A mean-field model for signal transmission in coupled neurons

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Structure of a Neuron

- Neuron components: dendrites, soma, axon, terminals, myelin sheath

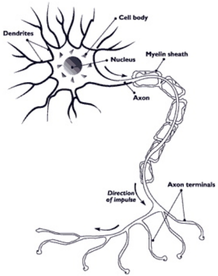

Figure: A Nerve cell[1]

[1] http://teens.drugabuse.gov/mom/tg_nerves.php

Myelin

- Most neuron axons sheathed in myelin
- Prevents crosstalk
- Disrupted myelin = bad news
- Justifies neglect of interactions

but...

- Cardiac tissue
- Olfactory system
- Ephaptic interactions

Hodgkin-Huxley Model

- Squid giant axon
- Eliminate x
- Four coupled ODEs

$$C_m \frac{dv}{dt} = -\bar{g}_K n^4 (v - v_K) - \bar{g}_{Na} m^3 h (v - v_{Na}) - \bar{g}_L (v - v_L) + I_{ext}$$
$$\frac{dm}{dt} = \alpha_m (1 - m) - \beta_m m$$
$$\frac{dn}{dt} = \alpha_n (1 - n) - \beta_n n$$
$$\frac{dh}{dt} = \alpha_h (1 - h) - \beta_h h$$

- Agrees well with physical experiment
- Difficult to analyze

FitzHugh-Nagumo Model

- Two coupled ODEs

$$V_i = \begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

$$\frac{\partial}{\partial t} V_i = f^{FHN}(V_i) = \begin{pmatrix} 0 & -1 \\ -c & b \end{pmatrix} V_i + \begin{pmatrix} v_i(v_i - a)(1 - v_i) \\ 0 \end{pmatrix}$$

- Qualitatively similar to HH model
- Phase-plane analysis
- Polynomial

Self Dynamics - Behavior of FHN Neuron

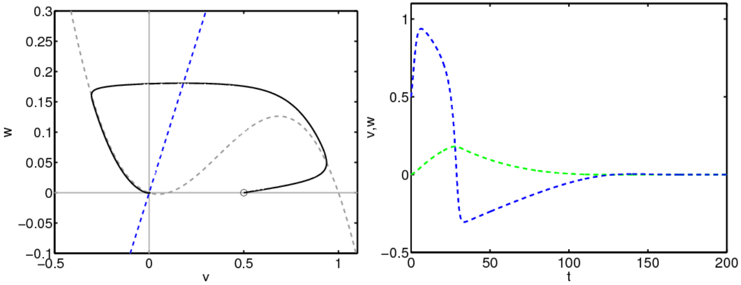

Figure: FHN Self Dynamics, Phase Plane and Time Domain

- No interaction term
- Represents the behavior of the neuron in isolation
- *in vitro* conditions or Myelenated neurons

Interaction Model

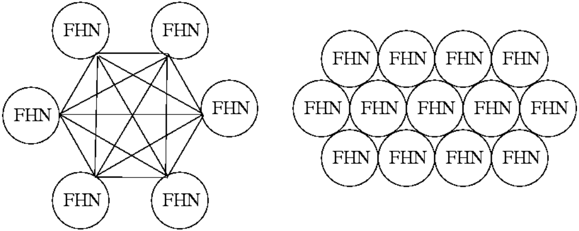

Figure: A network of FHN neurons

- Consider only pairwise interactions
- What do we want to measure?

Model with Interactions

$$\frac{dV_i}{dt} = f_{FHN}(V_i) + \sum_{j=1, j\neq i}^{N} e_{ij}(v_i, v_j, t)$$

If e is time-independent and analytic,

$$\frac{dV_i}{dt} = f_{FHN}(V_i) + \sum_{j=1, j\neq i}^{N} \sum_{m=1}^{p_1} \sum_{n=1}^{p_2} a_{ijmn} v_i^m v_j^n$$

$$\frac{dV_i}{dt} = f_{FHN}(V_i) + \sum_{m=1}^{p_1} \sum_{n=1}^{p_2} v^m A_{mn} v^n$$

Averaged Model

- Perturbation expansion with $v_i = \overline{v} + \epsilon\xi_i$, $w_i = \overline{w} + \epsilon\omega_i$,

$$\frac{d\overline{v}}{dt}(t) = (\overline{v}(\overline{v} - a)(1 - \overline{v})) - \overline{w} + \frac{1}{N}\sum_m \sum_n \vec{1}^T A_{mn} \vec{1}\,\overline{v}^{m+n}$$

$$\frac{d\overline{w}}{dt}(t) = -c\overline{v} + b\overline{w}$$

$$\frac{d\overline{\xi}}{dt}(t) = -\overline{\omega} + \frac{1}{N}\sum_m \sum_n (m \cdot \vec{1}^T A_{mn}\xi + n \cdot \xi^T A_{mn}\vec{1})$$

$$\frac{d\overline{\omega}}{dt}(t) = -c\overline{\xi} + b\overline{\omega}$$

Constraints by Geometry

- In order for $\frac{1}{N}\vec{1}^T A_{mn}\vec{1}$ terms not to dominate nor to be negligible, restrictions on A_{mn}

Name	Condition
Constant	$a_{ij} \sim O(\frac{1}{N})$
Identical rows/columns	$\sum_j a_{ij} \sim O(1)$
Sparse K nonzero elements	$\sum_{k=1}^{K} a_{(ij)_k} \sim O(N)$

FHN with Interactions

Implications:

- The mean interaction term will be at most twice the degree of the per-neuron interaction term
- FHN model is already cubic polynomial
- FHN + linear or bilinear interactions $\Rightarrow$ modified FHN

Experimental setup:

- Identical or near-identical initial conditions
- Full interaction (cocktail party)
- Linear and bilinear interactions

FHN with linear interactions

Apply linear interactions to FHN model neuron:

$$\frac{dv_i}{dt} = a_3 v_i^3 + a_2 v_i^2 + a_1 v_i - w_i + \sum_{j=1, j\neq i}^{N} \frac{a_{int}}{N} v_j$$

$$\frac{dw_i}{dt} = b v_i - c w_i$$

Prediction for the mean field:

$$\frac{d\overline{v}}{dt} = a_3 \overline{v}^3 + a_2 \overline{v}^2 + (a_1 + a_{int})\overline{v} - \overline{w}$$

$$\frac{d\overline{w}}{dt} = b\overline{v} - c\overline{w}$$

Parametrized Behavior of FHN Neuron

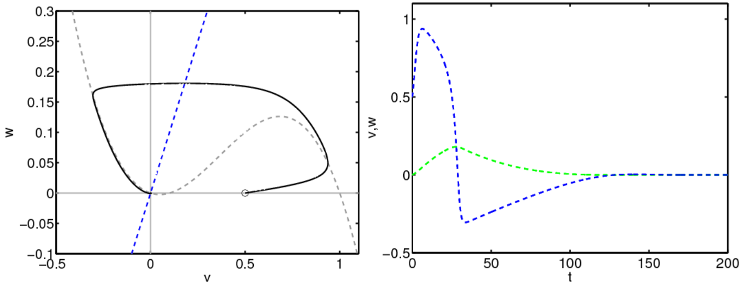

Figure: FHN Self Dynamics, Phase Plane and Time Domain

Linear FHN: Elongated Spike

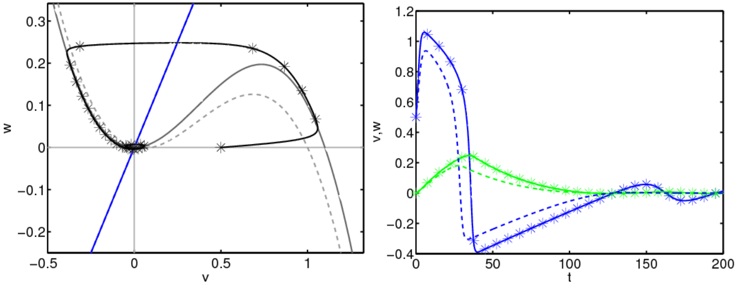

Figure: Elongated Spike, Phase Plane and Time Domain

Linear FHN: Subthreshold Oscillations

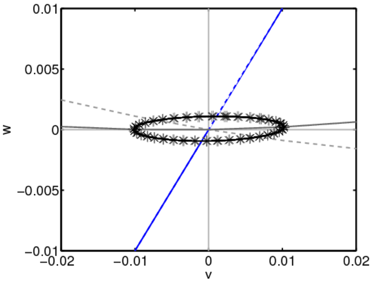

Figure: Subthreshold Oscillations, Phase Plane and Time Domain

Linear FHN: Relaxation-Oscillation

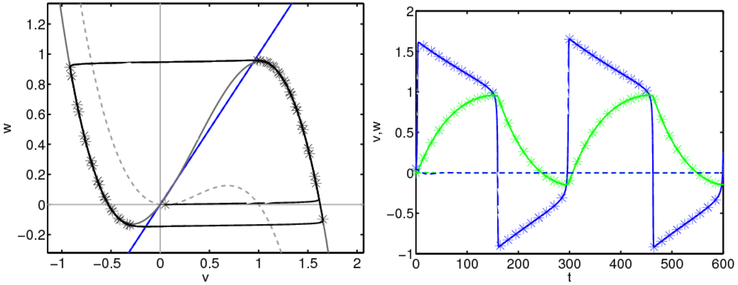

Figure: Relaxation-Oscillations, Phase Plane and Time Domain

FHN with bilinear interactions

Apply bilinear interactions to FHN model neuron:

$$\frac{dv_i}{dt} = a_3 v_i^3 + a_2 v_i^2 + a_1 v_i - w_i + \sum_{j=1, j\neq i}^{N} \frac{a_{int}}{N} v_i v_j$$

$$\frac{dw_i}{dt} = b v_i - c w_i$$

Prediction for the mean field:

$$\frac{d\overline{v}}{dt} = a_3 \overline{v}^3 + (a_2 + a_{int}) \overline{v}^2 + a_1 \overline{v} - \overline{w}$$

$$\frac{d\overline{w}}{dt} = b\overline{v} - c\overline{w}$$

Bilinear FHN: Suppression

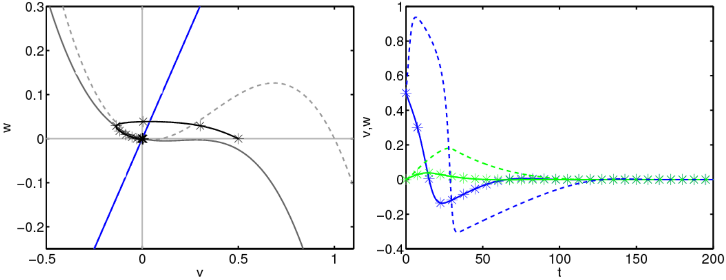

Figure: Bilinear Suppression, Phase Plane and Time Domain

Bilinear FHN: Bistability

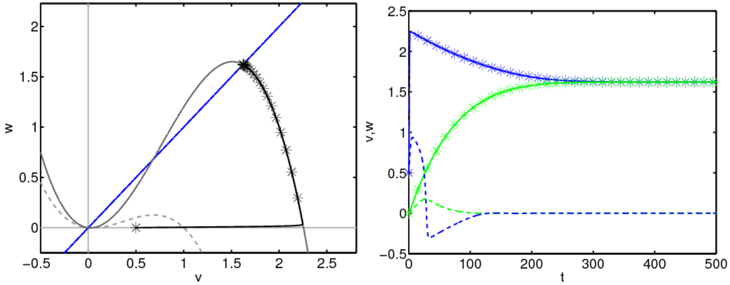

Figure: Bilinear bistability, Phase Plane and Time Domain

Summary

- Small interaction terms effect qualitative changes in nerve response
- Add space variable?
- Criteria for synchronization?
- Relax assumptions
- Agreement with experiment?

For more information

Eli Shlizerman, Konrad Schroder and J. Nathan Kutz. *Neural activity measures and their dynamics*. Forthcoming.