Aircraft Routing Under The Risk Of Detection

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Introduction

Find an aircraft’s optimal trajectory in 3-D space while minimizing risk of detection by radar/sensor

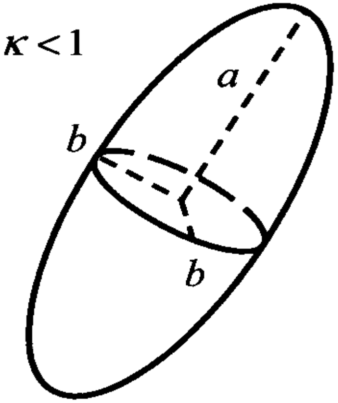
- Constraint on trajectory length
- Variable **Radar Cross-Section (RCS)** of aircraft
- Radar vs. Sensor ($n = 4$ vs. $n = 2$)
- Number of detecting installations
- Risk of detection: proportional to RCS of craft and reciprocal to n-th power of the distance between craft and radar/sensor

Assumptions

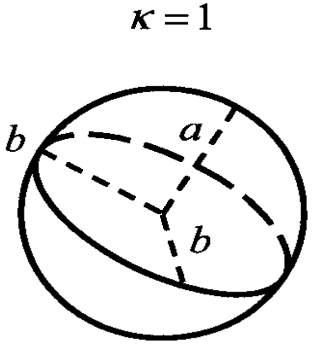
- Risk of detection independent of aircraft speed
- No uncertainty in aircraft detection or radar locations
- Does not consider aircraft kinematics equations
- No change in aircraft control during flight

Formulating the Model - 1

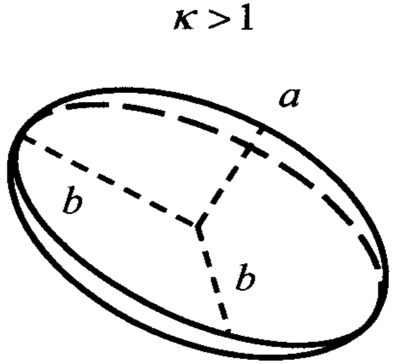
- Axisymmetrical ellipsoid with axes' length a, b, b
- Axis with length a is axis of ellipsoid symmetry – aligned with aircraft trajectory
- Introduce parameter k where k = b/a



elongated ellipsoid



sphere



compressed ellipsoid

Formulating the Model - 2

Vector r = (x, y, z) determines position of the ellipsoid geometrical center

Parameterize r as a function of length s: r(s) = (x(s), y(s), z(s))

Vector $\dot{\mathbf{r}}(s) = \frac{d}{ds}\mathbf{r}(s) = (\dot{x}(s), \dot{y}(s), \dot{z}(s))$

Use fact that $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

This implies that

$$\dot{\mathbf{r}}(s) \text{ must satisfy condition } \dot{\mathbf{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 1$$

Formulating the Model - 3

- Introduce vector q_i = (a_i, b_i, c_i), i = $\overline{1, N}$, determines the position of the i^{th} radar
- Distance from aircraft to i^{th} installation, denoted as $\|r_i(s)\|$

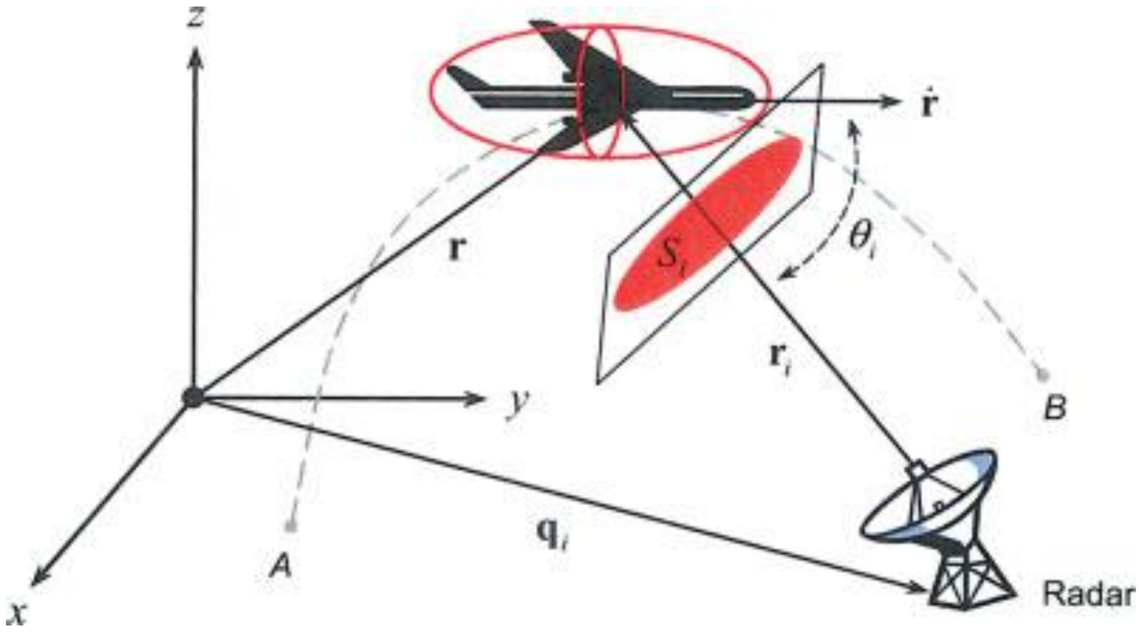
$$\|r_i(s)\| = \sqrt{(x-a_i)^2+(y-b_i)^2+(z-c_i)^2}$$

- The RCS of the aircraft exposed to the i^{th} radar at point (x, y, z) is proportional to the area S_i of the ellipsoid's projection onto the plane orthogonal to vector r_i
- The constant coefficient σ_i depends on the radar's technological characteristics

Formulating the Model - 4

Combining this knowledge we get the following formula:

$$\mathrm{RCS}_i = \sigma_i \, S_i$$



Formulating the Model - 5

The magnitude of the ellipsoid projection area is given by the formula:

$$S_i = \pi\, b \sqrt{a^2 \sin^2 \theta_i + b^2 \cos^2 \theta_i},$$

Rewrite the formula for RCS:

$$\mathrm{RCS}_i = \sigma_i\, \pi \left(\frac{a^2 + b^2}{2}\right) \frac{2\kappa}{1+\kappa^2} \sqrt{1 + (\kappa^2 - 1)\left(\frac{\mathbf{r}_i \cdot \dot{\mathbf{r}}}{\|\mathbf{r}_i\|}\right)^2}$$

$$\kappa \in [0, +\infty).$$

Formulating the Model - 6

$$\mathrm{RCS}_i/\|\mathbf{r}_i\|^n = \sigma_i \pi \left(\frac{a^2+b^2}{2}\right) \frac{2\kappa}{1+\kappa^2} \times \frac{\sqrt{1+(\kappa^2-1)\left(\frac{\mathbf{r}_i\cdot\dot{\mathbf{r}}}{\|\mathbf{r}_i\|}\right)^2}}{\|\mathbf{r}_i\|^n}$$

Reduces to:

$$C(\mathbf{r}_i, \dot{\mathbf{r}}) = \frac{2\kappa\sigma_i}{1+\kappa^2} \frac{\sqrt{\|\mathbf{r}_i\|^2 + (\kappa^2-1)(\mathbf{r}_i\cdot\dot{\mathbf{r}})^2}}{\|\mathbf{r}_i\|^{n+1}}$$

Formulating the Model - 7

Risk of detection by N radars at some point r = (x, y, z) is the sum of the risk functions (previous slide) for all i:

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \sum_{i=1}^{N} C(\mathbf{r}_i, \dot{\mathbf{r}})$$

$$= \frac{2\kappa}{1+\kappa^2} \sum_{i=1}^{N} \sigma_i \frac{\sqrt{\|\mathbf{r}_i\|^2 + (\kappa^2 - 1)(\mathbf{r}_i \cdot \dot{\mathbf{r}})^2}}{\|\mathbf{r}_i\|^{n+1}}$$

The Final Model

Finally, we obtain the risk of detection as:

$$\mathcal{F}(\mathbf{r}, \dot{\mathbf{r}}) = \int_0^l L\left(\mathbf{r}(s), \dot{\mathbf{r}}(s)\right)\, ds$$

The complete problem can now be formulated as:

$$\begin{aligned} \min_{P} \quad & \mathcal{F}(\mathbf{r}, \dot{\mathbf{r}}) \\ \text{s.t.} \quad & \dot{\mathbf{r}}^2 = 1, \\ & \mathbf{r}(0) = \mathbf{r}_A,\ \mathbf{r}(l) = \mathbf{r}_B, \quad l \leq l_* \end{aligned}$$

Approach 1: Calculus of Variations

Case: Single Radar

Procedure: Reduce the minimization problem to a vectorial differential equation to get analytical solution, reduce this to a nonlinear first order differential equation.

Solution: The solution to this differential equation is expressed by a quadrature

General Case: Minimization of a Functional with Non-Holonomic Constraint and movable End Point.

Consider the following general Calculus of Variations Problem:

$$\min_{\mathbf{r}} \Phi(\mathbf{r}, \dot{\mathbf{r}}, l), \tag{36}$$

$$\Phi(\mathbf{r}, \dot{\mathbf{r}}, l) = \int_0^l L(\mathbf{r}(s), \dot{\mathbf{r}}(s))ds, \tag{37}$$

$$\mathbf{r}(0) = \mathbf{r}_1, \quad \mathbf{r}(l) = \mathbf{r}_2, \tag{38}$$

$$\phi(\dot{\mathbf{r}}(s)) = 0, \tag{39}$$

$$l \leq l_*, \tag{40}$$

where $\mathbf{r}(s) = (x(s), y(s), z(s))$ and $\dot{\mathbf{R}}(s) = (\dot{x}(s), \dot{y}(s), \dot{z}(s))$.

Derivation Continued - 1

We begin with the Lagrangian multiplier method

Let $\Phi(L, \phi, \lambda, l) = \int_0^l (L(\mathbf{r}, \dot{\mathbf{r}}) + \lambda(s)\phi(\dot{\mathbf{r}}))ds$ be a Lagrangian for (36)–(39)

By definition, the variation of the Lagrangian can be expressed by:

$$\delta\Phi = \int_0^l (\delta L(\mathbf{r}, \dot{\mathbf{r}}) + \lambda\, \delta\phi(\dot{\mathbf{r}}) + \phi\delta\lambda)ds + (L + \lambda\,\phi)|_{s=l}\,\delta l$$

$$= \int_0^l \left(\frac{\partial L}{\partial \mathbf{r}} \cdot \delta\mathbf{r} + \frac{\partial L}{\partial \dot{\mathbf{r}}} \cdot \delta\dot{\mathbf{r}} + \lambda\, \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \cdot \delta\dot{\mathbf{r}} + \phi\delta\lambda \right) ds + (L + \lambda\phi)|_{s=l}\,\delta l$$

$$= \int_0^l \left[\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{ds}\frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{d}{ds}\left(\lambda \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \right) \right) \cdot \delta\mathbf{r} + \phi\delta\lambda \right] ds$$

$$+ \left[\left(\frac{\partial L}{\partial \dot{\mathbf{r}}} + \lambda\, \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \right) \cdot \delta\mathbf{r} \right] \Bigg|_{s=l} + (L + \lambda\phi)|_{s=l}\,\delta l,$$

Derivation Continued - 2

- We get that $\delta\mathbf{r}(l+\delta l) \equiv \delta\mathbf{r}(l) + \dot{\mathbf{r}}\delta l = 0$
- This implies that $\delta\mathbf{r}(l) = -\dot{\mathbf{r}}\delta l$
- From here the variation becomes:

$$\delta\Phi = \int_0^l \left[\left(\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{ds}\frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{d}{ds}\left(\lambda\,\frac{\partial \phi}{\partial \dot{\mathbf{r}}}\right)\right)\cdot\delta\mathbf{r} + \phi\delta\lambda\right] ds$$

$$+ \left[L + \lambda\,\phi - \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} + \lambda\,\frac{\partial \phi}{\partial \dot{\mathbf{r}}}\right)\cdot\dot{\mathbf{r}}\right]\Bigg|_{s=l}\delta l$$

Derivation Continued - 3

Constraint (39) is relaxed, the three variations are independent. Thus, there necessary conditions for an extremum simply reduce to the constraint itself and the equations:

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{ds}\frac{\partial L}{\partial \dot{\mathbf{r}}} - \frac{d}{ds}\left(\lambda \frac{\partial \phi}{\partial \dot{\mathbf{r}}}\right) = 0, \tag{41}$$

and

$$\left[L - \left(\frac{\partial L}{\partial \dot{\mathbf{r}}} + \lambda \frac{\partial \phi}{\partial \dot{\mathbf{r}}}\right) \cdot \dot{\mathbf{r}}\right]\Bigg|_{s=l} = 0. \tag{42}$$

Derivation Continued - 4

After integration:

$$L - \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} - \lambda \left(\dot{\mathbf{r}} \cdot \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \right) = C, \tag{43}$$

We can solve for the Lagrange multiplier from this expression and see that it has the following form:

$$\lambda(s) = \left(L - \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} + c_L \right) \Big/ \left(\dot{\mathbf{r}} \cdot \frac{\partial \phi}{\partial \dot{\mathbf{r}}} \right), \tag{44}$$

Derivation Completed

Substituting
$$\lambda(s) = \left(L - \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} + c_L\right) \Big/ \left(\dot{\mathbf{r}} \cdot \frac{\partial \phi}{\partial \dot{\mathbf{r}}}\right), \tag{44}$$

into the first of our two necessary conditions for an extremum, we get the vectorial differential equation:

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}} + \frac{\frac{\partial \phi}{\partial \dot{\mathbf{r}}}}{\dot{\mathbf{r}} \cdot \frac{\partial \phi}{\partial \dot{\mathbf{r}}}}\left(L - \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} + c_L\right)\right) = 0. \tag{45}$$

The non-holonomic constraint:

$$\phi(\dot{\mathbf{r}}(s)) = 0 \tag{39}$$

And the boundary conditions:

$$\mathbf{r}(0) = \mathbf{r}_1, \quad \mathbf{r}(l) = \mathbf{r}_2, \tag{38}$$

Specific Case Examined - 1

In our case, the single-radar system, an optimal solution to the optimization problem will satisfy the vectorial differential equation:

$$\sum_{i=1}^{N} \sigma_i \left(\left(\frac{\mathbf{r}_i}{\mathbf{r}_i \cdot \dot{\mathbf{r}}} - \dot{\mathbf{r}} \right) \dot{g}_i - \ddot{\mathbf{r}}\, g_i \right) = 0, \tag{7}$$

with boundary conditions

$$\mathbf{r}(0) = \mathbf{r}_A, \quad \mathbf{r}(l) = \mathbf{r}_B, \quad l \leq l_*, \tag{8}$$

and the constraint

$$\phi(\dot{\mathbf{r}}) = \dot{\mathbf{r}}^2 - 1 = 0. \tag{9}$$

Specific Case Examined - 2

Here, we have that g_i is defined by

$$g_i(\mathbf{r}_i, \dot{\mathbf{r}}, \lambda_L) = \frac{1}{\|\mathbf{r}_i\|^{n-1}\sqrt{\|\mathbf{r}_i\|^2 + (\kappa^2 - 1)\,(\mathbf{r}_i \cdot \dot{\mathbf{r}})^2}} + \lambda_L$$

Using constraint 9 from the previous slide, we get the relation:

$$\frac{\partial \phi}{\partial \dot{\mathbf{r}}} \Big/ (\dot{\mathbf{r}} \cdot \frac{\partial \phi}{\partial \dot{\mathbf{r}}}) \equiv \dot{\mathbf{r}}$$

In turn, plugging this into the derived vectorial differential equation, obtain:

$$\frac{\partial L}{\partial \mathbf{r}} - \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}} + \dot{\mathbf{r}}\left(L - \dot{\mathbf{r}} \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}} + c_L\right)\right) = 0$$

Specific Case Examined - 3

- Single radar – located at the origin of the system
- $(a_1, a_2, a_3) = (0, 0, 0)$, r_1 = r, σ_1 = 1
- Our equation 7 becomes: $\left(\frac{\mathbf{r}}{\mathbf{r}\cdot\dot{\mathbf{r}}} - \dot{\mathbf{r}}\right)\dot{g} - \ddot{\mathbf{r}}\, g = 0.$
- Similarly, functions $L(\mathbf{r}, \dot{\mathbf{r}}), g(\mathbf{r}, \dot{\mathbf{r}}, \lambda_L)$ become

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{2\kappa}{1+\kappa^2} \frac{\sqrt{\|\mathbf{r}\|^2 + (\kappa^2 - 1)(\mathbf{r}\cdot\dot{\mathbf{r}})^2}}{\|\mathbf{r}\|^{n+1}},$$

$$g\left(\mathbf{r}, \dot{\mathbf{r}}, \lambda_L\right) = \frac{1}{\|\mathbf{r}\|^{n-1}\sqrt{\|\mathbf{r}\|^2 + \left(\kappa^2 - 1\right)\left(\mathbf{r}\cdot\dot{\mathbf{r}}\right)^2}} + \lambda_L$$

Specific Case Examined - 4

Obtain:

$$\frac{d}{ds}([\mathbf{r} \times \dot{\mathbf{r}}]\, g) = 0$$

Which is equivalent to having the first integral of:

$$[\mathbf{r} \times \dot{\mathbf{r}}]\, g = \mathbf{C}$$

Where C = (C_1, C_2, C_3), constant vector.

Conclude that C $\cdot$ r = 0 and optimal trajectory is planar curve in 3-D space!!

Specific Case Examined - 5

- Next step: Introduce a system of polar coordinates (ρ, φ) in the trajectory's plane to obtain (1st Order ODE):

$$\frac{1}{\rho^{n-2}\sqrt{\kappa^2(\rho'_\psi)^2 + \rho^2}} + \frac{\lambda_L \rho^2}{\sqrt{(\rho'_\psi)^2 + \rho^2}} = C, \tag{11}$$

with boundary conditions

$$\rho(\psi_A) = \rho_A, \quad \rho(\psi_B) = \rho_B, \tag{12}$$

and the constraint on trajectory length

$$\int_{\psi_A}^{\psi_B} \sqrt{(\rho'_\psi)^2 + \rho^2}\, d\psi = l_*, \tag{13}$$

Specific Case Examined – 6

The analytical solution to this equation is given by a quadrature:

$$\psi(\rho) = \psi_A \pm \int_{\rho_A}^{\rho} \frac{d\tau}{\sqrt{(\upsilon^*(\tau, \lambda_L, C))^2 - \tau^2}}$$

Where υ^* is a positive root of a quartic equation:

$$f(\upsilon) = \rho^{n-2}\left(C\upsilon - \lambda_L \rho^2\right)\sqrt{\kappa^2\upsilon^2 + (1-\kappa^2)\rho^2} - \upsilon = 0$$

Calculus of Variations Conclusions

- Optimal risk is more sensitive to variation of the shape of the ellipsoid (k), than to the variation of a trajectory's total length, l_*
- Optimal trajectories for different k (esp. for k > 1) are very similar. Means that variation of ellipsoid shape has no strong effect on geometry of optimal trajectory
- When close by, optimal trajectory is more sensitive to a radar, than to a sensor

Approach 2: Network Optimization

Advantages:

- Finds globally optimal solutions, not locally optimal ones
- Can be generalized to several detecting installations

Method:

- Approximates admissible domain for trajectory by a 3-D network and represents the trajectory by a path in network
- Reduced to Constrained Shortest Path Problem (CSPP)
- Use Label Setting Algorithm (LSA) and path smoothing to solve
- CSPP is NP-complete → no exact polynomial algorithms to be expected (but answer can be verified quickly)

Path Smoothing and Diagram

- Undirected Network is formed: G = (N, A), where N = {1,…,n} is the set of nodes and A is the set of undirected arcs (Figure 1)
- Path smoothing means that in a path, we choose only those arcs which produce the angle with a preceding arc not greater than α (Figure 2)

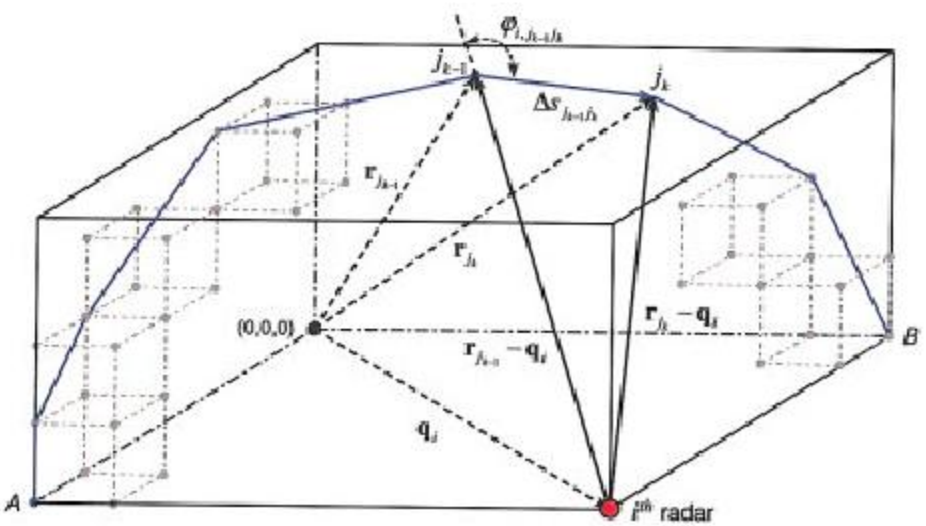


Figure 1

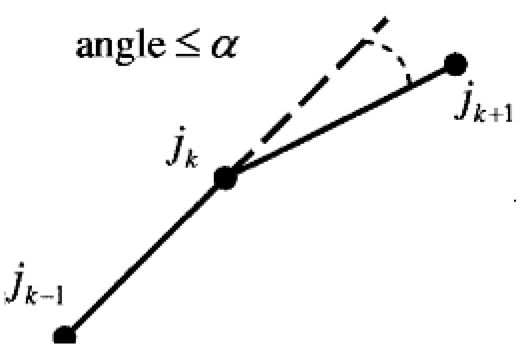


Figure 2

Discrete Optimization Conclusions

- Optimal trajectories obtained by analytical and discrete optimization approaches are very close – validates both approaches
- In case of single radar, optimal risk increases with parameter k. In examples with two/three radars, optimal risk for a spherical RCS is greater than that for elongated and compressed ellipsoids. Thus, in case of several radars, a sphere, being uniformly exposed to all radars, may accumulate greatest total risk along trajectory
- Cases with several radars, optimal trajectories with the same constraint on length but different ellipsoid shapes are relatively close to each other. Implies that in general, ellipsoid shape has no strong effect on geometry of optimal trajectory

Summary

- Formulated model for trajectory which minimizes detection
- Derived vectorial differential equation for minimization of a functional with Non-Holonomic constraint and movable end point
- Analyzed the quadrature as a solution to this equation
- Looked at the discrete optimization approach to solving the problem and then analyzed the solutions this method returned
- COMPARISON? – Both methods work and are validated!

References

Zabarankin, M., Uryasev, S., and Murkey, R. 2006. Aircraft routing under risk of detection. *Naval Research Logistics*, 53, 728-747.

Thank You!