The Swinging Spring: Regular and Chaotic Motion

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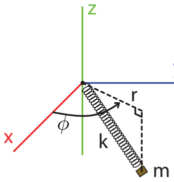
Outline of Talk

- Introduction to Problem
- The Basics: Hamiltonian, Equations of Motion, Fixed Points, Stability
- Linear Modes
- The Progressing Ellipse and Other Regular Motions
- Chaotic Motion
- References

Introduction

The swinging spring, or elastic pendulum, is a simple mechanical system in which many different types of motion can occur. The system is comprised of a heavy mass, attached to an essentially massless spring which does not deform. The system moves under the force of gravity and in accordance with Hooke's Law.



The Basics

We can write down the equations of motion by finding the Lagrangian of the system and using the Euler-Lagrange equations. The Lagrangian, L is given by

$$L = T - V$$

where T is the kinetic energy of the system and V is the potential energy.

The Basics

In Cartesian coordinates, the kinetic energy is given by the following:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the potential is given by the sum of gravitational potential and the spring potential:

$$V = mgz + \frac{1}{2}k(r - l_0)^2$$

where m is the mass, g is the gravitational constant, k the spring constant, r the stretched length of the spring ($\sqrt{x^2 + y^2 + z^2}$), and l_0 the unstretched length of the spring.

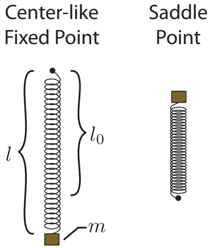
The Basics

The equations of motion are then given by:

$$\begin{aligned}
\ddot{x} &= -\frac{k}{m}\left(\frac{r-l_0}{r}\right)x \\
\ddot{y} &= -\frac{k}{m}\left(\frac{r-l_0}{r}\right)y \\
\ddot{z} &= -\frac{k}{m}\left(\frac{r-l_0}{r}\right)z - g
\end{aligned}$$

The Basics

This system has two fixed points: one linear center where the bob is hanging straight down $(x, y, z) = (0, 0, -l)$ and one saddle-type fixed point where the bob is poised just above where it is attached.



This talk will not consider the saddle-type fixed point.

The Basics

There are two constants of motion, the total energy E of the system, and the total angular momentum, h:

$$E = T + V$$

$$h = x\dot{y} - y\dot{x}$$

With only two first integrals and three spacial coordinates, the system is not integrable.

Linear Modes

- Consider very small amplitude motion about the fixed point. Linearizing about the fixed point $(0, 0, -l)$ (i.e. $r \approx l$), we obtain the equations for small oscillations:

$$\ddot{x} = -\frac{g}{l}x, \quad \ddot{y} = -\frac{g}{l}y \quad \ddot{z} = -\frac{k}{m}z$$

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- All three equations are easily solved and are simply sums of sines and cosines. The x and y components trace out an ellipse and have frequency $\omega_R = \sqrt{\frac{g}{l}}$ which is the same as a rigid pendulum.
- The vertical height varies sinusoidally with frequency $\omega_Z = \sqrt{\frac{k}{m}}$ which is that of a spring oscillating in one dimension.

Linear Modes

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- Example: $\epsilon = 2$, the bob will rotate around twice in the $x - y$ projection before returning to the initial position. For ϵ irrational, the motion will be quasi-periodic.
- Since the equations are completely decoupled in this approximation, we can expect no exchange of energy between swinging and springing modes. In other words, swinging does not induce springing and vice versa.

Elliptical Motion

Assuming small amplitude motion and dropping all terms of third order order or higher, the equations of motion become the following:

$$\begin{aligned}
\ddot{x} + \omega_R^2 x &= \lambda xz \\
\ddot{y} + \omega_R^2 y &= \lambda yz \\
\ddot{z} + 4\omega_R^2 z &= \frac{1}{2}\lambda(x^2 + y^2)
\end{aligned}$$

where $\lambda = l_0\omega_Z^2/l$ and it is assumed $\omega_Z = 2\omega_R$.

Elliptical Motion

Seek solutions which at lowest order are periodic and elliptical in the $x - y$ projection:

$$\begin{aligned} x &= \epsilon(A\cos\omega t) + \epsilon^2 x_2 + \cdots \\ y &= \epsilon(B\cos\omega t) + \epsilon^2 y_2 + \cdots \\ z &= \epsilon(C\cos 2\omega t) + \epsilon^2 z_2 + \cdots \end{aligned}$$

where ϵ is a small parameter, A, B, C constants, and $\omega = \omega_0 + \epsilon\omega_1 + \cdots$

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Elliptical Motion

- For $A = 0$ or $B = 0$, the solutions will be approximately cup shaped or cap shaped.
- For $C = 0$, we will have at lowest order that the elastic pendulum sweeps out a cone shape with the height of the bob approximately constant.
- There are no solutions to the case where $A, B, C \neq 0$. Instead change to a rotating coordinate frame to analyze.

The Progressing Ellipse

Using rotating coordinates α, β, and γ, with Θ varying with time and assuming the $\alpha - \beta$ axis rotating with constant angular velocity $\dot{\Theta} = \Omega$ we have:

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Denoting the rotation matrix as R, $\vec{\alpha} = (\alpha, \beta, \gamma)^T$ and $\vec{x} = (x, y, z)^T$, we have that $\ddot{\vec{\alpha}} = R\ddot{\vec{x}} + 2\dot{R}\dot{\vec{x}} + \ddot{R}\vec{x}$.

The Progressing Ellipse

Differentiating as required above, we get the following set of equations.

$$
\begin{aligned}
\ddot{\alpha} &+ (\omega_R^2 - \Omega^2)\alpha - 2\Omega\dot{\beta} = \lambda\alpha\gamma \\
\ddot{\beta} &+ (\omega_R^2 - \Omega^2)\beta + 2\Omega\dot{\alpha} = \lambda\beta\gamma \\
\ddot{\gamma} &+ 4\omega_R^2\gamma = \frac{1}{2}\lambda(\alpha^2 + \beta^2)
\end{aligned}
$$

Again, seek solutions of the form:

$$
\begin{aligned}
\alpha &= \epsilon(A\cos\omega t) + \epsilon^2\alpha_2 + \cdots \\
\beta &= \epsilon(B\cos\omega t) + \epsilon^2\beta_2 + \cdots \\
\gamma &= \epsilon(C\cos 2\omega t) + \epsilon^2\gamma_2 + \cdots
\end{aligned}
$$

The Progressing Ellipse

- Assuming small rotation, i.e. $\Omega = \epsilon\Omega_1$ and plugging the form of the solution into the differential equations, and requiring growing terms in the second order equations of ϵ to go to zero (and after a lot of algebra), a set of algebraic equations for Ω_1, ω_1, A, B, and C is obtained.
- For fixed values of A and B, the equations can be solved explicitly for Ω_1, ω_1 and C, with two possible solutions.

Simulating the Progressing Ellipse

Parameter values used are the same as used by Lynch (2002) and are as follows:

- mass: $m = 1\text{kg}$
- equilibrium stretched length: $l = 1\text{m}$
- gravitational constant: $g = \pi^2$ m s^{-2}
- spring constant: $k = 4\pi^2$ kg s^{-1}

All simulations were done in **MATLAB** using **ode45** with an absolute error tolerance of 10^{-6}.

Simulating the Progressing Ellipse

Initial values used are as follows:

- $A = 0.01$
- $B = 0.005$
- $C \approx 0.0237$ is given by the following formula derived by Lynch(2002):

$$C = \pm \frac{A^2 - B^2}{2\sqrt{2(A^2 + B^2)}}$$

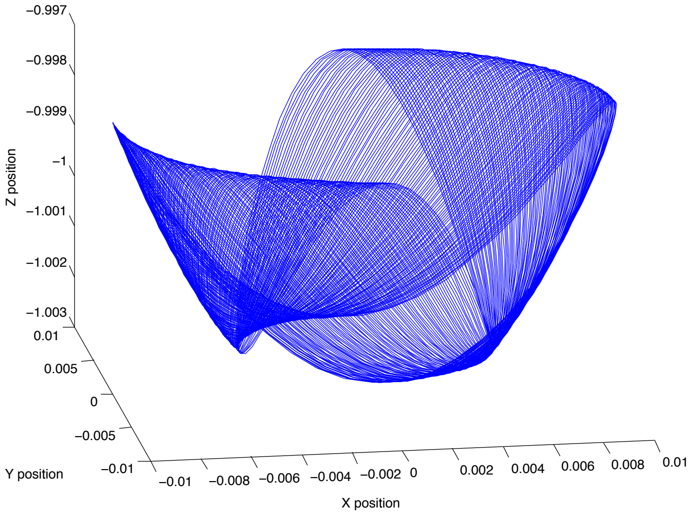
- $\omega = \omega_R + \omega_1$ where ω_1 is given by:

$$\omega_1 = \mp \frac{3\sqrt{2(A^2 + B^2)}}{16l} \omega_R$$

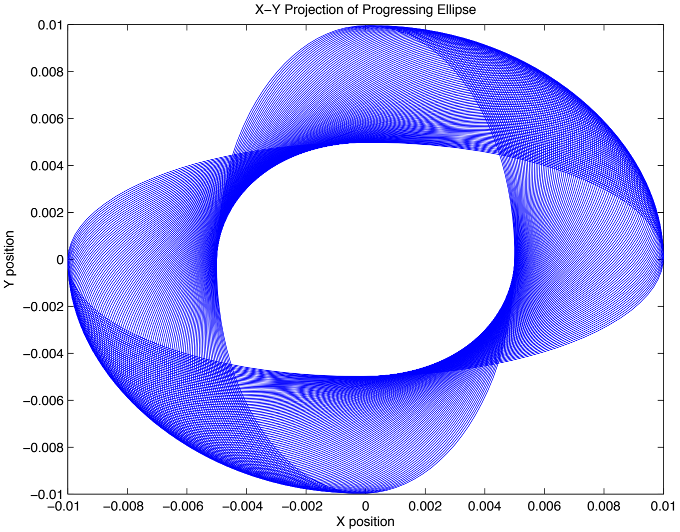
Simulating the Progressing Ellipse

The initial position was set to be $\vec{x} = (A, 0, C - l)^T$ and the initial velocity was set to be $\dot{\vec{x}} = (0, \omega B, 0)^T$. The 3-D image of the movement through 211 seconds (corresponding to 90° of precession) is shown:

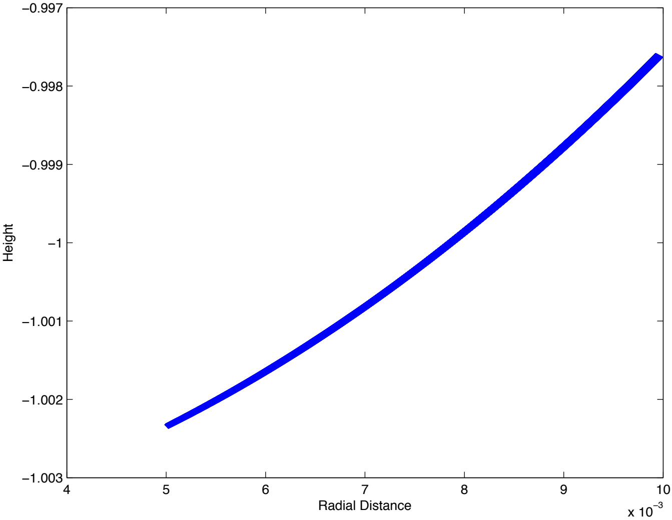
Simulating the Retrogressing Ellipse



Simulating the Progressing Ellipse



Simulating the Progressing Ellipse



Another Type of Motion?

- Question: Can we have any other types of motion?

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- Answer: Why, yes, yes we can!

Precession of the Swing Plane

- If the bob is started with almost entirely vertical oscillations, gradually the vertical oscillations subside and a swinging motion occurs.
- Swinging motion subsides and is replaced by a springing motion as before, and the process repeats.
- The motion appears planar (but is really elliptical).
- The "swing plane" rotates each time we return to swinging motion.

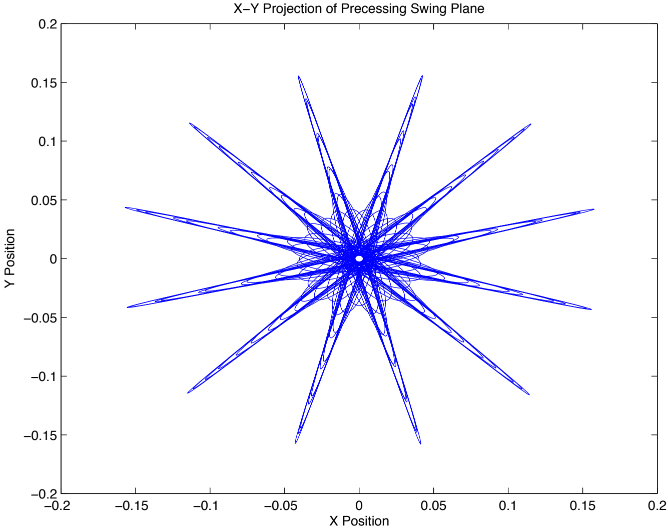
Simulating Precession of the Swing Plane

Initial conditions used leading to a precessing swing plane are as follows:

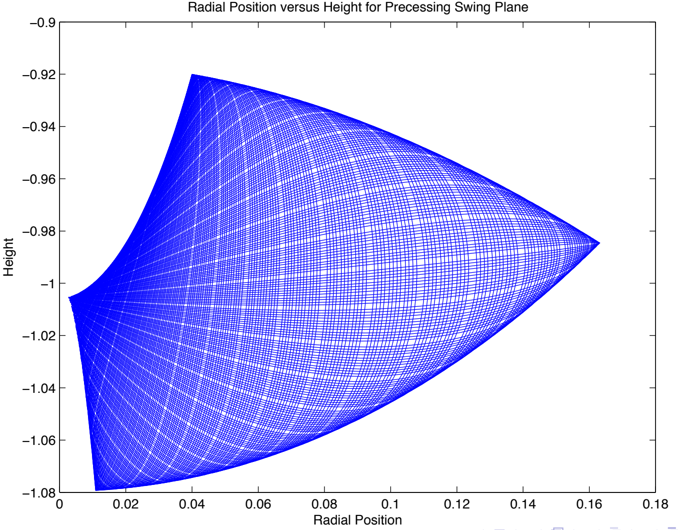
- $x_0 = 0.04\text{m}$, $y_0 = 0$
- $z_0 = -l + 0.08\text{m}$, note this corresponds to 8cm of compression.
- $\dot{x}_0 = 0$, $\dot{y}_0 = 0.03427\text{m/s}$, and $\dot{z}_0 = 0$.

For the complete derivation, see Lynch (2002).

Precession of Swing Plane



Precession of Swing Plane



Chaotic Motion

Can we find chaotic motion in this system? (Yes of course, else this would not be a very interesting system.)

Main characteristics of chaotic motion:

- Sensitive dependence on initial conditions - nearby trajectories diverge exponentially fast - i.e. positive Lyapunov exponent.
- Aperiodic long-term behavior - not all trajectories settle down to fixed points, or periodic/quasi periodic orbits.
- Trajectories densely fill the space.

Chaos in 2 Space Dimensions

Let's consider a simpler version of our original equations - restrict the motion to be planar.

Hamiltonian in dimensionless coordinates after rescaling length, time, and energy:

$$H = \frac{1}{2}(p_1^2 + p_2^2) + fq_2 + \frac{1}{2}\left(1 - f - \sqrt{q_1^2 + (1 - q_2^2)}\right)^2$$

where $f = \left(\frac{\omega_R}{\omega_Z}\right)^2$.

Chaos in 2 Space Dimensions

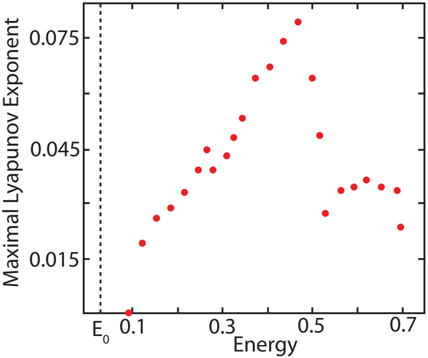


Figure : Graph of maximal Lyapunov exponent, adapted from Núñez-Yépez et al (1989).

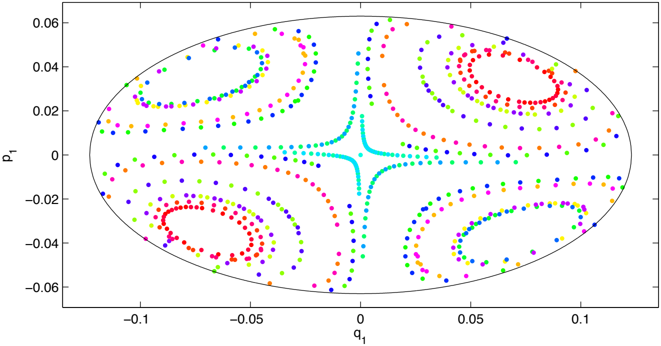
Chaos in 2 Space Dimensions

The positive Lyapunov exponent is a good indicator for sensitive dependence on initial conditions, but what about the other indicators of chaos?

Instead, let's look at some Poincaré sections.

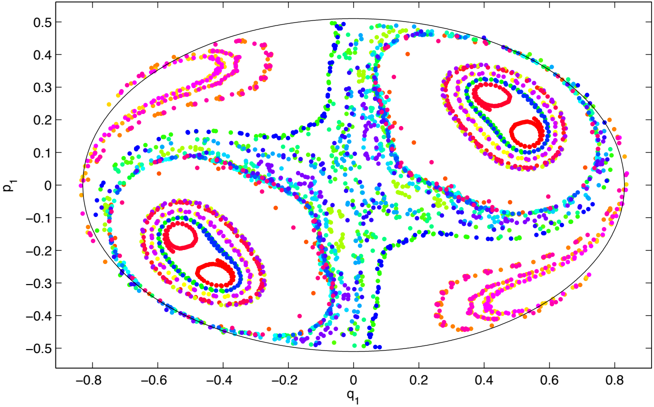
Chaos in 2 Space Dimensions

Low energy:



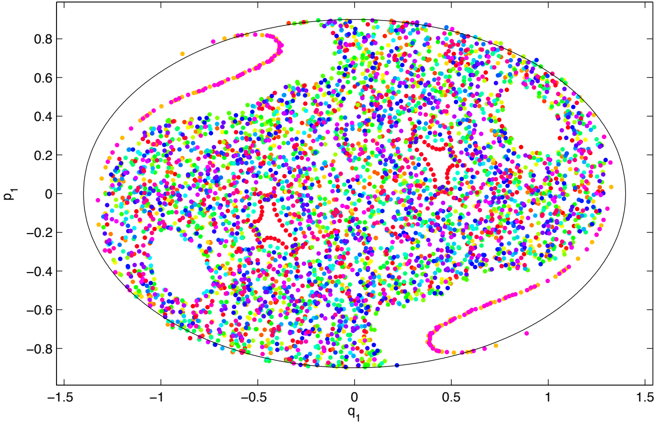
Chaos in 2 Space Dimensions

Slightly higher energy:



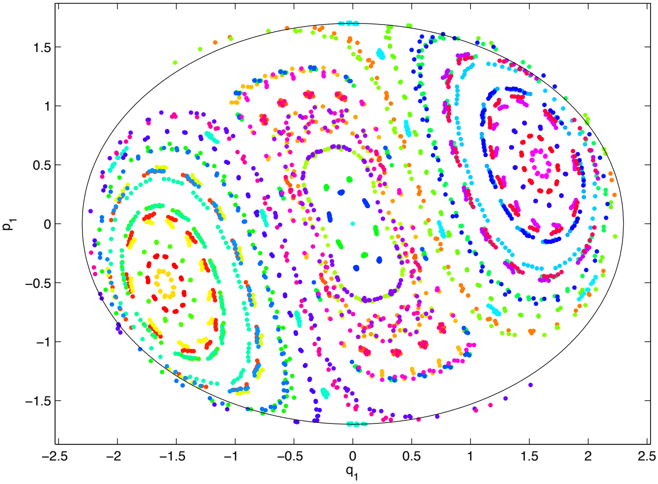
Chaos in 2 Space Dimensions

A little higher:



Chaos in 2 Space Dimensions

Cranked up all the way to 11:

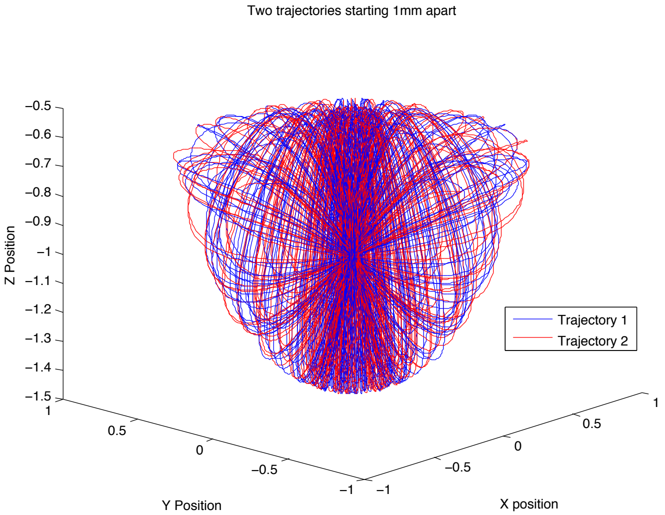


Chaos in 2 Space Dimensions

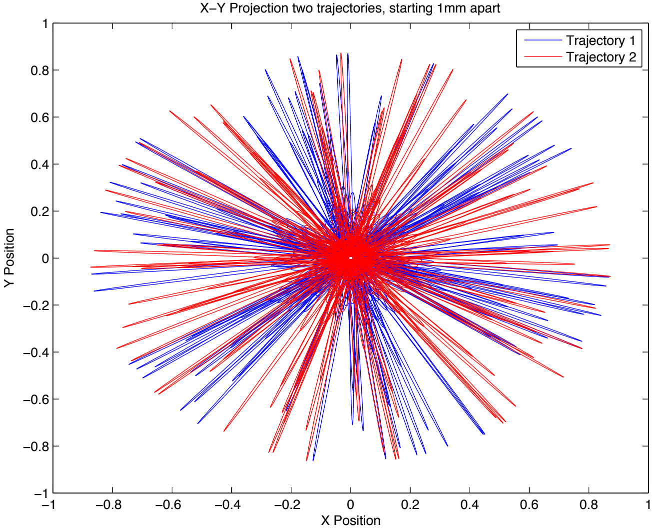
- For certain energy values, periodic or quasi periodic regular motion vanishes and trajectories tend to fill a 2-D space instead of a smooth curve.
- For very low energy, virtually all trajectories demonstrate regular motion.
- At high energies, there also appears to be regular motion.

What do these chaotic trajectories really look like back in 3-D?

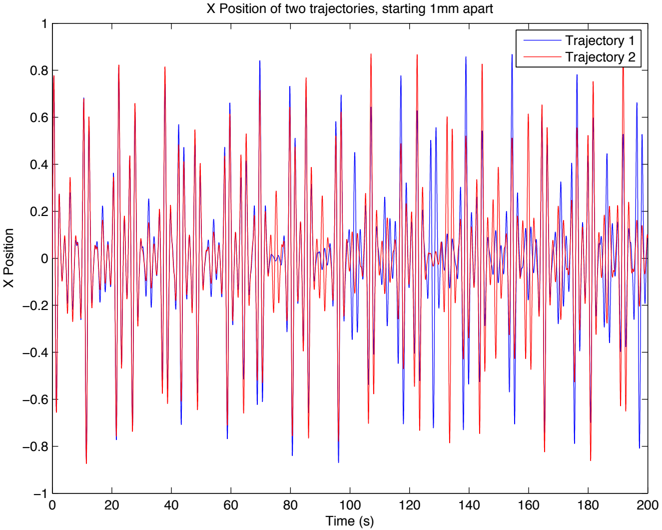
Chaotic Motion in 3-D



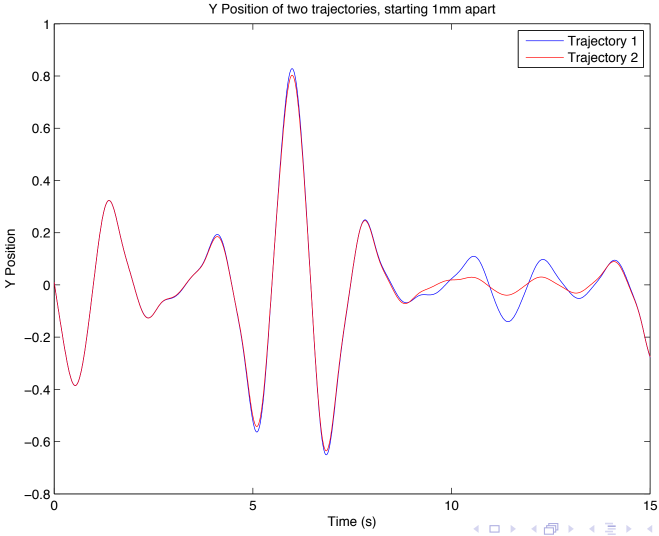
Chaotic Motion in 3-D



Chaotic Motion in 3-D



Chaotic Motion in 3-D



Conclusions

- Highly complex dynamics can occur from what seems like a very simple physical system.
- There are many qualitatively different types of regular motion.
- For certain energy values, the system demonstrates all the hallmarks of chaos.

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Special Thanks To...

Special thanks to my lovely fiancé Michael for debugging my code.

Figure : Polly the cat, who also helped with the making of this presentation.