A Martingale Approach for Fractional Brownian Motions

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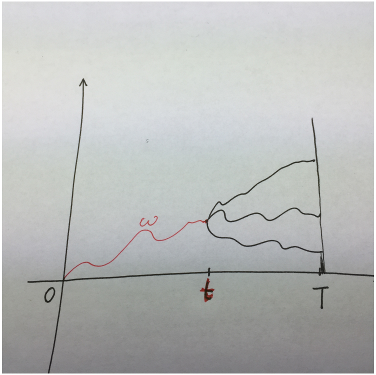
Outline

1. Introduction
2. Heat equation
3. Functional Itô formula
4. Nonlinear extension

Forward versus backward problems



Robust hedging ?

- Given S on $[0, T]$ and a terminal payoff ξ (at T)
- Complete market (linear case) :

$$Y_0 = \boldsymbol{E}^{\mathbb{P}}[\xi] \quad = \quad \left\{ y : \exists Z \text{ s.t. } y + \int_0^T Z_t dS_t = \xi, \ \mathbb{P}\text{-a.s.} \right\}$$

- Incomplete market : $\mathbb{P} \in \mathcal{P}$ (semimartingale measures)

$$Y_0 = \sup_{\mathbb{P} \in \mathcal{P}} \boldsymbol{E}^{\mathbb{P}}[\xi]$$

$$= \inf \left\{ y : \exists Z \text{ s.t. } y + \int_0^T Z_t dS_t \geq \xi, \ \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P} \right\}$$

- Beyond semimartingale framework ?

$$Y_0 = \inf \left\{ y : \exists Z \text{ s.t. } y + \int_0^T Z_t(\omega) dS_t(\omega) \geq \xi(\omega), \right.$$

$$\left. \text{for "all" rough paths } \omega \right\}$$

Rough price versus rough volatility

- Rough price S : $Z_t dS_t$?

- Rough volatility : $dS_t = S_t[b_t dt + \sigma_t dB_t]$ and σ is rough
  - ◇ See Huimeng's talk yesterday
  - ◇ See the recent work El Euch-Rosenbaum (2017)

- A natural model : σ driven by a fractional Brownian motion B^H

- Goal : characterize $Y_t := \boldsymbol{E}_t\big[\xi \mid \mathcal{F}_t^{B,B^H}\big]$?
  - ◇ σ (hence B^H) can be observed
  - ◇ To focus on the main idea we will assume ξ is $\mathcal{F}_T^{B^H}$-measurable and consider $Y_t = \boldsymbol{E}_t\big[\xi \mid \mathcal{F}_t^{B^H}\big]$

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Fractional Brownian Motion

- Let B^H be a fBM with $0 < H < 1$:
  - ◇ $B_t^H - B_s^H \sim \text{Normal}(0, (t-s)^{2H})$
  - ◇ $B^H = B$ when $H = \frac{1}{2}$
- Representation : $B_t^H = \int_0^t K(t,s) dW_s$
  - ◇ $K(t,s) \sim (t-s)^{2H-1}$, which blows up at $t = s$ when $H < \frac{1}{2}$
  - ◇ $\mathbb{F} := \mathbb{F}^{B^H} = \mathbb{F}^W$
- Two main features :
  - ◇ B^H is not Markovian $(H \neq \frac{1}{2})$
  - ◇ B^H is not a semimartingale $(H < \frac{1}{2})$

A Heat equation

- Let $\xi := g(B^H_T)$ and $V_t := \boldsymbol{E}_t[g(B^H_T)]$.

- Denote

$$v(t,x) := \boldsymbol{E}\left[g(x + B^H_T - B^H_t)\right] = \int_{\mathbb{R}} g(y) p_H(T-t, y-x) dy$$

$$\text{where} \quad p_H(t,x) := \frac{1}{\sqrt{2\pi} t^H} e^{-\frac{x^2}{2t^{2H}}}.$$

- Heat equation :

$$\partial_t p_H(t,x) - H t^{2H-1} \partial_{xx} p_H(t,x) = 0$$

$$\partial_t v(t,x) + H t^{2H-1} \partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- $V_0 = v(0,0)$

A Heat equation

- Let $\xi := g(B^H_T)$ and $V_t := \boldsymbol{E}_t[g(B^H_T)]$.

- Denote

$$v(t,x) := \boldsymbol{E}\Big[g(x + B^H_T - B^H_t)\Big]$$

- Heat equation :

$$\partial_t v(t,x) + Ht^{2H-1}\partial_{xx} v(t,x) = 0, \quad v(T,x) = g(x).$$

- However, $v(t, B^H_t)$ is not a martingale :

$V_0 = v(0, B^H_0)$, $V_T = v(T, B^H_T)$, but $V_t \neq v(t, B^H_t)$ for $0 < t < T$.

An alternative heat equation

- Let $\xi := g(B^H_T)$ and $V_t := \boldsymbol{E}_t[g(B^H_T)]$.

- Note

$$\begin{aligned} V_t &= \boldsymbol{E}_t\Big[g\big(\textstyle\int_0^T K(T,r)dW_r\big)\Big] \\ &= \boldsymbol{E}_t\Big[g\big(\textstyle\int_0^t K(T,r)dW_r + \int_t^T K(T,r)dW_r\big)\Big] \\ &= v\big(t, \textstyle\int_0^t K(T,r)dW_r\big), \end{aligned}$$

where $v(t,x) := \boldsymbol{E}\Big[g\big(x + \int_t^T K(T,r)dW_r\big)\Big]$

- Martingale property : $v\big(t, \int_0^t K(T,r)dW_r\big)$ is a martingale
- Heat equation :

$$\partial_t v(t,x) + \frac{1}{2}K^2(T,t)\partial_{xx}v(t,x) = 0, \quad v(T,x) = g(x).$$

A closer look

- $\Theta^t_T := \int_0^t K(T,r)dW_r = \boldsymbol{E}_t[B^H_T]$ is $\mathcal{F}_t$-measurable
  - ◇ Θ^t_T is the forward variance and is observable in market
- Three ways to express V_t :

$$V_t = v_1(t, B^H_{t\wedge\cdot}) = v_2(t, W_{t\wedge\cdot}) = v(t, \Theta^t_T)$$

  - ◇ v_1 could be smooth but B^H is not a semimartingale
  - ◇ W is a martingale (of course) but v_2 is not continuous
  - ◇ v has desired regularity and $t \mapsto \Theta^t_T$ is a martingale

An extension

- Denote $V_t := \boldsymbol{E}_t\Big[g(B^H_T) + \int_t^T f(s, B^H_s)ds\Big]$.
- By previous computation :

$$\begin{aligned} V_t &= \boldsymbol{E}_t[g(B^H_T)] + \int_t^T \boldsymbol{E}_t[f(s, B^H_s)]ds \\ &= v(T, g; t, \boldsymbol{E}_t[B^H_t]) + \int_t^T v(s, f(s,\cdot); t, \boldsymbol{E}_t[B^H_s])ds \\ &= u(t, \{\boldsymbol{E}_t[B^H_s]\}_{t\le s\le T}) \end{aligned}$$

- Note : u is path dependent
  - ◇ If $H = \frac{1}{2}$, $\boldsymbol{E}_t[B_s] = B_t$, so $V_t = u(t, B_t)$ is state dependent
  - ◇ In more general cases,

$$V_t = u\Big(t, \{B^H_s\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B^H_s]\}_{t\le s\le T}\Big).$$

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The canonical setup

- Recall

$$V_t = u\Big(t, \{B_s^H\}_{0\le s\le t} \otimes_t \{\boldsymbol{E}_t[B_s^H]\}_{t\le s\le T}\Big).$$

- For $t \in [0,T]$, $\omega \in \mathbb{D}^0([0,t))$, and $\theta \in C^0([t,T])$, define :

$$(\omega \otimes_t \theta)_s := \omega_s \mathbf{1}_{[0,t)}(s) + \theta_s \mathbf{1}_{[t,T]}(s), \quad 0 \le s \le T.$$

- The canonical space :

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0,T], \omega \in \mathbb{D}^0([0,t)), \theta \in C^0([t,T])\Big\};$$

$$\Lambda_0 := \Big\{(t, \omega \otimes_t \theta) \in \Lambda : \omega \in C^0([0,t]), \omega_0 = 0, \theta_t = \omega_t\Big\}.$$

Continuous mapping

- Recall

$$\Lambda := \Big\{(t, \omega \otimes_t \theta) : t \in [0, T], \omega \in \mathbb{D}^0([0, t)), \theta \in C^0([t, T])\Big\}.$$

- The metric :

$$\mathbf{d}((t, \omega \otimes_t \theta), (t', \omega' \otimes_{t'} \theta'))$$

$$:= \sqrt{|t - t'|} + \sup_{0 \le s \le T} \big|(\omega \otimes_t \theta)_s - (\omega' \otimes_{t'} \theta')_s\big|.$$

- $C^0(\Lambda)$: continuous mapping $u : \Lambda \to \mathbb{R}$
- $C^0_b(\Lambda)$: bounded $u \in C^0(\Lambda)$

Path derivatives

- Time derivative :

$$\partial_t u(t, \omega \otimes_t \theta) := \lim_{\delta \downarrow 0} \frac{u(t+\delta, \omega \otimes_t \theta) - u(t, \omega \otimes_t \theta)}{\delta}.$$

◇ $\partial_t u$ is the right time derivative !

- First order spatial derivative : Fréchet derivative with respect to θ

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[u(t, \omega \otimes_t (\theta + \varepsilon \eta)) - u(t, \omega \otimes_t \theta) \Big],$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta \in C^0([t, T])$.

Path derivatives (cont)

- Second order spatial derivative : bilinear operator on $C^0([t, T])$:

$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta_1, \eta_2)\rangle$$

$$:= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[\langle \partial_\theta u(t, \omega \otimes_t (\theta + \varepsilon\eta_1)), \eta_2\rangle - \langle \partial_\theta u(t, \omega \otimes_t \theta), \eta_2\rangle\Big].$$

for all $(t, \omega \otimes_t \theta) \in \Lambda$, $\eta_1, \eta_2 \in C^0([t, T])$.

- Define the spaces $C^{1,2}(\Lambda)$ and $C^{1,2}_b(\Lambda)$ in obvious sense

Functional Ito formula : $H \geq \frac{1}{2}$

- Regular case : $K(t,t)$ is finite and thus

$$s \in [t, T] \mapsto K^t_s := K(s,t) \text{ is in } C^0([t,T]).$$

- Denote : $X_s := B^H_s,\ 0 \leq s \leq t\,; \quad \Theta^t_s := \boldsymbol{E}_t[B^H_s],\ t \leq s \leq T$

- Functional Ito formula :

$$\begin{aligned} & du(t, X \otimes_t \Theta^t) \\ = \quad & \partial_t u(\cdot)dt + \langle \partial_\theta u(\cdot), K^t \rangle dW_t + \frac{1}{2}\langle \partial^2_{\theta\theta} u(\cdot), (K^t, K^t) \rangle dt. \end{aligned}$$

◇ If $H = \frac{1}{2}$, $K = 1$, this is exactly Dupire's functional Ito formula

Functional Ito formula : $H < \frac{1}{2}$

- $K(s,t) \sim (s-t)^{H-\frac{1}{2}}$, $\partial_s K(s,t) \sim (s-t)^{H-\frac{3}{2}}$, $0 \le t < s \le T$

- For some $\alpha > \frac{1}{2} - H$, for any $(t, \omega \otimes_t \theta) \in \Lambda_0$, any $t < t_1 < t_2 \le T$, any $\eta \in C^0([t,T]$ with support in $[t_1, t_2]$,

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), \eta \rangle \le C[t_2 - t_1]^\alpha \|\eta\|_\infty,$$
$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (\eta, \eta) \rangle \le C[t_2 - t_1]^{2\alpha} \|\eta\|^2_\infty.$$

  ◇ Roughly speaking, we want $\partial_{\theta_t} u(t, \omega \otimes_t \theta) = 0$.

- Denote $K^{t,\delta}_s := K^t_{(t+\delta)\vee s}$. Then the following limits exist :

$$\langle \partial_\theta u(t, \omega \otimes_t \theta), K^t \rangle := \lim_{\delta \to 0} \langle \partial_\theta u(t, \omega \otimes_t \theta), K^{t,\delta} \rangle;$$
$$\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t) \rangle := \lim_{\delta \to 0} \langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^{t,\delta}, K^{t,\delta}) \rangle.$$

- Functional Ito formula still holds

Linear path dependent PDE

- $V_t := \boldsymbol{E}_t\Big[g(B^H_T) + \int_t^T f(s, B^H_s)ds\Big] = u(t, X \otimes_t \Theta^t)$

- $V_t + \int_0^t f(s, B^H_s)ds$ is a martingale

- Linear PPDE :

$$\partial_t u(t, \omega \otimes_t \theta) + \frac{1}{2}\langle \partial^2_{\theta\theta} u(t, \omega \otimes_t \theta), (K^t, K^t)\rangle + f(t, \omega_t) = 0,$$
$$u(T, \omega) = g(\omega_T).$$

- Theorem. Assume f and g are smooth, then the above PPDE has a unique classical solution u.

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Nonlinear dynamics

- Forward dynamics : Volterra SDE

$$X_t = x + \int_0^t b(t; r, X_\cdot)dr + \int_0^t \sigma(t; r, X_\cdot)dW_r$$

- Backward dynamics : BSDE

$$Y_t = g(X_\cdot) + \int_t^T f(s, X_\cdot, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

◇ The backward one itself is time consistent. If we consider Volterra type of BSDEs, see a series of works by Jiongmin Yong.

- $Y_t = u(t, X \otimes_t \Theta^t)$, where

$$\Theta^t_s := x + \int_0^t b(s; r, X_\cdot)dr + \int_0^t \sigma(s; r, X_\cdot)dW_r.$$

Nonlinear PPDE

- Representation : $u(t,\omega\otimes_t\theta) := Y_t^{t,\omega\otimes_t\theta}$, where

$$X_s^{t,\omega\otimes_t\theta} = \theta_s + \int_t^s b(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dr + \int_t^s \sigma(s;r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta})dW_r$$

$$Y_s^{t,\omega\otimes_t\theta} = g(\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}) - \int_s^T Z_r^{t,\omega\otimes_t\theta}dW_r + \int_s^T f(r,\omega\otimes_t X_\cdot^{t,\omega\otimes_t\theta}, Y_r^{t,\omega\otimes_t\theta}, Z_r^{t,\omega\otimes_t\theta})dr.$$

- Semilinear PPDE : $\varphi_s^{t,\omega} := \varphi(s;t,\omega)$, $t\le s\le T$, for $\varphi = b,\sigma$,

$$\partial_t u + \frac{1}{2}\langle\partial^2_{\theta\theta}u,(\sigma^{t,\omega},\sigma^{t,\omega})\rangle + \langle\partial_\theta u, b^{t,\omega}\rangle + f\left(t,\omega,u,\langle\partial_\theta u,\sigma^{t,\omega}\rangle\right) = 0,$$

$$u(T,\omega) = g(\omega).$$

Further research

- Controlled problems (fully nonlinear PPDE)
  - See Huimeng's talk yesterday
- Viscosity solution
- Efficient numerical algorithms

Thank you very much for your attention !

Welcome to the 9th WCMF in Los Angeles in 2018 !