1)\theta_i / n)} - \theta_i (\delta_i - 1) \frac{\sigma_i}{\sigma_i^2 + \nu_i^2 (1 + (\delta_i - 1)\theta_i / n)} \frac{\varphi_n}{1 + \psi_n}$$

- Competition does not always increase the risky allocation it depends on whether $\delta_i \lesseqgtr 1$ (nirvana slns, etc.)

Single stock

- Assume that for all $i = 1, \ldots, n$ we have $\mu_i = \mu > 0$, $\sigma_i = \sigma > 0$, and $\nu_i = 0$, with μ and σ independent of i.

- Define the constants

$$\overline{\delta} := \frac{1}{n} \sum_{i=1}^{n} \delta_i \qquad \text{and} \qquad \overline{\theta(\delta - 1)} := \frac{1}{n} \sum_{i=1}^{n} \theta_i(\delta_i - 1)$$

- There exists unique constant equilibrium, given by

$$\pi^{i,*} = \delta_i^{ef} \frac{\mu}{\sigma^2} \qquad \text{with} \qquad \delta_i^{ef} := \delta_i - \frac{\theta_i(\delta_i - 1)\overline{\delta}}{1 + \overline{\theta(\delta - 1)}}$$

Passing to the limit as $n \uparrow \infty$
The mean field game under CRRA risk preferences

The mean field game

- Recall that the type vector of agent i is $\zeta_i := (x_0^i, \delta_i, \theta_i, \mu_i, \nu_i, \sigma_i)$
- It induces an empirical measure, which is the probability measure on

$$\mathcal{Z}^p := (0,\infty) \times (0,\infty) \times [0,1] \times (0,\infty) \times [0,\infty) \times [0,\infty)$$

given by

$$m_n(A) = \frac{1}{n}\sum_{i=1}^{n} 1_A(\zeta_i), \quad \text{for Borel sets } A \subset \mathcal{Z}^p$$

- Assume that m_n has a weak limit m
- Let $\zeta = (\xi, \delta, \theta, \mu, \nu, \sigma)$ denote a r.v. with this distribution m
- Then, the strategy $\pi^{i,*}$ "should" converge to

$$\lim_{n\to\infty} \pi^{i,*} = \delta_i \frac{\mu_i}{\sigma_i^2+\nu_i^2} + \theta_i \frac{\sigma_i}{\sigma_i^2+\nu_i^2} \frac{\varphi}{1-\psi},$$

where

$\varphi := \lim_{n\uparrow\infty} \varphi_n = \mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2+\nu^2}\right]$ and $\psi := \lim_{n\uparrow\infty} \psi_n = \mathbb{E}\left[\theta(\delta-1)\frac{\sigma^2}{\sigma^2+\nu^2}\right]$

Definition of the mean field game

- The representative agent's wealth process solves

$$dX_t = \pi_t X_t(\mu dt + \nu dW_t + \sigma dB_t), \qquad X_0 = \xi$$

- Let $\overline{X}$ be an $\mathcal{F}_T^{\mathrm{MF}}$ -mble rv, representing the geometric mean wealth among the continuum of agents
- Representative agent aims to maximize the expected payoff

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[U(X_T \overline{X}^{-\theta}; \delta)\right]$$

- Recall that in the n -player game, the aggregate wealth is the geometric mean, $\overline{X}_T = \left(\prod_{k=1}^{n} X_T^k\right)^{1/n}$
- A "geometric mean" of a measure m on $(0, \infty)$ is defined as

$$\exp\left(\int_{(0,\infty)} \log y \; dm(y)\right)$$

- When m is the empirical measure of n points $(y_1, \ldots, y_n)$, this reduces to the usual definition $(y_1 y_2 \cdots y_n)^{1/n}$

Definition of the mean field game (cont.)

- Let arbitrary strategy $\pi^* \in \mathcal{A}_{\mathrm{MF}}$
- Consider the $\mathcal{F}_T^B$-mble rv

$$\overline{X} := \exp \mathbb{E}[\log X_T^* \,|\, \mathcal{F}_T^B]$$

  where $(X_t^*)_{t\in[0,T]}$ is the wealth process using π^*

- Then, π^* is a mean field equilibrium if it is optimal for the optimization problem

$$\sup_{\pi \in \mathcal{A}_{\mathrm{MF}}} \mathbb{E}\left[U(X_T \overline{X}^{-\theta}; \delta)\right]$$

  corresponding to this choice of $\overline{X}$

- A constant MFE is a MFE π^* which is constant in time, i.e., $\pi_t^* = \pi_0^*$ for all $t \in [0, T]$.
- A constant MFE π^* is then the $\mathcal{F}_0^{\mathrm{MF}}$ -measurable random variable π_0^*

Solving the mean field game

- Assume that, a.s., $\delta > 0$, $\theta \in [0,1]$, $\mu > 0$, $\sigma \geq 0$, $\nu \geq 0$, and $\sigma + \nu > 0$

- Define the constants

$$\varphi := \mathbb{E}\left[\delta \frac{\mu\sigma}{\sigma^2 + \nu^2}\right] \qquad \text{and} \qquad \psi := \mathbb{E}\left[\theta(\delta - 1)\frac{\sigma^2}{\sigma^2 + \nu^2}\right]$$

- There always exists a unique constant MFE,

$$\pi^* = \delta \frac{\mu}{\sigma^2 + \nu^2} - \theta(\delta - 1)\frac{\sigma}{\sigma^2 + \nu^2}\frac{\varphi}{1 + \psi}$$

- Competition does not always increase the stock allocation unless $\delta < 1$

Single stock case

- Suppose (μ, σ) are deterministic, with $\nu \equiv 0$ and $\mu, \sigma > 0$
- Define the constants

$$\overline{\delta} := \mathbb{E}[\delta] \qquad \text{and} \qquad \overline{\theta(\delta - 1)} := \mathbb{E}[\theta(\delta - 1)]$$

- Then, there exists a unique constant MFE, given by

$$\pi^* = \delta^{ef} \frac{\mu}{\sigma^2} \qquad \text{with} \quad \delta^{ef} = \delta - \frac{\theta(\delta - 1)\overline{\delta}}{1 + \overline{\theta(\delta - 1)}}$$