Mathematical Models of the Evolution of Surface Waves on Deep Water
Crystal Lee
Derivation of 1D NLS
Numerical Solutions of 2D NLS
Solitary Waves
Perturbed Solitary Wave
Unperturbed vs Perturbed Solitary Wave

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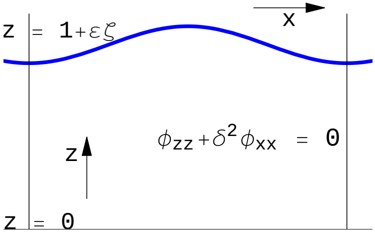
Derivation of 1D NLS
Governing Equations



$$\phi_{zz} + \delta^2 \phi_{xx} = 0 \quad \text{on } 0 < z < 1 + \epsilon\zeta$$

$$\phi_z = \delta^2(\zeta_t + \epsilon\phi_x\zeta_x) \quad \text{on } z = 1 + \epsilon\zeta$$

$$\phi_t + \zeta + \frac{\epsilon}{2}\left(\frac{1}{\delta^2}\phi_z^2 + \phi_x^2\right) = \frac{T\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} \quad \text{on } z = 1 + \epsilon\zeta$$

$$\phi_z = 0 \quad \text{on } z = 0$$

Derivation of 1D NLS

- Introduce the transformation

$$\xi = x - c_p(k)t \qquad \eta = \epsilon(x - c_g(k)t) \qquad \tau = \epsilon^2 t$$

$$k = \frac{2\pi}{\lambda}$$

- Look for an asymptotic solution of the form

$$\phi \sim \sum_{n=0}^{\infty} \epsilon^n \phi_n(\xi, \eta, \tau, z) \qquad \zeta \sim \sum_{n=0}^{\infty} \epsilon^n \zeta_n(\xi, \eta, \tau)$$

Derivation of 1D NLS

- Solution takes the form

$$\phi_0 = f_0(\eta, \tau) + F_0(z, \eta, \tau)e^{ik\xi} + F_0^*(z, \eta, \tau)e^{-ik\xi}$$

$$\zeta_0 = A_0(\eta, \tau)e^{ik\xi} + A_0^*(\eta, \tau)e^{-ik\xi}$$

and

$$\phi_n = \sum_{m=0}^{n+1} F_{nm}(z, \eta, \tau)e^{ikm\xi} + F_{nm}^*(z, \eta, \tau)e^{-ikm\xi}$$

$$\zeta_n = \sum_{m=0}^{n+1} A_{nm}(\eta, \tau)e^{ikm\xi} + A_{nm}^*(\eta, \tau)e^{-ikm\xi}.$$

Derivation of 1D NLS

- Finally,

$$-2ikc_pA_\tau + \alpha A_{\eta\eta} + \beta|A|^2A = 0,$$

where

$$\alpha = c_g^2 - (1 - \delta k\tanh\,\delta k)\text{sech}^2\delta k$$

and

$$\beta = \frac{k^2}{c_p^2}\Big(\frac{1}{2}(1 + 9\coth^2\delta k - 13\text{sech}^2\delta k - 2\tanh^4\delta k)$$
$$- (2c_p + c_g\text{sech}^2\delta k)^2(1 - c_g^2)^{-1}\Big)$$

Numerical Solutions of 2D NLS

- Determine the numerical solution to

$$iu_\tau + \alpha u_{\xi\xi} + \beta u_{\eta\eta} + \gamma |u|^2 u = 0$$

with IC

$$u(\xi, \eta, 0) = f(\xi, \eta)$$

and periodic BCs in ξ and η on $[0, L]$.

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$$iu_\tau + \alpha u_{\xi\xi} + \beta u_{\eta\eta} + \gamma|u|^2 u = 0$$
$$u = u(\xi,\eta,\tau) = u_R(\xi,\eta,\tau) + iu_I(\xi,\eta,\tau)$$

$$\text{Re:} \; -u_{I_\tau} + \alpha u_{R_{\xi\xi}} + \beta u_{R_{\eta\eta}} + \gamma u_R^3 + \gamma u_I^2 u_R = 0$$
$$\text{Im:} \; u_{R_\tau} + \alpha u_{I_{\xi\xi}} + \beta u_{I_{\eta\eta}} + \gamma u_I u_R^2 + \gamma u_I^3 = 0$$

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$$u(\xi,\eta,0) = f(\xi,\eta)$$
$$u(\xi,\eta,0) = u_R(\xi,\eta,0) + iu_I(\xi,\eta,0)$$

$$\text{Re: } u_R(\xi,\eta,0) = \text{Re}(f(\xi,\eta))$$
$$\text{Im: } u_I(\xi,\eta,0) = \text{Im}(f(\xi,\eta))$$

Numerical Solutions of 2D NLS
BCs

$$u(0,\eta,\tau) = u(L,\eta,\tau)$$
$$u(\xi,0,\tau) = u(\xi,L,\tau)$$

$$\text{Re:}\ \ u_R(0,\eta,\tau) = u_R(L,\eta,\tau)$$
$$u_R(\xi,0,\tau) = u_R(\xi,L,\tau)$$
$$\text{Im:}\ \ u_I(0,\eta,\tau) = u_I(L,\eta,\tau)$$
$$u_I(\xi,0,\tau) = u_I(\xi,L,\tau)$$

- Let

$$\tau = \epsilon^2 \sqrt{gk} t$$

$$\xi = \epsilon k \left(x - \frac{g + 3k^2 T}{2\sqrt{gk + k^3 T}} t\right)$$

$$\eta = \frac{\epsilon k y}{b}.$$

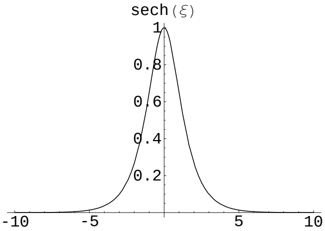
Then, the surface displacement is given by

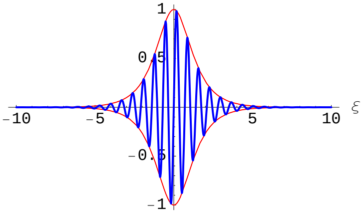
$$\zeta(x, y, t) = \frac{-2\epsilon\sqrt{gk}}{k\sqrt{k(g + k^2 T)}} \Big(u_R(\xi, \eta, \tau)\sin(kx - \sqrt{k(g + k^2 T)} t) + u_I(\xi, \eta, \tau)\cos(kx - \sqrt{k(g + k^2 T)} t) \Big)$$

Solitary Waves

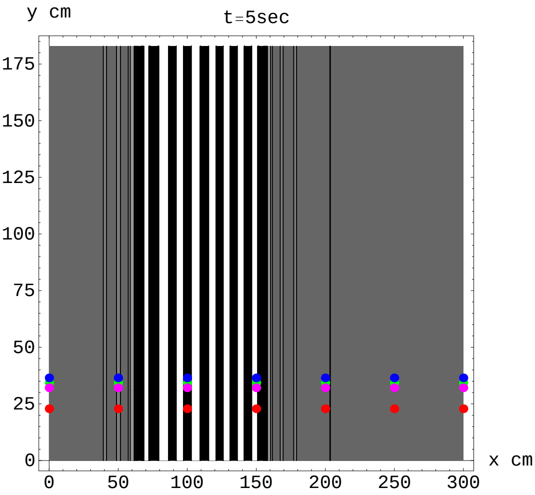
- A soliton solution of the 1D NLS equation is given by

$$u(\xi,\tau) = \pm\sqrt{2\frac{\alpha}{\gamma}}\operatorname{sech}(\xi)e^{i\alpha\tau}.$$

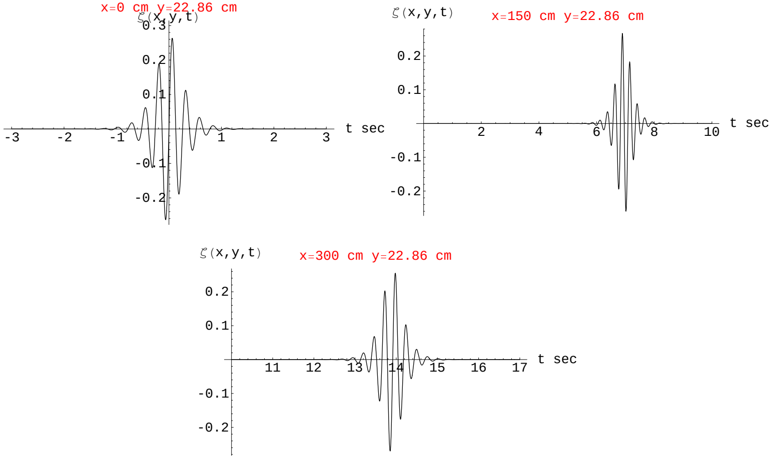


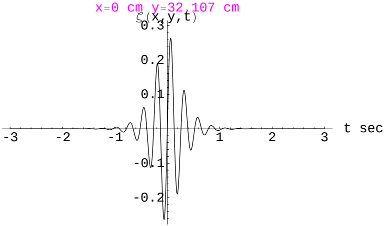


Solitary Waves
Solitary Wave IC

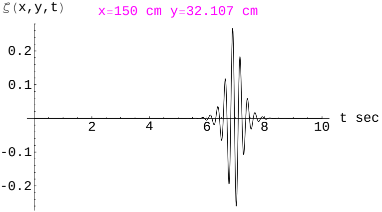

y cm
t=5sec
175
150
125
100
75
50
25
0
0
50
100
150
200
250
300
x cm

Solitary Waves
Solitary Wave IC

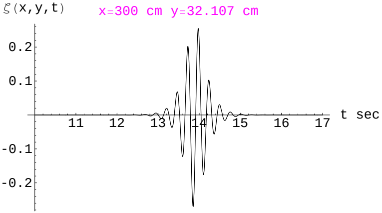




x=0 cm y=32.107 cm
ζ(x,y,t)
0.3
0.2
0.1
-0.1
-0.2
-3
-2
-1
1
2
3
t sec



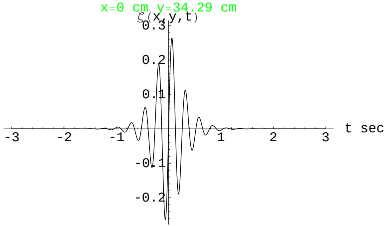
ζ(x,y,t)
x=150 cm y=32.107 cm
0.2
0.1
-0.1
-0.2
2
4
6
8
10
t sec

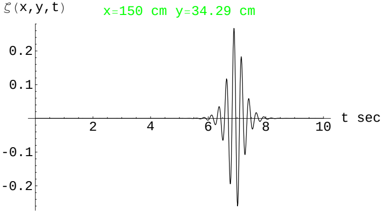


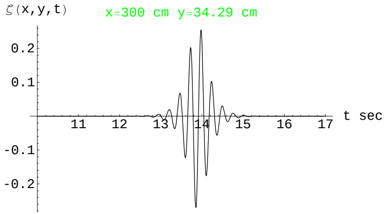
ζ(x,y,t)
x=300 cm y=32.107 cm
0.2
0.1
-0.1
-0.2
11
12
13
14
15
16
17
t sec

Solitary Waves

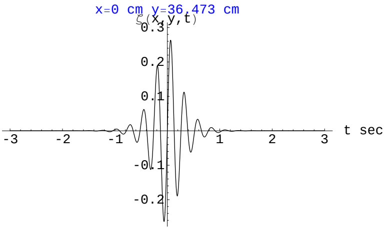
Solitary Wave IC

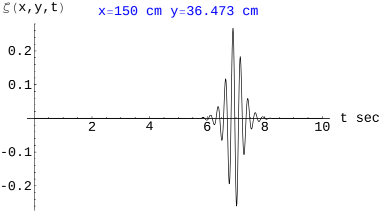


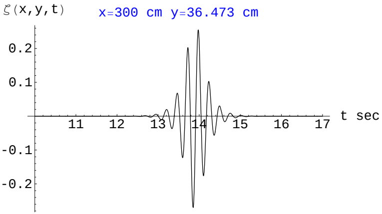




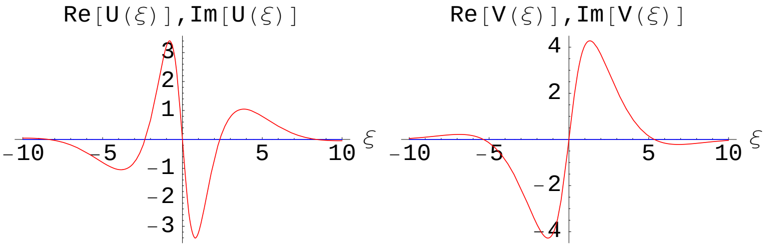
Solitary Waves
Solitary Wave IC







Perturbed Solitary Wave



Perturbed Solitary Wave

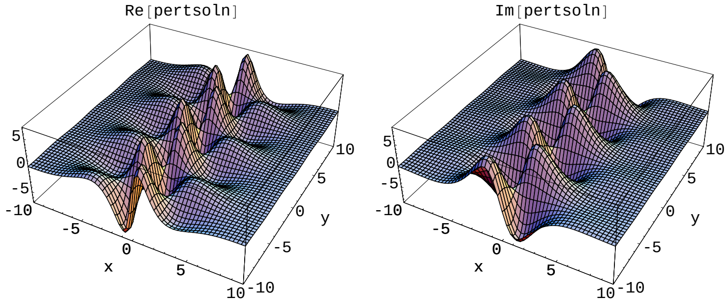
- Consider the following IC.

$$\text{pertsoln}(\xi, \eta, 0, \text{eps}) = \sqrt{2\frac{\alpha}{\gamma}}\text{sech}\ \xi$$

$$+\ \text{eps}((\Re[U(\xi)] + i\Im[U(\xi)])e^{i\rho(\eta-\frac{\pi}{2})} + (\Re[U(\xi)] - i\Im[U(\xi)])e^{-i\rho(\eta-\frac{\pi}{2})})$$

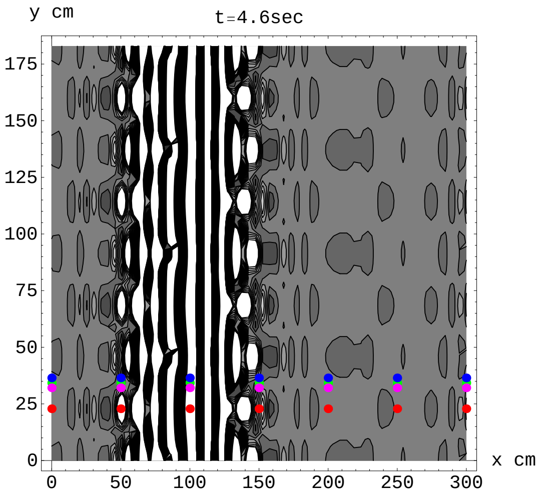
$$+\ i\text{eps}((\Re[V(\xi)] + i\Im[V(\xi)])e^{i\rho(\eta-\frac{\pi}{2})} + (\Re[V(\xi)] - i\Im[V(\xi)])e^{-i\rho(\eta-\frac{\pi}{2})})$$

Perturbed Solitary Wave

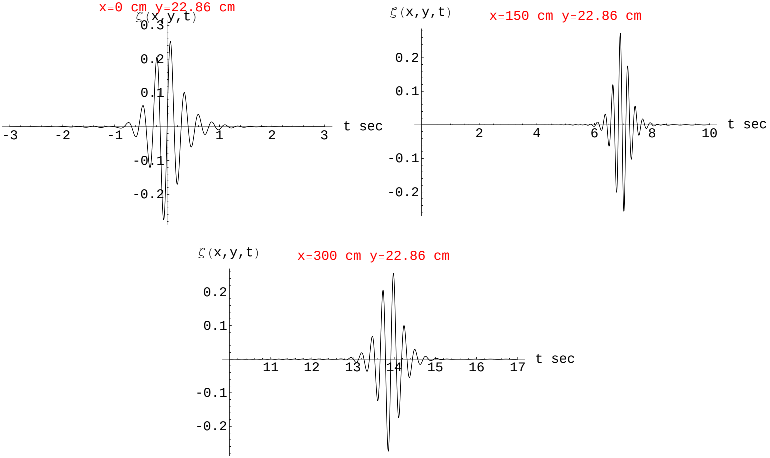


Perturbed Solitary Wave

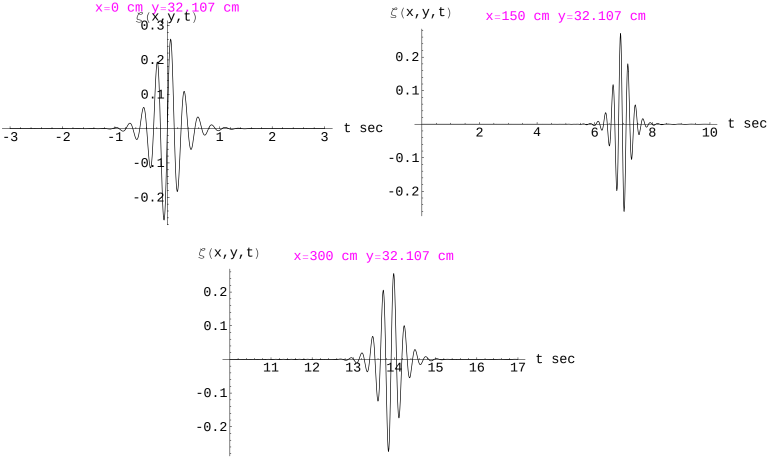
Perturbed Solitary Wave



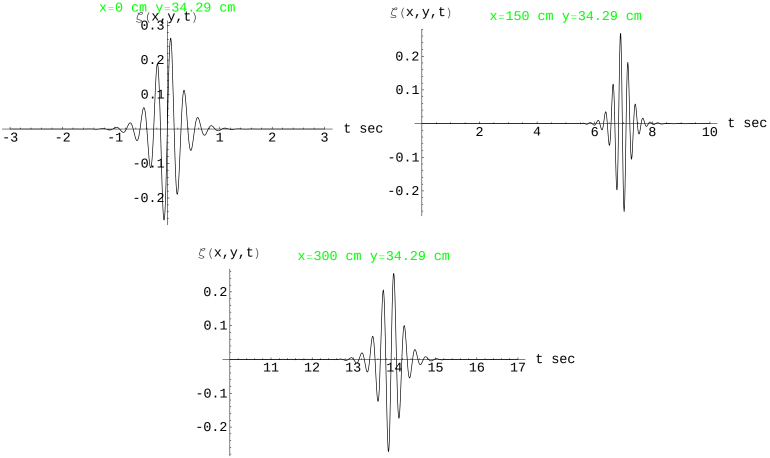
Perturbed Solitary Wave



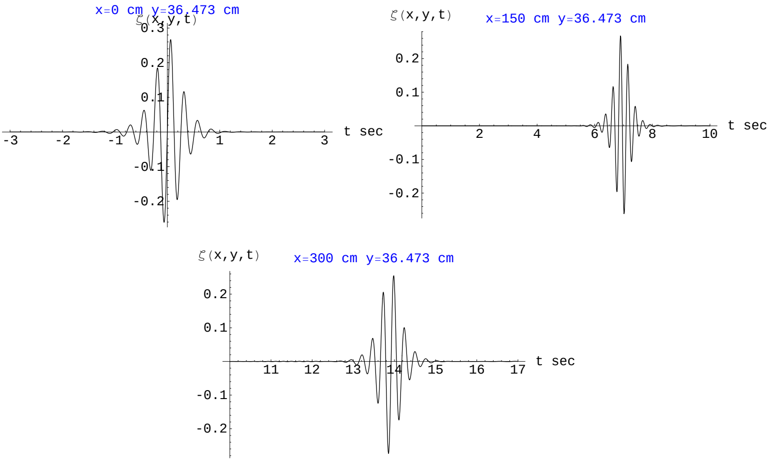
Perturbed Solitary Wave



Perturbed Solitary Wave

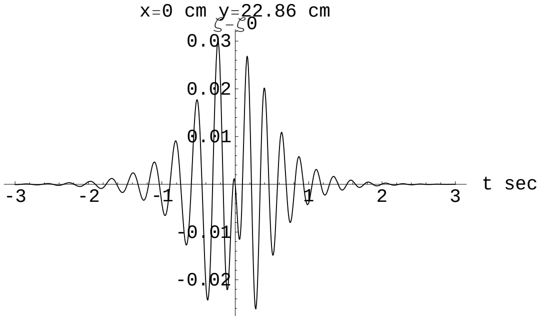


Perturbed Solitary Wave



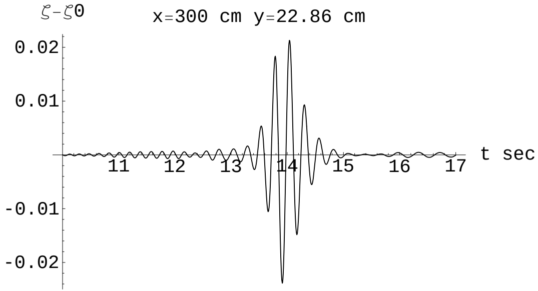
Unperturbed vs Perturbed Solitary Wave

- At $x = 0$ cm, $y = 22.86$ cm,



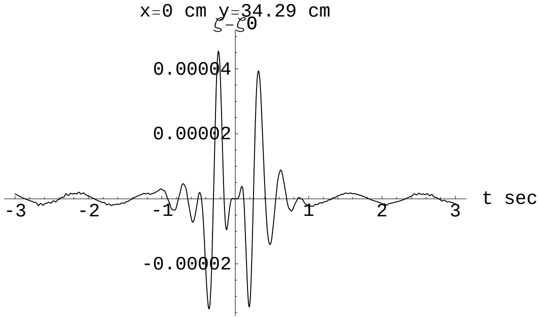
Unperturbed vs Perturbed Solitary Wave

- At $x = 300$ cm, $y = 22.86$ cm,



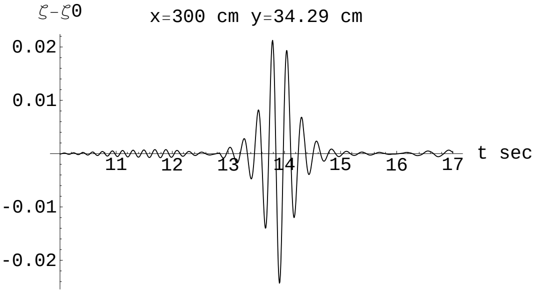
Unperturbed vs Perturbed Solitary Wave

- At $x = 0$ cm, $y = 34.29$ cm,



Unperturbed vs Perturbed Solitary Wave

- At $x = 300$ cm, $y = 34.29$ cm,



Results

- The soliton solution is not stable with respect to the perturbation.

References

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