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HIDDEN MARKOV MODEL

Outline

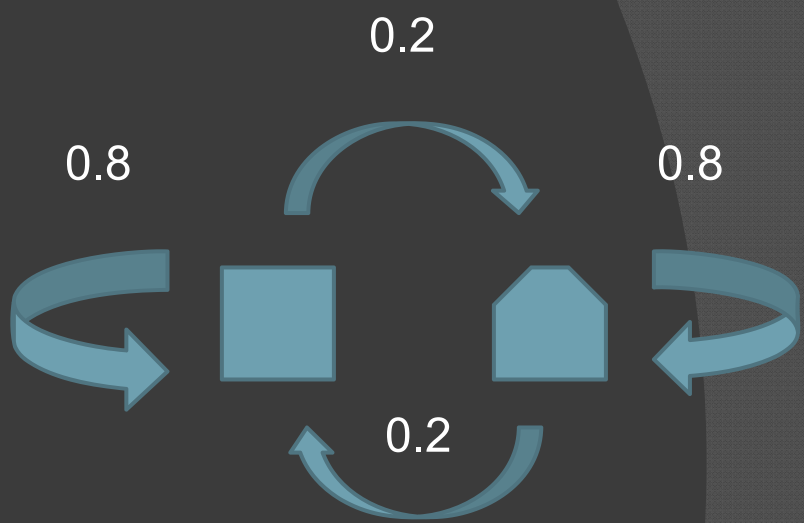
- An example of gambling
- Hidden Markov model (HMM), the description
- Three basis problems of HMMs
- Viterbi path, how to infer the hidden states from observations

Dealer's cheating

- A simple gamble
  - The dealer tosses dice
  - The gamblers bid on 'big' or 'small'
- The dealer can cheat
  - by switching the die between a fair one and an unfair one
  - but he cannot switch dice too frequently

Dealer's cheating

	fair	unfair
1	1/6	0
2	1/6	0
3	1/6	0
4	1/6	1/3
5	1/6	1/3
6	1/6	1/3

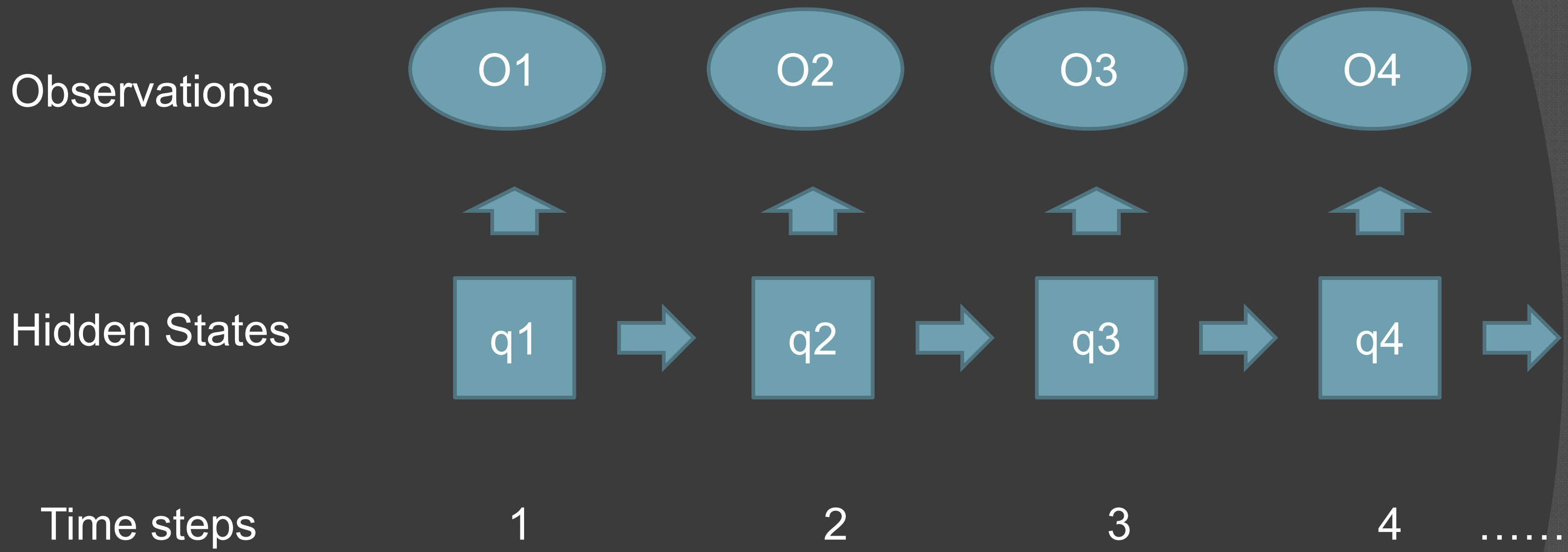


Transition probability between two states, fair and unfair

- Observation sequence

3 6 6 5 4, 6 4 4 6 6, 6 5 1 3 3, 4 1 4 5 5

Model description



Model description

- Hidden Markov Model (HMM)
  - N, the number of states, S_i
  - M, the number of observation symbols, v_k
  - A = {a_{ij}}, the state transition probability

$$a_{ij} = Pr\{ q_{t+1} = S_j \mid q_t = S_i \}$$

  - B = {$b_j(k)$}, the observation symbol probability distribution

$$b_j(k) = Pr\{ v_k \text{ at } t \mid q_t = S_j \}$$

  - $\pi = \{\pi_i\}$, the initial state distribution
- Notation $\lambda = (A, B, \pi)$

Three basic problems

1. Evaluating the probability of a observation sequence

   compute $P(O|\lambda)$ efficiently

2. Uncover the hidden states, given the observation and the model

   find a states sequence to maximize $P(Q|O, \lambda)$

3. Optimize the model parameters to best explain the observations

   find $\lambda = (A, B, \pi)$ to maximize $P(O|\lambda)$

Viterbi Algorithm

- Find the single best state sequence

  Maximize $P(Q|O,\lambda)$

  equivalent to: Maximize $P(Q, O|\lambda)$

- Scoring quantity

  $$\delta_t(i) = \max_{q_1 q_2 \ldots q_{t-1}} P(q_1 q_2 \ldots q_t = i, O_1 O_2 \ldots O_t \mid \lambda)$$

- Recursive relation

  $$\delta_{t+1}(j) = [\max_i \delta_t(i) \, a_{ij}] \, b_j(O_{t+1})$$

Viterbi Algorithm

$$\delta_{t+1}(j) = \max_{q_1 \ldots q_{t-1}, i} P(q_1 \ldots q_{t-1}, q_t = i, q_{t+1} = j, O_1 \ldots O_{t+1} \mid \lambda)$$

$$= \max_{q_1 \ldots q_{t-1}, i} P(q_1 \ldots q_t = i, O_1 \ldots O_t \mid \lambda) P(q_{t+1} = j, O_{t+1} \mid q_t = i, \lambda)$$

$$= \max_{q_1 \ldots q_{t-1}, i} P(q_1 \ldots q_t = i, O_1 \ldots O_t \mid \lambda) P(q_{t+1} = j \mid q_t = i, \lambda)$$

$$P(O_{t+1} \mid q_{t+1} = j, \lambda)$$

$$= \{\max_i [\max_{q_1 \ldots q_{t-1}} P(q_1 \ldots q_t = i, O_1 \ldots O_t \mid \lambda) a_{ij}]\}$$

$$P(O_{t+1} \mid q_{t+1} = j, \lambda)$$

$$= [\max_i \delta_t(i) a_{ij}] b_j(O_{t+1})$$

Viterbi Algorithm

- Dynamic programming method
  - Recursive relation

    $$\delta_{t+1}(j) = [\max \delta_t(i)\ a_{ij}]\ b_j(O_{t+1})$$

  - Score for each step, from t=1 to T
  - Trace back to get the best path

Dealer's cheating problem

- $N = 2$, $M = 6$
- A, $a_{11} = a_{22} = 0.9$, $a_{12} = a_{21} = 0.1$
- B, $b_1(k) = 1/6$, $k=1,2,\ldots,6$

  $b_2(k) = 0$, $k=1,2,3$; $b_2(k) = 1/3$, $k=4,5,6$
- π, $\pi_1=1$, $\pi_2=0$

Dealer's cheating problem

- Observation sequence

3 6 6 5 4, 6 4 4 6 6, 6 5 1 3 3, 4 1 4 5 5

- Viterbi path

3 6 6 5 4, 6 4 4 6 6, 6 5 1 3 3, 4 1 4 5 5

Pretty Good!

Thank you!