Pole Dynamics

AMATH 573: Final Project

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Introduction

Pole Dynamics:

- Soliton solutions in terms of complex x
- Look for poles of the solution
- Understand the solution by examining the motion of the poles in the complex plane.

One Soliton Solution

One soliton solution in terms of complex x

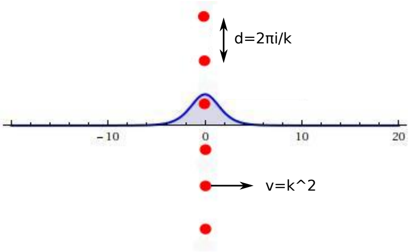
$$u = \partial_x^2 \ln\left(1 + e^{kx - k^3 t}\right)$$

It has poles when,

$$1 + e^{kx - k^3 t} = 0$$

Pole dynamics given by

$$x_n(t) = \frac{i\pi(1+2n)}{k} + k^2 t$$



Velocity proportonial to k^2
Pole spacing proportonial to $1/k$

Analogously, two-soliton pole dynamics are obtained by solving for complex x

$$1 + e^{k_1 x - k_1^3 t} + e^{k_2 x - k_2^3 t} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 e^{k_1 x - k_1^3 t + k_2 x - k_2^3 t} = 0$$

Analytical work done by Thickstun.

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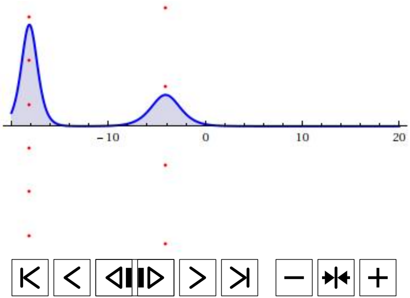
Analytical work done by Thickstun.

...we can solve numerically!

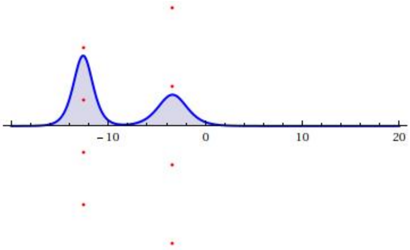
Two Soliton Solution

For $k_1 = 1.8$ and $k_2 = 1$:



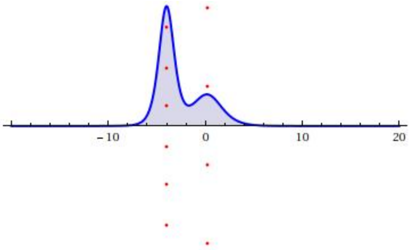
Two Soliton Solution

For $k_1 = 1.5$ and $k_2 = 1$:



Two Soliton Solution

For $k_1 = 2$ and $k_2 = 1$:



More complicated

Things can get more complicated

- KdV solutions with a finite number of poles
- KdV solutions in term of elliptic functions
- Periodic solutions
- Benjamin-Ono pole expansion

Pole Expansion

Let's try with the Bejamin-Ono equation

$$u_t + 2uu_x + \mathbf{H}(u_{xx})$$

Propose finite pole expansion inspired by the one-soliton solution

$$u = \sum_{j=1}^{N} \frac{i}{x - x_j(t)} + \sum_{j=1}^{N} \frac{-i}{x - x_j^*(t)}$$

Substituting...

...and after lots of algebra we get a N-body problem

$$i\frac{dx_j}{dt} + 2\left(\sum_{\substack{k=1\\k\neq j}}^{N}\frac{1}{x_j - x_k} - \sum_{k=1}^{N}\frac{1}{x_j - x_k^*}\right) = 0$$

How can we solve this problem?

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How can we solve this problem?

- Ask the dark soliton for help...

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How can we solve this problem?

- Ask the dark soliton for help...
- ...or try to use what we learned in class

Hamiltonians and Lax Pairs

Hamiltonian System

Let's take another time derivative, after more algebra...

$$\frac{d^2x_j}{dt^2} = 8\left(\sum_{k\neq j}\frac{1}{(x_j - x_k)^3}\right)$$

This is a N-body problem with a inverse square potential; therefore, the Hamiltonian should be the "kinetic energy" plus the potential.

$$H = \frac{1}{2}\sum_j\left(\frac{dx_j}{dt}\right)^2 + 2\left(\sum_j\sum_{k\neq j}\frac{1}{(x_j - x_k)^2}\right)$$

Notice we duplicated the dimension of the system!

Integration with Lax Pairs

Let's look for a Lax Pair L,A:

$$L\psi = \lambda\psi$$
$$\psi_t = A\psi_t$$

Such that the compatibility condition $L_t = [A, L]$ is our N-body problem eq.

It is trivial that the Lax Pair is...

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$$L_{kj} = \delta_{kj}\frac{dx_j}{dt} + i\frac{1-\delta_{kj}}{x_k - x_j}$$
$$A_{kj} = -i\delta_{kj}\sum_{l\neq k}\frac{1}{(x_k - x_l)^2} + i\frac{1-\delta_{kj}}{(x_k - x_l)^2}$$

N Conserved quantities

Let's find the conserved quantities, for instance the trace of L

$$\mathrm{tr}[L] = \lambda_1 + \lambda_2 + \ldots$$

We usually assume $\lambda_t = 0$, therefore

$$\frac{d\mathrm{tr}[L]}{dt} = 0$$

First conserved quantity! By induction

$$\frac{d\mathrm{tr}[L^n]}{dt} = 0$$

We have N conserved quantites. We still need N more.

2N conserved quantities

Using $X_t = \delta_{kj} x_j$ we can rewrite the compatibility condition as

$$X_t = [A, X] + L$$

With this, it's easy to prove that

$$\mathrm{tr}[XL^{n-1}] - \mathrm{tr}[L^n]t = c_n$$

So, now we have 2N conserved quantities!!! We can solve the problem.

Explicit solution

Let's try to solve the problem in the most general way.

Let's consider

$$K(t) = U^{-1}(t)X(t)U(t)$$

With

$$\frac{dU}{dt} = AU \qquad UU^{\dagger} = I \qquad U(t_0) = I$$

Using the compatibility condition and that A is anti-hermitian, we can prove that

$$\frac{dK}{dt} = U^{-1}LU$$

Explicit Solution

Furthermore, we can derive again and obtain

$$\frac{d^2K}{dt^2} = 0$$

therefore

$$K(t) = C_1 + (t - t_0)C_2$$

Determinig the constants matrices

$$K(t) = X(t_0) + (t - t_0)L(t_0)$$

Explicit solution

Remebering that $K(t) = U^{-1}(t)X(t)U(t)$

$$X(t) = U(t)K(t)U^{-1}(t)$$

Since $X = \delta_{jk}x_j$, the eigenvalues of X are given by

$$0 = \det(\lambda I - X) = \prod_{k=1}^{N}(\lambda - x_k(t))$$

Therefore, the eigenvalues of X are the poles $x_k(t)$

X and K have the same eigenvalues (X is diagonal)

We already know K!!!

Summary

KdV

- Make x complex in soliton solution
- Look for poles
- Solve numerically or analitically

Benjamin-Ono

- Propose pole expansion (inspired by one-soliton solution)
- Substitute to obtain N-body problem
- Derivate once more to have a Hamiltonian system
- Find Lax pairs
- Integrate the system finding 2N conserved quantities
- Find an explicit solution

References

References:

- Thickstun W.R., A System of Particles Equivalent to Solitons, Journal of Mathematical Analysis and Applications, 1976.
- Ablowitz M. J., Segur H., Solitons and the Inverse Scattering Transform, SIAM, 1981.
- Bona J. L., Weissler F. B., Pole dynamics of interacting solitons and blowup of complex-valued solutions of KdV, IOP Publishing, 2008.