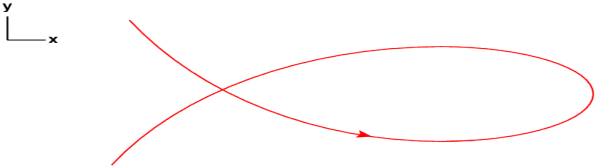
Melnikov's Method



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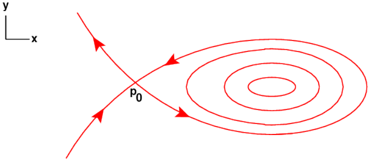
May 30, 2013

Consider a Hamiltonian flow

- $\dot{q} = JDH(q)$
  - $J =$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

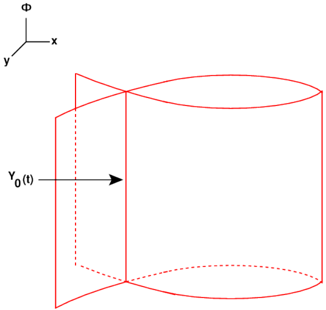
  - $DH = (\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y})$
  - flow is C^2
  - hyperbolic fixed point of saddle type
  - family of periodic curves inside the homoclinic orbit



Now perturb that system!

- append a $\frac{2\pi}{\omega}$ periodic function to that system.
- $\dot{q} = JDH(q) \boxed{\; + \;\; \varepsilon g(q, t, \varepsilon)}$
- Melnikov: A method to determine if this system is chaotic.

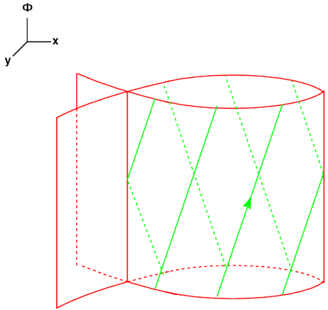
Get rid of the time dependence



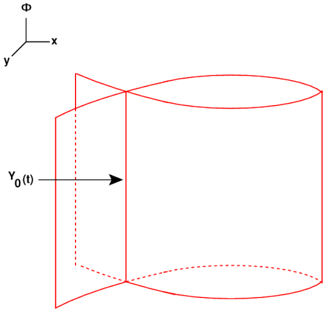
$$\begin{cases} \dot{q} = JDH(q) + \varepsilon g(q, \phi, \varepsilon) \\ \dot{\phi} = \omega \end{cases}$$

- for $\varepsilon = 0$, stable and unstable manifolds coincide to form a 2-D surface, Γ
- $\gamma_0(t) = (p_0, \phi(t))$
  $\phi(t) = \omega t + \phi_0$

A typical trajectory

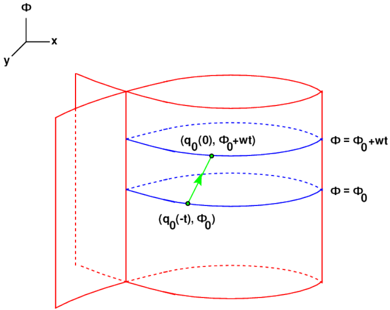


A paramaterization for Γ



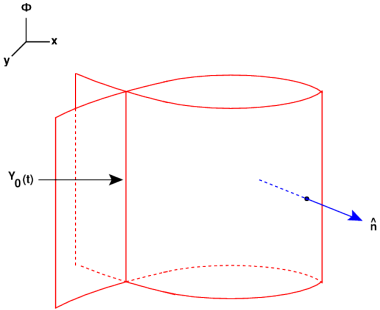
- $\Gamma = \{(q,\phi) \in \mathbb{R}^2 \times S^1 \,|\, q = q_0(-t_0),\ t_0 \in \mathbb{R},\ \phi = \phi_0 \in (0, 2\pi]\}$
- $(q_0(-t_0), \phi_0) \in \Gamma$ is unique

Interpreting the parameterization



- t_0 is the time of flight from a point $q_0(-t)$ to the point $q_0(0)$ on the homoclinic connection
- ϕ_0 is a horizontal cross section

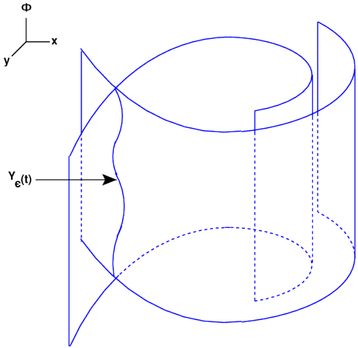
A vector normal to Γ



- For any point $p \in \Gamma$, there is a vector normal to the surface through p : $\hat{n}_p$

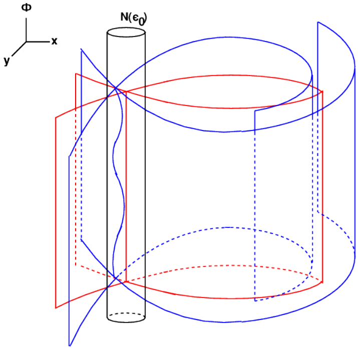
$$\begin{aligned} \hat{n}_p &= \left.(-\dot{y}, -\dot{x}, 0)\right|_p \\ &= \left.\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, 0\right)\right|_p \end{aligned}$$

Consider $\varepsilon \neq 0$



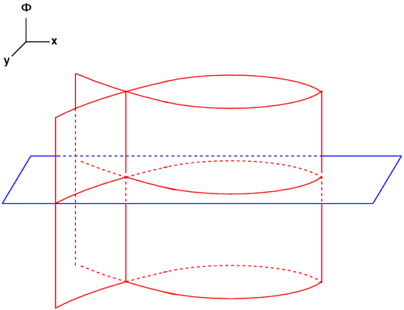
- Γ wiggles about
- $W^u \neq W^s$
- $\gamma_0(t)$ becomes $\gamma_\varepsilon(t)$
- System is non autonomous

For small ε, things are not so bad



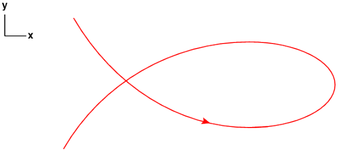
- $\gamma_\varepsilon(t)$ is still periodic, with same stability as unperturbed $\gamma_0(t)$
- $\gamma_\varepsilon(t) = \gamma_0(t) + O(\varepsilon)$
- $W^s_{loc}(\gamma_\varepsilon(t))$ is ε close to $W^s_{loc}(\gamma_0(t))$
- $W^u_{loc}(\gamma_\varepsilon(t))$ is ε close to $W^u_{loc}(\gamma_0(t))$

Look at a horizontal slice



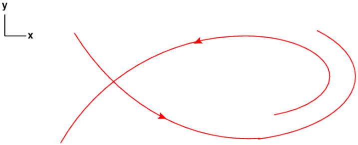
- Fix $\phi = \phi_0$
- $\Sigma^{\phi_0} \equiv \left\{ (p, \phi) \in \mathbb{R}^2 \middle| \phi = \phi_0 \right\}$

A cross section is only as good as the flow through it



Unperturbed flow

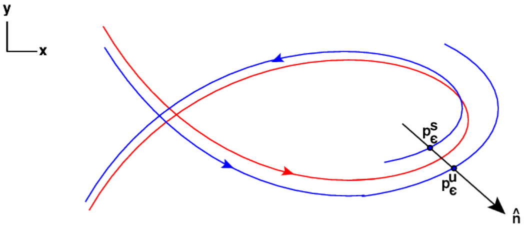
- stable manifold $W^s(\gamma_0(t))$
- unstable manifold $W^u(\gamma_0(t))$



Perturbed flow

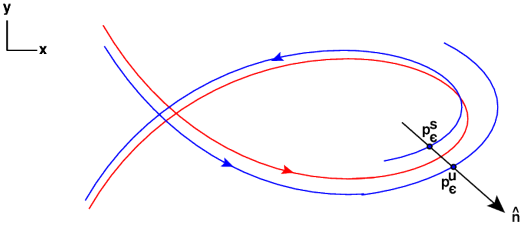
- stable manifold $W^s(\gamma_\varepsilon(t))$
- unstable manifold $W^u(\gamma_\varepsilon(t))$

Look at points intersecting $\hat{n}$



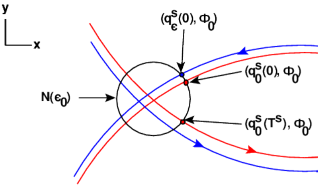
- $\{p_\varepsilon^s\} \equiv W^s(\gamma_\varepsilon(t)) \cap \hat{n}$
- $\{p_\varepsilon^u\} \equiv W^u(\gamma_\varepsilon(t)) \cap \hat{n}$

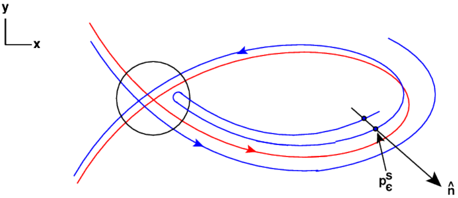
Pick the p_ε^s closest to $\gamma_\varepsilon(t)$



- $W^s(\gamma_\varepsilon(t))$ can intersect $\hat{n}$ more than once; pick the one that crosses $\hat{n}$ last in forward time
- $W^u(\gamma_\varepsilon(t))$ can intersect $\hat{n}$ more than once; pick the one that crosses $\hat{n}$ last in backwards time

Consequences of the choice of p^s_ε





Lemma:

Take a point $\bar{p}^s_\varepsilon$ that is *not* closest to $\gamma_\varepsilon(t)$. Let $(q^s_\varepsilon(t), \phi_0)$ be a trajectory such that $(q^s_\varepsilon(0), \phi(0)) = \bar{p}^s_\varepsilon$. For ε sufficiently small, $(q^s_\varepsilon(t > 0), \phi_0)$ must pass through the ε-neighborhood $N(\varepsilon_0)$ before intersecting again with $\hat{n}$.

proof:

$|(q^s_\varepsilon(t), \phi(t)) - (q^s_0(t), \phi(t))| = O(\varepsilon)$ for $t \geq 0$.

Then, by Gronwall's inequality:

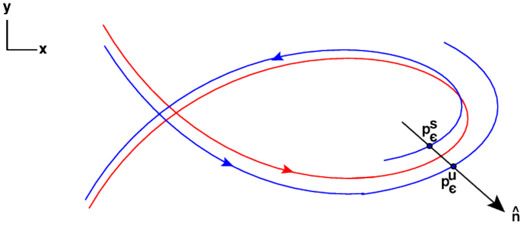
$|(q^s_\varepsilon(t), \phi(t)) - (q^s_0(t), \phi(t))| = O(\varepsilon)$ for $t \in [T^s, 0]$.

$\exists \bar{t} > 0$ such that $(q^s_\varepsilon(\bar{t}), \phi_0) \in W^s(\gamma_\varepsilon(t)) \cap \hat{n}$.

Then for some $t \in (0, \bar{t})$, $(q^s_\varepsilon(t), \phi(t))$ must have entered $N(\varepsilon_0)$.

A similar result holds for the unstable manifold.

Introduce a distance function



$$\text{let } d(p_0, \varepsilon) \equiv |p_\varepsilon^u - p_\varepsilon^s|$$

- write it (for convenience) as $\frac{(p_\varepsilon^u - p_\varepsilon^s)\cdot(DH(q_0(-t_0)),0)}{\|DH(q_0(-t_0))\|}$
- use the parameterization $p_\varepsilon^s = (q_\varepsilon^s, \phi_0)$

$$\boxed{d(t_0, \phi_0, \varepsilon) = \frac{DH(q_0(-t_0)) \cdot (q_\varepsilon^u - q_\varepsilon^s)}{\|DH(q_0(-t_0))\|}}$$

Taylor expand $d(t_0, \phi_0, \varepsilon)$ about $\varepsilon = 0$

$$d(t_0, \phi_0, \varepsilon) = d(t_0, \phi_0, 0) + \varepsilon \frac{\partial d}{\partial \varepsilon}(t_0, \phi_0, 0) + O(\varepsilon^2)$$

- $d(t_0, \phi_0, 0) = 0$
- $\frac{\partial d}{\partial \varepsilon} = \frac{DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\varepsilon^u}{\partial \varepsilon}|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s}{\partial \varepsilon}|_{\varepsilon=0} \right)}{\|DH(q_0(-t_0))\|}$
- We care about when $d(t_0, \phi_0, \varepsilon) = 0$

$$\text{define } \boxed{M(t_0, \phi_0) = DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\varepsilon^u}{\partial \varepsilon}|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s}{\partial \varepsilon}|_{\varepsilon=0} \right)}$$

Problem

As it is written, this requires knowing the solution to the perturbed problem. Melnikov does something clever to get around this

- introduce the time dependent Melnikov function

$$M(t : t_o, \phi_0) = \underbrace{DH(q_0(t - t_0))}_{\text{evolves via unperturbed flow}} \cdot \underbrace{\left(\frac{\partial q_\varepsilon^u(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s(t)}{\partial \varepsilon}\Big|_{\varepsilon=0} \right)}_{\text{evolves via perturbed flow}}$$

- derive an ODE for $M(t :, t_0, \phi_0)$
- $+$ a great deal of cumbersome math $+$ notation $+$ solve an ODE
- the details are omitted, and in the end result in

$$\boxed{M(t_0, \phi_0) = \int_{-\infty}^{\infty} DH(q_0(t)) \cdot g(q_0(t), \omega t - \omega t_0 + \phi_0, 0) dt}$$

Zeros of the Melnikov function imply chaos

By either the Moser's Theorem or the Smale-Birkhoff Homoclinic Theorem, neither of which will be proven here, a system exhibits chaos if there are zero's of the Melnikov function.

Moser's/Smale-Birkhoff rely on the flow posessing a hyperbolic periodic point, and having the stable and unstable manifolds of this point intersecting transversely.

If these conditions are met, then there exists an invariant Cantor set that is topologically conjugate to a shift map... ie: the flow is chaotic.

Application: Epidemiology

Consider an SIR model:

S $\equiv$ fraction of population who is susceptible
I $\equiv$ fraction of population who is infected
R $\equiv$ fraction of population who has recovered

System written as:

$$\begin{cases} \dot{S} = -B(I,t)S + \mu - \mu S \\ \dot{I} = B(I,t) - (\gamma + \mu)I \\ \dot{R} = \gamma I - \mu R \end{cases}$$

With:

$$\begin{cases} B(I,t) = \beta(t) I^p \\ \beta(t) = \frac{p^{2p}}{(p-1)^{2p-1}} \mu(1 + \varepsilon^2 b_1 + \varepsilon^4 b2 + \varepsilon^5 b_3 \sin(\mu\varepsilon\Omega t) \\ 2pb_2 - (p-1)b_1^2 > 0 \\ \gamma = \frac{\mu(1+\varepsilon^2 b_1)}{p-1} \end{cases}$$

Application: Epidemiology

Since $S(t) + I(t) + R(t) = 1$, the flow can be written:

$$\begin{cases} \dot{I} = \beta(t) I^p (1 - I - R) - (\gamma + \mu) I \\ \dot{R} = \gamma I - \mu R \end{cases}$$

After some tedious scaling and transformations, the equations are in the desired form for the Melnikov method:

$$\begin{cases} \frac{dq}{d\tau} = w \\ \frac{dw}{d\tau} = -c^2 + q^2 + \boxed{\varepsilon \left(b_1 w + \frac{2(p-1)}{p} w + \frac{p^3}{2(1-p)} b_3 \sin(\Omega \tau) \right) + O(\varepsilon^2)} \end{cases}$$

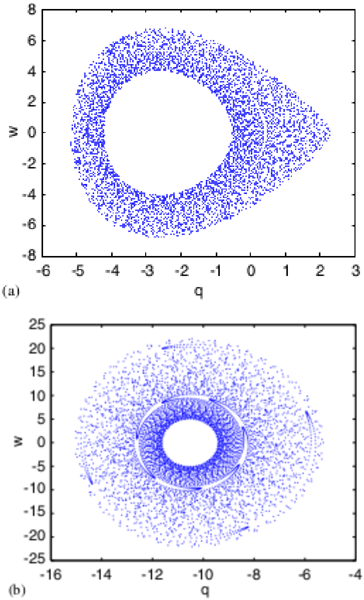
The unperturbed system has Hamiltonian

$$H(q, w) = \frac{1}{2} w^2 + c^2 q - \frac{1}{3} q^3$$

The perturbed system has Melnikov function

$$\int_{-\infty}^{\infty} w_h(\tau) \left(b_1 w_h(\tau) + \frac{2(p-1)}{p} q_h(\tau) w_h(\tau) + \frac{p^3}{2(1-p)} b_3 \sin(\Omega(\tau + \tau_0)) \right) d\tau$$

Application: Epidemiology



Conclusion: Stable and unstable manifolds in the Poincaré map intersect transversely if $|b_3| > b_c$

$$b_c \approx \frac{2(p-1)^2(2c)^{\frac{7}{2}}}{7\pi\Omega^2 p^4}\sinh\left(\frac{\Omega\pi}{(2c)^{\frac{1}{2}}}\right)$$

Poincaré map shown for 2 sets of parameter values and initial conditions.

How tedious is tedious?

"In this comment, we show that a technical carelessness which might lead to the unjustified result occurs when the authors in [1] substituted the pertubed degenerate points into Eq.(8). Unfortunately, these related inaccuracies might result in the unjustified main results in [1]."

Huaqing Li, Xiaofeng Liao, Li Xiao. Comments on: "Melnikov analysis of chaos in a general epidemiological model" [Nonlinear Anal. RWA8 (2007) 20]". Nonlinear Analysis: Real World Applications, v13 n1 (201202): 39-41

A generalization of the method

Theorem:

Consider the system

$$\dot{x} = f(x) + \varepsilon g(x,t), \qquad x = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$$

- $f(x), g(x,t) \in C^2$
- homoclinic orbit $h(t)$ exists to a hyperbolic fixed point when $\varepsilon = 0$
- $g(h(t),t)$ is bounded
- $\frac{\partial g}{\partial t}(h(t),t)$ is bounded
- $\nabla g(h(t),t)$ is bounded

Then if $M(t_0) = \int_{-\infty}^{\infty} f(h(t-t_0)) \wedge g(h(t-t_0),t)dt$ has a simple zero at some t_0, then the flow is equal to a chaotic system for $t \in (-T,T)$, for T as large as one wants.

proof sketch:

The nonperiodic system in the time interval $(-T,T)$ is extended to a t-periodic vector field on which the Melnikov theory applies. If T is choosen large enough that this new system exhibits chaos, so in turn does the original system during the arbitrarily large time interval $(-T,T)$.

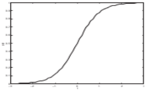
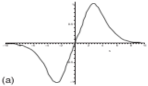
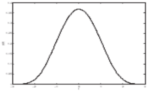
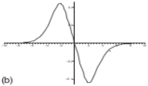
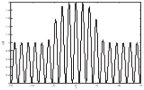
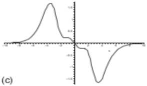
A generalization of the method: Gylden's problem

Consider the system

$$\begin{cases} \frac{du}{d\tau} = v \\ \frac{dv}{d\tau} = u - u^3 + \varepsilon u p\left(\frac{p_\theta^4}{16}t(\tau)\right) \end{cases}$$

Has Melnikov function

$$M(\tau_0) = \int_{-\infty}^{\infty} -\sqrt{2}\text{sech}(\tau)\tanh(\tau) p\left(\frac{p_\theta^4}{16}t(\tau+\tau_0)\right) d\tau$$

Behaviour	$p(t)$	$p(t)$	$M(\tau_0)$
Increase	$\frac{1+\tanh(t)}{2}$		 (a)
Blink	$1 + e^{-\cosh(t)}$		 (b)
Flare	$(1 + e^{-\lambda\cosh(t)})\cos^2(wt)$		 (c)

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- Castilho C. and Marchesin M. "A practical use of the Melnikov homoclinic method". Journal of Mathematical Physics 50, 112704. (2009)

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