Pole Dynamics

AMATH 573: Final Project

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Pole Dynamics:

- Soliton solutions in terms of complex $x$
- Look for poles of the solution
- Understand the solution by examining the motion of the poles in the complex plane.
One soliton solution in terms of complex $x$

\[ u = \partial_x^2 \ln \left( 1 + e^{kx - k^3 t} \right) \]

It has poles when,

\[ 1 + e^{kx - k^3 t} = 0 \]

Pole dynamics given by

\[ x_n(t) = \frac{i\pi(1 + 2n)}{k} + k^2 t \]

Velocity proportional to $k^2$

Pole spacing proportional to $1/k$
Analogously, two-soliton pole dynamics are obtained by solving for complex $x$

$$1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x - k_1^3 t + k_2 x - k_2^3 t} = 0$$

Analytical work done by Thickstun.
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...we can solve numerically!
For $k_1 = 1.8$ and $k_2 = 1$: 

\[ \begin{array}{c}
\text{\includegraphics[width=\textwidth]{plot.png}}
\end{array} \]
For $k_1 = 1.5$ and $k_2 = 1$:
For $k_1 = 2$ and $k_2 = 1$:
Things can get more complicated

- KdV solutions with a finite number of poles
- KdV solutions in term of elliptic functions
- Periodic solutions
- Benjamin-Ono pole expansion
Let’s try with the Benjamin-Ono equation

\[ u_t + 2uu_x + H(u_{xx}) \]

Propose finite pole expansion inspired by the one-soliton solution

\[ u = \sum_{j=1}^{N} \frac{i}{x - x_j(t)} + \sum_{j=1}^{N} \frac{-i}{x - x^*_j(t)} \]

Substituting...
...and after lots of algebra we get a N-body problem

\[ i \frac{dx_j}{dt} + 2 \left( \sum_{k=1}^{N} \frac{1}{x_j - x_k} - \sum_{k=1}^{N} \frac{1}{x_j - x_k^*} \right) = 0 \]

How can we solve this problem?
...and after lots of algebra we get a N-body problem

\[ i \frac{d}{dt} \mathbf{j} + 2 \left( \sum_{k=1 \atop k \neq j}^{N} \frac{1}{x_j - x_k} - \sum_{k=1}^{N} \frac{1}{x_j - x_k^*} \right) = 0 \]

How can we solve this problem?

- Ask the dark soliton for help...
...and after lots of algebra we get a N-body problem

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How can we solve this problem?

- Ask the dark soliton for help...
- ...or try to use what we learned in class

Hamiltonians and Lax Pairs
Let’s take another time derivative, after more algebra...

\[ \frac{d^2 x_j}{dt^2} = 8 \left( \sum_{k \neq j} \frac{1}{(x_j - x_k)^3} \right) \]

This is a N-body problem with a inverse square potential; therefore, the Hamiltonian should be the “kinetic energy” plus the potential.

\[ H = \frac{1}{2} \sum_j \left( \frac{dx_j}{dt} \right)^2 + 2 \left( \sum_j \sum_{k \neq j} \frac{1}{(x_j - x_k)^2} \right) \]

Notice we duplicated the dimension of the system!
Integration with Lax Pairs

Let’s look for a Lax Pair $L, A$:

\[
L \psi = \lambda \psi \\
\psi_t = A \psi_t
\]

Such that the compatibility condition $L_t = [A, L]$ is our N-body problem eq.

It is trivial that the Lax Pair is...
Integration with Lax Pairs

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It is trivial that the Lax Pair is...

\[
L_{kj} = \delta_{kj} \frac{dx_j}{dt} + i \frac{1 - \delta_{kj}}{x_k - x_j}
\]

\[
A_{kj} = -i \delta_{kj} \sum_{l \neq k} \frac{1}{(x_k - x_l)^2} + i \frac{1 - \delta_{kj}}{(x_k - x_l)^2}
\]
Let's find the conserved quantities, for instance the trace of $L$

$$\text{tr}[L] = \lambda_1 + \lambda_2 + \ldots$$

We usually assume $\lambda_t = 0$, therefore

$$\frac{d\text{tr}[L]}{dt} = 0$$

First conserved quantity! By induction

$$\frac{d\text{tr}[L^n]}{dt} = 0$$

We have $N$ conserved quantities. We still need $N$ more.
Using $X_t = \delta_{kj} x_j$ we can rewrite the compatibility condition as

$$X_t = [A, X] + L$$

With this, it’s easy to prove that

$$\text{tr}[XL^{n-1}] - \text{tr}[L^n] t = c_n$$

So, now we have $2N$ conserved quantities!!! We can solve the problem.
Explicit solution

Let’s try to solve the problem in the most general way.

Let’s consider

\[ K(t) = U^{-1}(t)X(t)U(t) \]

With

\[ \frac{dU}{dt} = AU \quad \quad UU^\dagger = I \quad \quad U(t_0) = I \]

Using the compatibility condition and that \( A \) is anti-hermitian, we can prove that

\[ \frac{dK}{dt} = U^{-1}LU \]
Furthermore, we can derive again and obtain

\[
\frac{d^2 K}{dt^2} = 0
\]

therefore

\[
K(t) = C_1 + (t - t_0)C_2
\]

Determining the constants matrices

\[
K(t) = X(t_0) + (t - t_0)L(t_0)
\]
Remembering that \( K(t) = U^{-1}(t)X(t)U(t) \)

\[ X(t) = U(t)K(t)U^{-1}(t) \]

Since \( X = \delta_{jk}x_j \), the eigenvalues of \( X \) are given by

\[ 0 = \det(\lambda I - X) = \prod_{k=1}^{N} (\lambda - x_k(t)) \]

Therefore, the eigenvalues of \( X \) are the poles \( x_k(t) \)

\( X \) and \( K \) have the same eigenvalues (\( X \) is diagonal)

We already know \( K \)!!!
Summary

KdV
- Make $x$ complex in soliton solution
- Look for poles
- Solve numerically or analytically

Benjamin-Ono
- Propose pole expansion (inspired by one-soliton solution)
- Substitute to obtain N-body problem
- Derivate once more to have a Hamiltonian system
- Find Lax pairs
- Integrate the system finding $2N$ conserved quantities
- Find an explicit solution
References:

