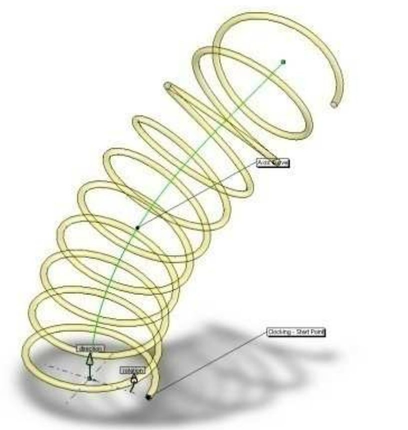
Minimizing Energy Functionals to Model Helical Structures



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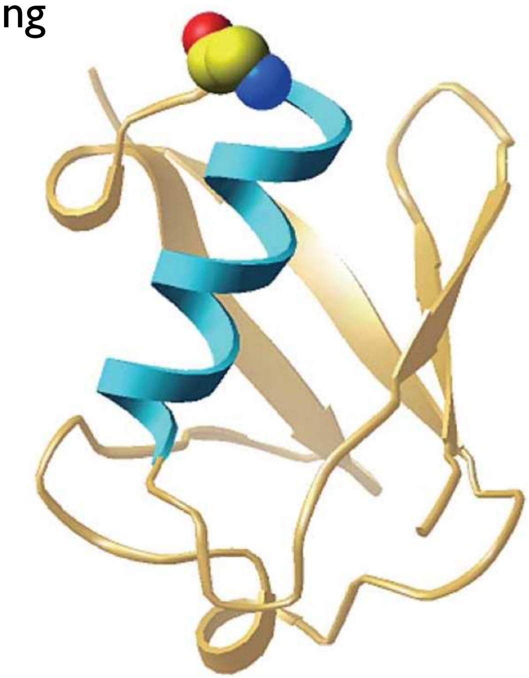
Motivation

- Modeling helices has generated much research from physics, biology, chemistry, and engineering
- Misfolding proteins is a major cause of many illnesses including Alzheimer's, mad cow, and Creutzfeldt-Jakob diseases
- Some helical models include the structure of nucleic acids and proteins, polymers, and the morphologies of calcites and silica-barium carbonate



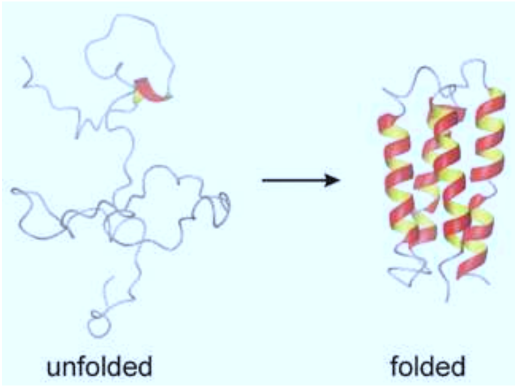
Energy Density Functions

- Proteins fold into minimum-energy structures

- Let us consider energy densities depending on curvature κ, torsion τ and their derivatives with respect to arc length, κ' and τ'

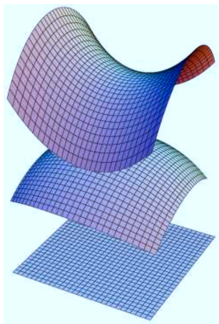
- We want to minimize the energy functional of the form

$$\int_a^b \mathcal{F}(\kappa, \tau, \kappa', \tau')|\boldsymbol{r}'|ds$$

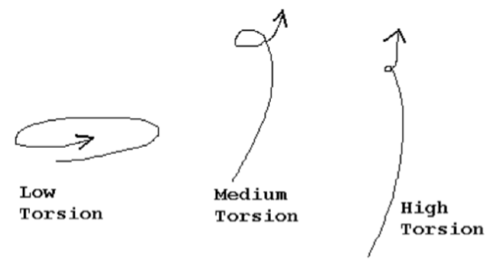


Curvature and Torsion

- Curvature and torsion are mathematically defined as



$$\kappa = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3}, \qquad \tau = \frac{\det(\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''')}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2}$$



- Encodes all geometric info for a 3-D curve up to rotations and translations.

- Frenet-Serret formulae describe how unit tangent, normal, and binormal vectors move along a curve:

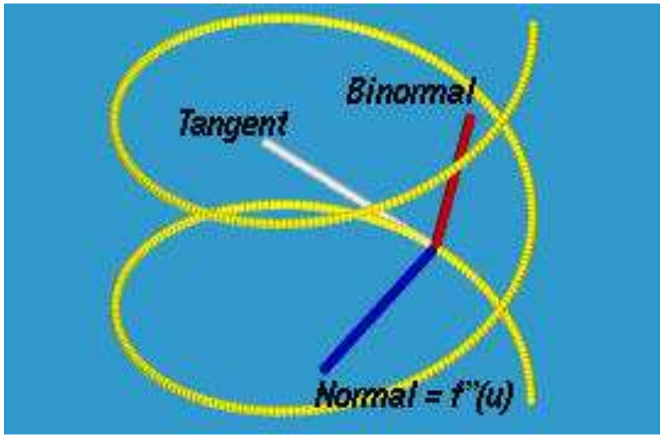
$$\boldsymbol{T}' = \kappa \boldsymbol{N}, \qquad \boldsymbol{N}' = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}, \qquad \boldsymbol{B}' = -\tau \boldsymbol{N}$$

Variations to the Curve

- Consider a variation to the curve of the form:

$$\tilde{\boldsymbol{r}}(s) = \boldsymbol{r}(s) + \varepsilon_1\eta_1(s)\boldsymbol{T}(s) + \varepsilon_2\eta_2(s)\boldsymbol{N}(s) + \varepsilon_3\eta_3(s)\boldsymbol{B}(s)$$

- Variations in tangential direction does not result in information about $\mathcal{F}(\kappa, \tau, \kappa', \tau')$ but variations in the normal and binormal directions gives us the Euler-Lagrange equations



Euler-Lagrange Equations

- (a)

$$\frac{d^2}{ds^2}\left[\frac{\partial\mathcal{F}}{\partial\kappa}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right]+\frac{2\tau}{\kappa}\frac{d^2}{ds^2}\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]$$
$$+\left(\frac{\tau'}{\kappa}-\frac{2\kappa'\tau}{\kappa^2}\right)\frac{d}{ds}\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]$$
$$+\left(\kappa^2-\tau^2\right)\left[\frac{\partial\mathcal{F}}{\partial\kappa}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right]+2\kappa\tau\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]$$
$$+\kappa\left(\kappa'\frac{\partial\mathcal{F}}{\partial\kappa'}+\tau'\frac{\partial\mathcal{F}}{\partial\tau'}-\mathcal{F}\right)=0$$

- (b)

$$-\frac{1}{\kappa}\frac{d^3}{ds^3}\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]+\frac{2\kappa'}{\kappa^2}\frac{d^2}{ds^2}\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]$$
$$+2\tau\frac{d}{ds}\left[\frac{\partial\mathcal{F}}{\partial\kappa}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right]$$
$$+\left(\frac{\tau^2}{\kappa}+\frac{\kappa''}{\kappa^2}-\frac{2\kappa'^2}{\kappa^3}-\kappa\right)\frac{d}{ds}\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]$$
$$+\tau'\left[\frac{\partial\mathcal{F}}{\partial\kappa}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right]-\kappa'\left[\frac{\partial\mathcal{F}}{\partial\tau}-\frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\tau'}\right)\right]=0$$

Simplifying the Equations

- Consider energy densities of the following forms:

(1) $\mathcal{F}(\kappa)$

(2) $\mathcal{F}(\kappa, \tau)$

(3) $\mathcal{F}(\kappa, \kappa')$

Energy Densities $\mathcal{F}(\kappa)$

- Euler-Lagrange equations reduce to

(1a) $$\frac{d^2}{ds^2}\left(\frac{d\mathcal{F}}{d\kappa}\right) + (\kappa^2 - \tau^2)\frac{d\mathcal{F}}{d\kappa} - \kappa\mathcal{F} = 0$$

(1b) $$\tau'\frac{d\mathcal{F}}{d\kappa} + 2\tau\frac{d}{ds}\left(\frac{d\mathcal{F}}{d\kappa}\right) = 0$$

- Consider the case $\tau = \tau_0 \neq 0$

eqn. (1b) becomes $$\frac{d}{ds}\left(\frac{d\mathcal{F}}{d\kappa}\right) = k'\frac{d^2\mathcal{F}}{d\kappa^2} = 0$$

(i) $\kappa = \kappa_0$ or

(ii) $\mathcal{F}(\kappa) = \alpha + \beta\kappa$

- Consider the case $\frac{\tau}{\kappa} = C$

Case (i) $\kappa = \kappa_0, \tau = \tau_0$

- Eqn. (1a) becomes $(\kappa_0^2 - \tau_0^2)\dfrac{d\mathcal{F}}{d\kappa} - \kappa_0\mathcal{F} = 0$

- Defines a class of energy densities of the form $\mathcal{F}(\kappa) = C_0 e^{\frac{\kappa_0\kappa}{(\kappa_0^2-\tau_0^2)}}$

- Circular helices are solutions and can be parameterized by arc length as

$$\boldsymbol{r}(s) = a\,Cos\left(\frac{s}{c}\right)\hat{\boldsymbol{\imath}} + a\,Sin\left(\frac{s}{c}\right)\hat{\boldsymbol{\jmath}} + \frac{bs}{c}\hat{\boldsymbol{k}}, \qquad a^2 + b^2 = c^2$$

for which $\kappa_0 = \dfrac{a}{c^2}, \qquad \tau_0 = \dfrac{b}{c^2}$

- For protein structures, practical values of the helix radius a, and the pitch $2\pi b$, are $a \approx 2.5$ Å and $2\pi b \approx 5.4$ Å

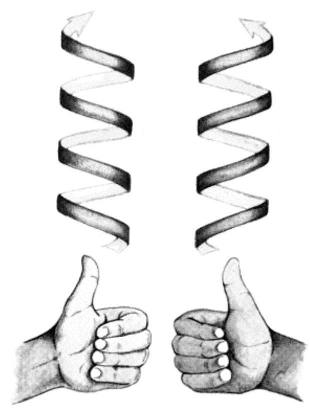
Case (ii) $\mathcal{F}(\kappa) = \alpha + \beta\kappa$

- Eqn. (1b) can be integrated to find that $\tau = \frac{C_1}{\left(\frac{d\mathcal{F}}{d\kappa}\right)^2}$

  and therefore $\tau = \frac{C_1}{\beta^2}$

- Plug the result for torsion into eqn. (1a) to find that $\kappa = -\frac{C_1^2}{\alpha\beta^3}$

- Note that for a positive curvature, alpha and beta must have opposite signs. Also, C_1 determines the sign of the torsion which indicates whether the helix is right-handed (+) or left-handed (-).



Generalized Helices with $\dfrac{\tau}{\kappa} = C$

- If we let $\tau = C_2\kappa$ and plug that into the integrated solution for torsion in case (ii), then we can solve for the energy density:

$$\mathcal{F}(\kappa) = \pm 2\sqrt{\frac{C_1\kappa}{C_2}} + C_0$$

- We may then rewrite eqn. (1a) as

$$\kappa'' - \frac{3}{2}\frac{(\kappa')^2}{\kappa} + 2(1 + C_2^2)\kappa^3 \pm 2C_0\sqrt{\frac{C_2}{C_1}}\kappa^{\frac{5}{2}} = 0$$

- If we let $\kappa = \kappa_0$, then we recover our circular helices where

$$\kappa = \frac{C_0^2 C_2}{C_1(1 + C_2^2)^2}, \qquad \tau = \frac{C_0^2 C_2^2}{C_1(1 + C_2^2)^2}$$

- However, if we look for conical helices and let $\kappa = \dfrac{\kappa_0}{s}, \tau = \dfrac{\tau_0}{s}$,

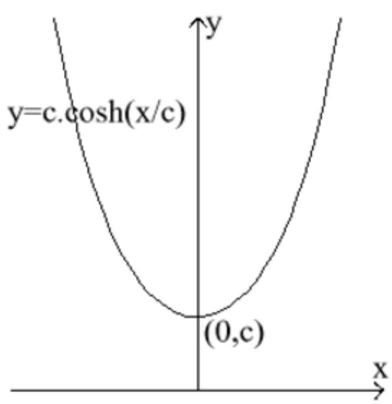
  then we see that such solutions are not permitted.

- The general solution is fairly complicated and basically describes a helix on the surface of a catenary cylinder:

$$\kappa = \frac{4C_0^2 C_2}{C_1(1+C_2^2)^2} t^{-2}, \qquad \tau = \frac{4C_0^2 C_2^2}{C_1(1+C_2^2)^2} t^{-2}$$

where t is related to arc length by

$$s = \frac{C_1(1+C_2^2)^{\frac{3}{2}}}{4C_2 C_0^2} \int_a^t (\tilde{c} - \tilde{t}^{-2} \pm \tilde{t}^{-1})^{-\frac{1}{2}}\, d\tilde{t}$$

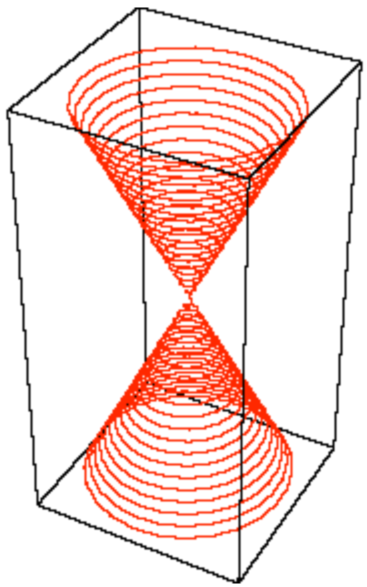


Energy Densities $\mathcal{F}(\kappa, \tau)$

- Consider the cases

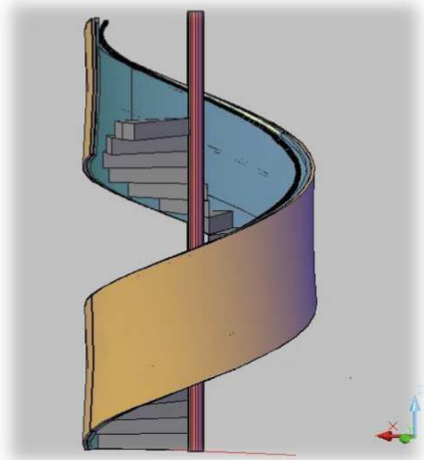
(i) $\mathcal{F}(\kappa, \tau) = \alpha + \beta\kappa + \gamma\tau$

(ii) Conical helices: $\kappa(s) = \dfrac{\kappa_0}{s}, \qquad \tau(s) = \dfrac{\tau_0}{s}$



Case (i) $\mathcal{F}(\kappa,\tau) = \alpha + \beta\kappa + \gamma\tau$

- Euler-Lagrange equations become



(2a) $-\alpha\kappa + \gamma\kappa\tau - \beta\tau^2 = 0$

(2b) $\beta\tau' - \gamma\kappa' = 0$

- Unique Solution: $\kappa = \dfrac{-C_0^2\beta}{\alpha + C_0\gamma}, \qquad \tau = \dfrac{\alpha C_0}{\alpha + C_0\gamma}$

- Substituting $\kappa = \kappa_0, \tau = \tau_0$ for more general circular helices yields a weak condition on the energy density

$$(\kappa_0^2 - \tau_0^2)\frac{\partial \mathcal{F}}{\partial \kappa} + 2\kappa_0\tau_0\frac{\partial \mathcal{F}}{\partial \tau} - \kappa_0\mathcal{F} = 0$$

Case (ii) $\kappa(s) = \dfrac{\kappa_0}{s}, \qquad \tau(s) = \dfrac{\tau_0}{s}$

- Consider a conical helix with the parameterization of

$$\boldsymbol{r}(t) = \alpha t\, Cos(\beta \log t)\hat{\boldsymbol{\imath}} + \alpha t\, Sin(\beta \log t)\hat{\boldsymbol{\jmath}} + \gamma t \hat{\boldsymbol{k}}$$

$$t = \frac{s}{\sqrt{\alpha^2(1+\beta^2)+\gamma^2}}$$

- The definitions of curvature and torsion tell us

$$\kappa_0 = \frac{\alpha\beta\sqrt{1+\beta^2}}{\sqrt{\alpha^2(1+\beta^2)+\gamma^2}}, \qquad \tau_0 = \frac{\beta\gamma}{\sqrt{\alpha^2(1+\beta^2)+\gamma^2}}$$

- Energy densities associated with this conical helix are of the form

$$\mathcal{F}(\kappa,\tau) = f_0\left(\frac{\tau}{\kappa}\right) + f_1\left(\frac{\tau}{\kappa}\right) Sin(\beta \log \kappa) + f_2\left(\frac{\tau}{\kappa}\right) Cos(\beta \log \kappa) + \kappa f_3\left(\frac{\tau}{\kappa}\right)$$

Energy Densities $\mathcal{F}(\kappa, \kappa')$

- Euler-Lagrange equations reduce to

(3a) $$\frac{d^2}{ds^2}\left[\frac{\partial\mathcal{F}}{\partial\kappa} - \frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right] + (\kappa^2 - \tau^2)\left[\frac{\partial\mathcal{F}}{\partial\kappa} - \frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right] + \kappa\left(\kappa'\frac{\partial\mathcal{F}}{\partial\kappa'} - \mathcal{F}\right) = 0$$

(3b) $$2\tau\frac{d}{ds}\left[\frac{\partial\mathcal{F}}{\partial\kappa} - \frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right] + \tau'\left[\frac{\partial\mathcal{F}}{\partial\kappa} - \frac{d}{ds}\left(\frac{\partial\mathcal{F}}{\partial\kappa'}\right)\right] = 0$$

- Consider the cases
  - (i) Circular helices: $\kappa = \kappa_0, \tau = \tau_0$
  - (ii) $\mathcal{F}(\kappa, \kappa') = \alpha + \beta\kappa + \gamma\kappa'$
  - (iii) Conical helices: $\kappa(s) = \frac{\kappa_0}{s}, \qquad \tau(s) = \frac{\tau_0}{s}$

Case (i) $\kappa = \kappa_0, \tau = \tau_0$

- Eqn. (3b) vanishes and eqn. (3a) becomes

$$(\kappa_0^2 - \tau_0^2)\frac{\partial \mathcal{F}}{\partial \kappa} - \kappa_0 \mathcal{F} = 0$$

- Defines a class of energy densities of the form

$$\mathcal{F}(\kappa, \kappa') = \psi(\kappa') e^{\frac{\kappa_0 \kappa}{(\kappa_0^2 - \tau_0^2)}}$$

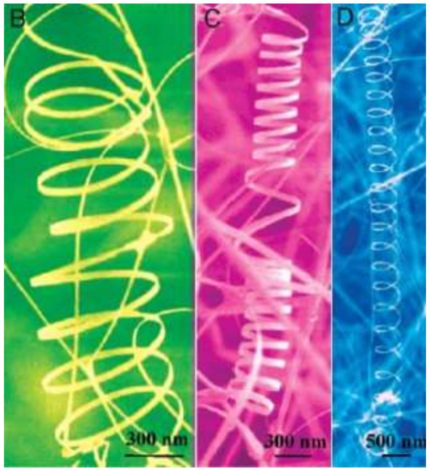
Case (ii) $\mathcal{F}(\kappa,\kappa') = \alpha + \beta\kappa + \gamma\kappa'$

- Eqn. (3b) can be integrated to find that $\tau = \dfrac{C_1}{\left[\dfrac{\partial\mathcal{F}}{\partial\kappa} - \dfrac{d}{ds}\left(\dfrac{\partial\mathcal{F}}{\partial\kappa'}\right)\right]^2}$

  and therefore $\tau = \dfrac{C_1}{\beta^2}$

- Plug the result for torsion into eqn. (3a) to find that $\kappa = -\dfrac{C_1^2}{\alpha\beta^3}$

- Same results as $\mathcal{F}(\kappa) = \alpha + \beta\kappa$



Case (iii) $\kappa(s) = \frac{\kappa_0}{s}, \qquad \tau(s) = \frac{\tau_0}{s}$

- From the integrated solution for torsion in case (ii) we know

$$\frac{\partial \mathcal{F}}{\partial \kappa} - \frac{d}{ds}\left(\frac{\partial \mathcal{F}}{\partial \kappa'}\right) = \pm\sqrt{\frac{C_1}{\tau}}$$

- Therefore eqn. (3a) can be rewritten as

$$\frac{d^2}{ds^2}\sqrt{\frac{C_1}{\tau}} + (\kappa^2 - \tau^2)\sqrt{\frac{C_1}{\tau}} \pm \kappa\left(\kappa' \frac{\partial \mathcal{F}}{\partial \kappa'} - \mathcal{F}\right) = 0$$

- Plugging in $\kappa = \frac{\kappa_0}{s}, \tau = \frac{\tau_0}{s}$ allows us to further rewrite eqn. (3a) as

$$\left(\kappa_0^2 - \tau_0^2 - \frac{1}{4}\right)\sqrt{\frac{C_1}{\tau_0}}\, s^{-\frac{3}{2}} \pm \frac{\kappa_0}{s}\left(\kappa' \frac{\partial \mathcal{F}}{\partial \kappa'} - \mathcal{F}\right) = 0$$

- A sufficient condition is $\kappa' \frac{\partial \mathcal{F}}{\partial \kappa'} - \mathcal{F} = C_2 s^{-\frac{1}{2}} = C_2 \kappa^{\alpha} (-\kappa')^{\beta}$
  and $\alpha + 2\beta = \frac{1}{2}$

- Solving the PDE from the condition above tells us

  $$\mathcal{F}(\kappa, \kappa') = \frac{C_2}{\beta - 1} \kappa^{\alpha} (-\kappa')^{\beta} + \kappa' \, \psi(\kappa), \qquad \beta \neq 1$$

  and for $\beta = 1$ the result is $\mathcal{F}(\kappa, \kappa') = C_2 \kappa^{-\frac{3}{2}} \kappa' log(-\kappa') + \kappa' \, \psi(\kappa)$

- Substituting back into eqn. (3a) we get a condition for C_2 :

$$\left(\kappa_0^2 - \tau_0^2 - \frac{1}{4}\right) \sqrt{\frac{C_1}{\tau_0}} + \kappa_0^{\alpha+\beta+1} C_2 = 0 \implies$$

$$C_1 = \frac{\kappa_0^{2(\alpha+\beta+1)} \tau_0 C_2^2}{\left(\kappa_0^2 - \tau_0^2 - \frac{1}{4}\right)^2}$$

Conclusions

- For energy densities $\mathcal{F}(\kappa)$ we found solutions that yield circular helices and complicated generalized helices but not conical helices.

- For energy densities $\mathcal{F}(\kappa,\tau)$ we found solutions that yield circular helices and conical helices.

- In a similar manner to the previous two classes of energy densities we found circular and conical helix solutions for energy densities $\mathcal{F}(\kappa,\kappa')$.

- General classes of energy densities that yield helical extremal curves could be of use in modeling protein structures and other helical objects.

References

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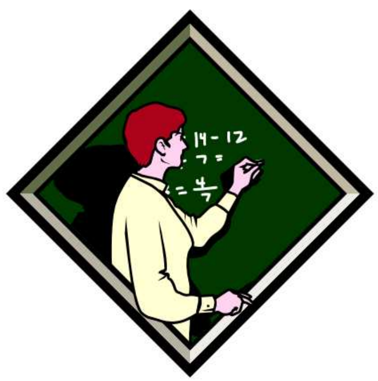

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