Ostragradskii's Theorem

Extending The Legendre Transformation

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Outline

- Consider the functional,

$$\mathcal{L}[\boldsymbol{u}] = \int_a^b L(u_1, u_1', \ldots, u_n, u_n') dx.$$

Canonical Variables

- The Legendre Transformation first introduces canonical variables,

$$q_i = u_i,$$
$$p_i = \frac{\partial L}{\partial q_i'}, \quad i = 1, \ldots, n$$

- And Hamiltonian,

$$H(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{n} p_i q_i' - L.$$

New System

- With this choice of variables and Hamiltonian, the Euler-Lagrange Equations are equivalent to the first-order system,

$$p_i' = -\frac{\partial H}{\partial q_i}$$
$$q_i' = \frac{\partial H}{\partial p_i}, \quad i = 1, \dots, n$$

New System

- Not all systems have a Lagrangian depending upon just first-order derivatives.
- To deal with these cases we can generalize the Lagendre Transformation to any number of derivatives.
- This is the Ostragradskii Transformation.

New Canonical Variables

- Consider the more general functional,

$$\mathcal{L}[\boldsymbol{u}] = \int_a^b L(u_1, \dots, u_1^{(N_1)}, \dots, u_k, \dots, u_k^{(N_k)})dx \qquad (1)$$

where N_i denotes the highest order of derivative for u_i present in $L(\boldsymbol{u})$.

- The canonical variables are, for $i = 1, \dots, k$ and $j = 1, \dots, N_i$

$$\begin{aligned} q_{ij} &= \frac{d^{(j-1)}}{dx^{(j-1)}} u_i, \\ p_{ij} &= \frac{\delta \mathcal{L}}{\delta u_i^{(j)}}. \end{aligned} \qquad (2)$$

Full Generality

- $\frac{\delta \mathcal{L}}{\delta u_i^{(j)}}$ denotes the variational derivative with respect to $u_i^{(j)}$:

$$\frac{\delta \mathcal{L}}{\delta u_i^{(j)}} = \frac{\partial L}{\partial u_i^{(j)}} - \frac{d}{dx}\frac{\partial L}{\partial u_i^{(j+1)}} + \ldots + (-1)^k \frac{d^k}{dx^k}\frac{\partial L}{\partial u_i^{(j+k)}}$$

  where $j + k = N_i$.
- Or,

$$\frac{\delta \mathcal{L}}{\delta u_i^{(j)}} = \sum_{k=0}^{N_i - j} (-1)^k \frac{d^k}{dx^k}\frac{\partial L}{\partial u_i^{(j+k)}}.$$

Full Generality

- Now the Hamiltonian becomes,

$$H(\boldsymbol{q}, \boldsymbol{p}) = \sum_{i=1}^{k} \sum_{j=1}^{N_i} p_{ij} q'_{ij} - L(\boldsymbol{q}, \boldsymbol{p}). \tag{3}$$

- ***Definition*[2] :** The Lagrangian $L(\boldsymbol{u})$ is said to be (strongly) non-singular if for $i = 1, \ldots, k$ and $j = 1, \ldots, 2N_i - 1$, the equations for the canonical variables can be solved as,

$$u_i^{(j)} = f_{ij}(\boldsymbol{q}, \boldsymbol{p}).$$

Full Generality

- ***Theorem* (*Ostragradskii* [2, 1]) :** If the Lagrangian, $L(\boldsymbol{u})$, is non-singular, then the first-order-order system,

$$q'_{ij} = \frac{\partial H}{\partial p_{ij}}, \quad p'_{ij} = -\frac{\partial H}{\partial q_{ij}},$$

obtained using the Hamiltonian (3) and canonical variables (2) is equivalent to the Euler-Lagrange equations for (1).

Proof

For the sake of notation, let,

$$\mathcal{L}[\boldsymbol{u}] = \int_a^b L(u_1, u_1', u_1'', u_2, u_2', u_2'')dx.$$

We then have the canonical variables,

$$q_{11} = u_1, \quad q_{21} = u_2, \quad q_{12} = u_1', \quad q_{22} = u_2'$$
$$p_{11} = \frac{\partial L}{\partial u_1'} - \frac{d}{dx}\frac{\partial L}{\partial u_1''}, \quad p_{12} = \frac{\partial L}{\partial u_1''}$$
$$p_{21} = \frac{\partial L}{\partial u_2'} - \frac{d}{dx}\frac{\partial L}{\partial u_2''}, \quad p_{22} = \frac{\partial L}{\partial u_2''}.$$

- The equations $p_{12} = \frac{\partial L}{\partial u_1''}$ and $p_{22} = \frac{\partial L}{\partial u_2''}$ imply,

$$p_{12} = g(u_1, u_1', u_1'', u_2, u_2', u_2'') = g(q_{11}, q_{12}, u_1'', q_{21}, q_{22}, u_2'')$$
$$p_{22} = r(u_1, u_1', u_1'', u_2, u_2', u_2'') = r(q_{11}, q_{12}, u_1'', q_{21}, q_{22}, u_2'')$$

- Then from non-singularity,

$$u_1'' = G(q_{11}, q_{12}, p_{12}, q_{21}, q_{22}, p_{22})$$
$$u_2'' = R(q_{11}, q_{12}, p_{12}, q_{21}, q_{22}, p_{22})$$

- The key observation is that this is independent of p_{11} and p_{21}.

- Then the Hamiltonian can be written as,

$$H = p_{11}u_1' + p_{12}u_1'' + p_{21}u_2' + p_{22}u_2'' - L(\mathbf{u}, \mathbf{u}', \mathbf{u}'')$$
$$= p_{11}q_{12} + p_{12}G + p_{21}q_{22} + p_{22}R - L(q_{11}, q_{21}, q_{12}, q_{22}, G, R)$$

- Then compute partials,

$$\frac{\partial H}{\partial q_{11}} = p_{12}\frac{\partial G}{\partial q_{11}} + p_{22}\frac{\partial R}{\partial q_{11}} - \frac{\partial L}{\partial q_{11}} - \frac{\partial L}{\partial u_1''}\frac{\partial G}{\partial q_{11}} - \frac{\partial L}{\partial u_2''}\frac{\partial R}{\partial q_{11}}$$

- Since, $p_{12} = \frac{\partial L}{\partial u_1''}$ and $p_{22} = \frac{\partial L}{\partial u_2''}$,

$$\frac{\partial H}{\partial q_{11}} = -\frac{\partial L}{\partial q_{11}}$$

- Similar cancelation occurs with respect to $q_{12},\ p_{11}$ and p_{12} and,

$$\frac{\partial H}{\partial q_{11}} = -\frac{\partial L}{\partial q_{11}}, \quad \frac{\partial H}{\partial q_{12}} = -\frac{\partial L}{\partial q_{12}} + p_{11}, \quad \frac{\partial H}{\partial p_{11}} = q_{12}, \quad \frac{\partial H}{\partial p_{12}} = G$$

  are the four partials.
- Analagous formulae result for $q_{21}, q_{22}, p_{21}, p_{22}$.
- Now, assume the Euler-Lagrange (E-L) equations hold, in particular,

$$\frac{\partial L}{\partial u_1} - \frac{d}{dx}\frac{\partial L}{\partial u_1'} + \frac{d^2}{dx^2}\frac{\partial L}{\partial u_1''} = 0$$

- The E-L equations together with the Hamiltonian partials give the Hamiltonian system.
- For example, the E-L equation for u_1 implies,

$$\frac{\partial L}{\partial u_1} = \frac{d}{dx}\left(\frac{\partial L}{\partial u_1'} - \frac{d}{dx}\frac{\partial L}{\partial u_1''}\right) = p_{11}' = -\frac{\partial H}{\partial q_{11}}$$

- Now need to show the Hamiltonian System implies that the Euler-Lagrange equations hold.

- Differentiating $L + H$,

$$\frac{\partial}{\partial q_{11}}(L+H) = \frac{\partial L}{\partial q_{11}} + \frac{\partial H}{\partial q_{11}} = 0$$

using $\frac{\partial L}{\partial q_{11}} = -\frac{\partial H}{\partial q_{11}}$.

- Then since $-\frac{\partial H}{\partial q_{11}} = p'_{11} = \frac{d}{dx}\left(\frac{\partial L}{\partial u'_1} - \frac{d}{dx}\frac{\partial L}{\partial u''_1}\right)$ and $q_{11} = u_1$,

$$\frac{\partial L}{\partial q_{11}} - p'_{11} = \frac{\partial L}{\partial u_1} - \frac{d}{dx}\left(\frac{\partial L}{\partial u'_1} - \frac{d}{dx}\frac{\partial L}{\partial u''_1}\right) = 0.$$

- Similarly for q_{21}, and the theorem is proved.

Consequences of having a Hamiltonian

- If $N_i = N \;\; \forall i$, then this converts an k-dimensional $2N$-order system to a $2Nk$-dimensional first-order system.
- Poisson Bracket:

$$f'(\boldsymbol{q}, \boldsymbol{p}) = \{f(\boldsymbol{q}, \boldsymbol{p}), H(\boldsymbol{q}, \boldsymbol{p})\}$$

- Conservation of the Hamiltonian:

$$H'(\boldsymbol{q}, \boldsymbol{p}) = \{H(\boldsymbol{q}, \boldsymbol{p}), H(\boldsymbol{q}, \boldsymbol{p})\} = 0$$

- Possible Casmirs:

$$C'(\boldsymbol{q}, \boldsymbol{p}) = \{C(\boldsymbol{q}, \boldsymbol{p}), H(\boldsymbol{q}, \boldsymbol{p})\} = 0$$

KP: $(-4u_t + uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$

- The Lagrangian,

$$L[\boldsymbol{u}] = -\frac{u_1''^2}{27} + \frac{u_2' u_1''}{9} - \frac{u_2'^2}{9} + \frac{u_1 u_1'^2}{9} - \frac{2u_1 u_2 u_1'}{9} - \frac{u_1^4}{81} + \frac{2u_1 u_2^2}{9},$$

appears in the study of the stationary KP hierarchy [1].

- The canonical variables are:

$$q_{11} = u_1, \quad q_{12} = u_1', \quad q_{21} = u_2, \quad q_{22} = u_2'$$

$$p_{11} = \frac{\partial L}{\partial u_1'} - \frac{d}{dx}\frac{\partial L}{\partial u_1''} = \frac{2u_1(u_1' - u_2)}{9} + \frac{2u_1'''}{27} - \frac{u_2''}{9}$$

$$p_{12} = \frac{\partial L}{\partial u_2''} = -\frac{2u_1''}{27} + \frac{u_2'}{9}$$

$$p_{21} = \frac{\partial L}{\partial u_2'} - \frac{d}{dx}\frac{\partial L}{\partial u_2''} = \frac{u_1''}{9} - \frac{2u_2'}{9}$$

- Substituting in the q_{ij}'s, the system for u_1''', u_1'' and u_2'' is

$$\begin{aligned} p_{11} &= \frac{2q_{11}(q_{12}-q_{21})}{9} + \frac{2u_1'''}{27} - \frac{u_2''}{9} \\ p_{12} &= -\frac{2u_1''}{27} + \frac{q_{22}}{9} \\ p_{21} &= \frac{u_1''}{9} - \frac{2q_{22}}{9} \end{aligned}$$

- The last two equations give,

$$q_{22} = -27p_{12} - 18p_{21}$$

- Using the E-L equation,

$$u_2'' = \frac{1}{2}\left(2u_1u_1' - 4u_1u_2 + u_1'''\right)$$

- Solving,

$$u_1'' = -54p_{12} - 27p_{21},$$
$$u_2'' = q_{11}q_{12} - 2q_{11}q_{21} + 26p_{11} - 3q_{11}q_{12}$$
$$u_1''' = 54p_{11} - 6q_{11}q_{12}$$

- Therefore,

$$H(\boldsymbol{q}, \boldsymbol{p}) = -27p_{12}^2 - 27p_{12}p_{21} - 9p_{21}^2 + p_{11}q_{12} + \frac{q_{11}^4}{81} - \frac{q_{11}q_{12}^2}{9} - \frac{2q_{21}^2q_{11}}{9} + \frac{2q_{11}q_{12}q_{21}}{9}$$

- Only depends on 6 variables $\Rightarrow$ 6-dimensional first-order system.

References

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- Dubrovin, B. A., Fomenko, A. T., and Novikov, S. P. 1985. *Modern Geometry – Methods and Applications. Part II: The Geometry and Topology of Manifolds*. Springer, New York.