Computations Related to the Construction of Finite Genus Solutions to the Kadomstev-Petviashvilli Equation

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Abstract

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Krichever's method of integrating certain partial differential equations using algebro-geometric techniques provides an explicit approach to the construction of finite-genus solutions to the Kadomtsev-Petviashvili (KP) equation. The closed-form expression that results can be used as an ansatz provided that the parameters of the ansatz have meaning in the context of this construction. The mathematical framework that is the basis for the `algcurves` package, a `Maple` package that provided computational tools for working with Riemann surfaces, is used to produce two procedures: the Krichever Construction Method (KCM), which encapsulates the construction of finite-genus solutions to KP using Krichever's method, and the Extended Dubrovin Method (EDM), which computes the parameters of the closed-form expression using no more than a Riemann matrix and a few parameters while extending ideas in Dubrovin's 1981 survey ([11, Chapter 4]). Moreover, an approach to computing the Riemann constant vector that modifies Patterson's work ([28]) is presented.

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ing and exploration, whenever I choose to be challenged in this way. I have learned from you that this gift, the love of mathematics, also comes with responsibility. That if your head is in the clouds, your feet better be planted firmly on a path to something or somewhere. How can one truly articulate their thankfulness for such profound gifts? It seems that all I can do is try to live up to the example you have set for me.

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DEDICATION

To my wife Thalia, who has been by my side this whole time; to Elijah and Fernando, who are my inspiration; and, to the self-proclaimed invisible youth walking our streets, if I am enough then so are you.

Chapter 1

MOTIVATION, HISTORY & BACKGROUND MATHEMATICS

In this thesis, I am primarily concerned with the explicit computation of finite-genus solutions of the Kadomtsev-Petviashvili equation.

1.1 The Kadomtsev-Petviashvili equation & its use in modeling shallow-water waves

The Kadomtsev-Petviashvili (KP) equation can be expressed as

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{1.1}$$

where u is a complex-valued function in the real variables x, y and t, and $\sigma^2 = \pm 1$. Typically, x and y represent spatial variables, while t is seen as a temporal variable. The KP equation models the time evolution of waves that are long (relative to depth) and have slow dependence in y. The parameter σ has physical relevance. When $\sigma^2 = -1$, the waves that are modeled have high surface tension and equation 1.1 is called the KP1 equation. Whenever $\sigma^2 = 1$, the equation is called the KP2 equation and it models waves for which surface tension does not contribute significantly to the physical behavior of the wave. See [8, 5, 6] for further discussion on the KP equation relevant to this thesis.

When Kadomtsev and Petviashvili derived these equations from the Korteweg-de Vries (KdV) equation,

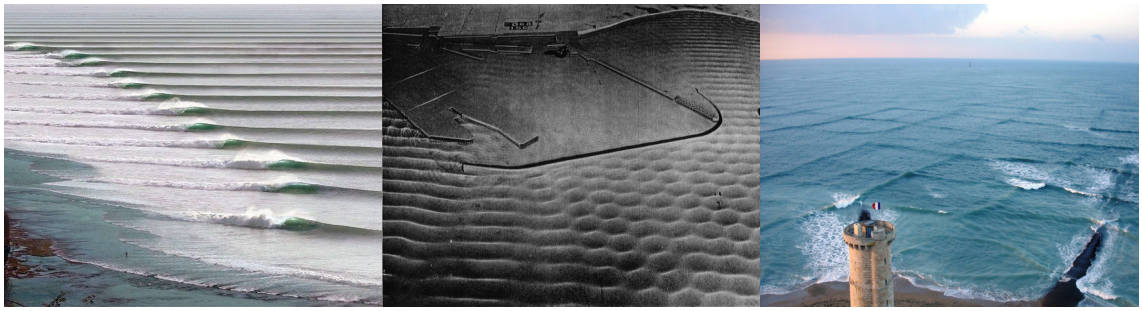
$$6uu_x + u_{xxx} = 4u_t, \tag{1.2}$$

they established the fact that KP1 is not stable to perturbations in the y coordinate whereas KP2 is stable to such perturbations [21]. In this paper, the equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1.3}$$

will be our primary focus. I will call it the KP equation and will no longer mention KP1 or KP2.

Figure 1.1: Pictures of physical waves that can be modeled by the KP2 equation



Our motivation is in computing finite-genus solutions that can model the evolution of water waves in shallow water. It is known that real solutions to equation 1.3 model such waves (see for example [12, 18, 19]).

1.2 A brief history of finite-genus solutions to KP

I discuss, briefly, the history that led to the development of Krichever's scheme of integrating certain partial differential equations using algebro-geometric methods [22]. I recommend [13] for a more detailed account. To begin I start with the history of finite-genus (or quasi-periodic) solutions to KdV. It is straightforward to check that the Lax equation,

$$\frac{\partial L}{\partial t} = [A, L],$$

where

$$L = \partial_x^2 + u \quad \text{and} \quad A = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}\partial_x u,$$

is equivalent to equation 1.2. The Lax pair L and A can be used to generate scattering data for the inverse scattering transform (IST). Gardner, Greene, Kruskal and Miura used what could be described as the inspiration of this machinery in 1967 to obtain explicit expressions

for solutions of the KdV equation [17]. The system

$$\begin{cases} L\,\Psi = \lambda\Psi, \\ \Psi_t = A\,\Psi, \end{cases}$$

along with the requirement that u vanish as $|x| \to \infty$ sufficiently rapidly, were used to produce the scattering data required for application of the Gel'fand-Levitan-Marchenko equation [25]. The methods of Gardner et al. provided mathematicians the ability to derive what are called *soliton* solutions to KdV. Mathematically, these solutions are localized and stable with respect to disturbances.[1]

Naturally, mathematicians sought to generalize these results and there were many directions to investigate. The summer of 1974 was particularly productive. Dryuma published, in [10], an explicit description of Lax pairs for KP,

$$L = \partial_x^2 + u \quad \text{and} \quad A = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}\left(u_x + w\right),$$

where $w = u_y$. The Lax pair/IST framework was generalized by Shabat and Zakharov ([31]). Considering solutions to 1.2 that are periodic as analogs to the soliton solutions, Novikov began to develop the framework for obtaining quasi-periodic solutions to KdV ([27]).

The characterization of quasi-periodic solutions to KdV became a focus for several mathematicians. Novikov solved the Cauchy problem for periodic solutions ([27]). He argued that the appropriate analog of solitons and N-solitons are the potentials that result in a finite number of gaps in the spectrum of L[2]. Dubrovin and Novikov provided methods for arriving at explicit expressions for quasi-periodic solutions ([15]). This was followed by Its and Matveev who found explicit expressions involving theta functions ([20]). For both groups of mathematicians, the key to finding expressions was inspiration received from the decade-old work of Akhiezer ([14]). Essentially, they constructed the eigenfunctions from initial data using Riemann surface theory.

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[1] A celebrated account by Scott Russell of what he calls the "Wave of Translation" details his surprise encounter with a soliton while he was watching a boat travel along a channel [29].

[2] L is self-adjoint and so the spectrum lies on the real line.

From this foundation, Krichever developed a scheme for obtaining quasi-periodic solutions to PDEs that can be expressed in Zakharov-Shabot form ([22, 23, 24]):

$$\left[L - \partial_y,\ A - \partial_t\right] = 0. \tag{1.4}$$

In particular, closed-form expressions of quasi-periodic solutions of the KP equation were derived. In Chapter 2, I compute so-called rank 1 solutions, following [11, Chapter 3].

1.3 Essential mathematical background

The purpose of this section is to record mathematical facts that will be used in the chapters that follow. In particular, I cite basic facts regarding compact Riemann surfaces and computational Riemann surfaces. For a more expository account of these topics consider [3, 11, 30]. I assume a working knowledge of differential manifold theory, real and complex analysis, and other first-year graduate-level courses.

1.3.1 Notation & conventions

In this thesis I will use the following notations to describe certain mathematical objects.

I use the following notation for subspaces of the Riemann sphere:

$$\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \text{ and } \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}.$$

Notation 1.1 (Set operations)**.** Let S be a set with topological structure, and let $E \subset F \subset S$. The smallest closed set to contain F is called the *closure* and is denoted by $\overline{F}$. An element of F that is not in E is in the *complement* of E, and the collection of such elements is denoted $S \setminus E$. The largest open set contained in F is called the *interior* of F and is denoted F°. The *boundary* of F is $\overline{F} \setminus F^\circ$ and is denoted ∂F.

Notation 1.2 (Vectors and Lists)**.** Vectors will be denoted by boldface symbols. I also differentiate a vector $\boldsymbol{L} \in \mathbb{C}^d$ from a list L of size d. The list L, is an ordered set and will not be used often. The primary difference between L and $\boldsymbol{L}$ is that I allow L to have arbitrary

objects as elements. For $i \in \{1, \ldots, d\}$, I denote the ith component of L and $\boldsymbol{L}$ by $L[i]$ and $\boldsymbol{L}[i]$ respectively.

I also need to consider complex projective space.

Notation 1.3 (Complex projective space)**.** Fix $r > 1$, then

$$\mathbb{C}\mathrm{P}^r = \left\{ (z_1 \ : \ z_2 \ : \ \ldots \ : \ z_{r+1}) : \begin{bmatrix} z_1 & z_2 & \cdots & z_{r+1} \end{bmatrix} \neq \mathbf{0} \right\}.$$

In many cases, the element $(z_1 \ : \ z_2 \ : \ \ldots \ : \ z_{r+1})$ of projective space can be treated as any one of its representatives: $(\lambda z_1, \ \lambda z_2, \ \ldots, \lambda z_{r+1})$ for any $\lambda \in \mathbb{C}^\times$. Also, whenever possible, instead of F defined on $\mathbb{C}\mathrm{P}^r$, I consider $\widetilde{F}$ defined on $\mathbb{C}^{r+1}$ and given by

$$\widetilde{F}(\boldsymbol{z}) = F(\boldsymbol{z}[1] \ : \ \boldsymbol{z}[2] \ : \ \ldots \ : \ \boldsymbol{z}[r+1]).$$

These facts follow from the fact that $\mathbb{C}\mathrm{P}^r$ is the quotient of $\mathbb{C}^r \setminus \{0\}$ by scalar multiplication.

Notation 1.4 (Curves & points on curves)**.** I will typically use f to denote a curve.

- The degree of f, with respect to z, is denoted by $\deg(f, z)$. The l-degree of f, with respect to z is defined to be the smallest power of z that appears in f and is denoted by $\mathrm{ldeg}(f, z)$. The total degree of f is denoted by $\deg(f)$.

- I write

  $$P = (u, \lambda)$$

  for points that satisfy

  $$f(u, \lambda) = 0.$$

  If a point is denoted P_*, or (u_*, λ_*), for some symbol $*$, this is used across the notation

  $$P_* = (u_*, \lambda_*).$$

- Given u_0 there may be more than one point P associated to u_0. I use the terminology P is rooted at u_0. If P has property p then I say there is a point, rooted at u_0, that exhibits property p.

- If z parameterizes a neighborhood of some point P on f then write

$$P(z) = (u(z), \lambda(z)).$$

- Irreducible planar algebraic curves (see Section 1.3.2 and Definition 1.7) are the primary interest. Unless otherwise mentioned, assume that a curve being considered is of this type.

1.3.2 Planar algebraic curves & the de-singularized Riemann surface

For our purposes, a planar algebraic curve f is a member of $\mathbb{C}[u, \lambda]$, and for simplicity I refer to f as a curve. Fix the curve f, and express it as an element of $(\mathbb{C}[u])[\lambda]$:

$$f(u, \lambda) = a_n(u)\lambda^n + \cdots + a_0(u), \text{ with } n > 0, \ a_n \neq 0. \tag{1.5}$$

For this thesis, consider λ implicitly defined by $u \in \mathbb{C}$. My first task is to provide definitions, notation, conventions, and results that will be used. From there, I define the de-singularized Riemann surface associated with f. Most of the ideas from this section are from [3, 9, 16].

Quite simply, we are interested in $f^{-1}(0) \subset \mathbb{C}^2$ and its closure in $\mathbb{C}^2_\infty$. The closure of $f^{-1}(0)$ can be obtained explicitly by analyzing the asymptotics of $f^{-1}(0)$ with the use of the homogenization of f.

Definition 1.5 (Homogenization of a curve)**.** The homogenization of f is a polynomial over $\mathbb{C}$ on $\mathbb{CP}^2$ that naturally coincides with f as follows. Let $F : \mathbb{C}^3 \setminus \{\mathbf{0}\} \to \mathbb{C}$ given by

$$F(u, \ \lambda, \ z) = z^N f\left(\frac{u}{z}, \ \frac{\lambda}{z}\right), \text{ where } N = \deg f. \tag{1.6}$$

Then F naturally represents the homogenization of f.

Let F and N be defined as above. The canonical embedding of $f^{-1}(0)$ in $\mathbb{CP}^2$ is defined by

$$(u, \ \lambda) \in f^{-1}(0) \mapsto (u \ : \ \lambda \ : \ 1).$$

On the slice $\mathbb{C} \times \mathbb{C} \times \{1\}$ of $\mathbb{C}^3$, F and f coincide:

$$F\left(u, \lambda,\ 1\right) = f\left(u, \lambda\right).$$

If $\boldsymbol{z} \in F^{-1}(0)$ then every multiple of $\boldsymbol{z}$ is also a member of $F^{-1}(0)$. Therefore, we partition the level set of F according to whether or not the 3rd component is zero. The vectors $\boldsymbol{z}$ with $\boldsymbol{z}[3] \in \mathbb{C}^{\times}$ can be scaled so that a multiple of $\boldsymbol{z}$ represents a member of the canonical embedding. Thus any $\boldsymbol{z} \in F^{-1}(0)$, with non-zero 3rd component, is associated with an element of $f^{-1}(0)$ represented by $\boldsymbol{z}/\boldsymbol{z}[3]$. The remaining vectors $\boldsymbol{z} \in F^{-1}(0)$, characterized by the fact that $\boldsymbol{z}[3]$ is zero, are called infinite points of f. In this case, we can determine the asymptotic behavior of $f^{-1}(0)$ as follows. The homogenization F reduces to the homogeneous polynomial $\tilde{f}$:

$$F\left(u, \lambda,\ 0\right) = \tilde{f}\left(u, \lambda\right) = \sum_{i=0}^{N} c_i u^i \lambda^{N-i},\ \text{where } \{c_i\}_{i=1}^{N} \subset \mathbb{C}.$$

Since

$$\tilde{f}\left(u, \lambda\right) = 0 \qquad \text{if and only if} \qquad \text{for } c \in \mathbb{C}^{\times}\ \tilde{f}\left(cu, c\lambda\right) = 0,$$

we can consider, for example,

$$\tilde{f}\left(1,\ \frac{\lambda}{u}\right) = 0.$$

This is a polynomial of one variable and the roots fix the ratio λ/u. Let $r \in \mathbb{C}$ be a root so that

$$\lambda_0(u) = ru \text{ for } u \in \mathbb{C},\ \text{and } \tilde{f}(u, \lambda_0(u)) = 0.$$

Then, since

$$\left(u\ :\ \lambda_0(u)\ :1\right) = \left(1\ :\ r\ :\frac{1}{u}\right)\ \text{and } F(1, r, 0) = 0,$$

we have

$$F(u, \lambda_0(u), 1) \to 0,\ \text{as } |u| \to \infty.$$

Alternatively, as $|u| \to \infty$, we have

$$f\left(u,\ \lambda_0(u)\right) = F\left(1,\ r,\ \frac{1}{u}\right) \to 0.$$

Thus, $\lim_{|u|\to\infty}(u, ru)$ is in the closure of $f^{-1}(0)$.

Ultimately, we would like to construct a Riemann surface from what has been explicitly produced, and since λ is viewed as an implicit function of u we work with the u-plane. The resultant of two curves will help us determine points that may need special treatment.

Definition 1.6 (Resultant of two curves)**.** Recall the curve f as given by equation 1.5. Let g be a curve of the form

$$g(u, \lambda) = b_m(u)\lambda^m + \cdots + b_0(u), \text{ for some } m > 1,\ b_m \neq 0.$$

The resultant of f and g, with respect to λ, is a polynomial in u, and I denote it by $R(f, g)$. It is defined by

$$R(f, g) = \det \begin{bmatrix} a_n & & \dots & \dots & & a_0 & & \\ & \ddots & & & & & \ddots & \\ & & a_n & & \dots & \dots & & a_0 \\ b_m & \dots & & \dots & b_0 & & & \\ & \ddots & & & & \ddots & & \\ & & b_m & \dots & & \dots & b_0 & \end{bmatrix},$$

and the following property holds:

$$R(f, g)(u) = 0 \text{ if and only if } f(u, \lambda) = g(u, \lambda) = 0, \text{ for some } \lambda.$$

Before we make use of the resultant, we define what it means for f to be irreducible.

Definition 1.7 (Irreducible curve)**.** The curve $f \in \mathbb{C}[u, \lambda]$ is *irreducible* if $f \notin \mathbb{C}$ and whenever

$$f = hg, \quad \text{for some } h, g \in \mathbb{C}[u, \lambda],$$

then either $h \in \mathbb{C}^\times$ or $g \in \mathbb{C}^\times$.

In what follows, we assume that f is irreducible. Our next objective is to analyze the structure of $\overline{f^{-1}(0)}$.

Definition 1.8 (Points of the curve)**.** Let

$$C_f = \left\{(u, \lambda) \in \mathbb{C}^2 : \ f(u, \lambda) = 0\right\}$$

and denote its closure, in $\mathbb{C}^2_\infty$, by Γ_f. We use the notation of equation 1.4. By calling P a point of f, we mean $P \in \Gamma_f$. If $P \in \partial C_f$ then we call P an infinite point of f or a point of f at ∞.

Proposition 1.3.1. *Let g be a curve. The points of g, Γ_g, form a compact set. Moreover, if g is irreducible then Γ_g is connected.*

Proof. Since $\Gamma_g \subset \mathbb{C}^2_\infty$ is closed by definition, compactness follows. For the rest of the claim see [16]. □

We consider $P \in \Gamma_f$ as dependent on u. We need to determine points

$$P(u) = (u, \lambda(u))$$

of f that are singular or that cause problems in our scheme. The u associated with such points can be handled separately from all others.

Definition 1.9 (Smooth & singular points on the curve)**.** Given the curve f, let (u, λ) be a point of the curve. Whenever

$$\begin{bmatrix} f_u(u, \lambda) & f_\lambda(u, \lambda) \end{bmatrix} \neq \mathbf{0},$$

we call (u, λ) a smooth point of f. If (u, λ) is not a smooth point of f then it is called a singular point of f.

Since we will use the u-plane to construct the surface, the associated projection map and local coordinate expressions will allow us to work primarily with the coordinate u.

Definition 1.10 (The projection map & u-coordinates)**.** Let $\pi : \Gamma_f \to \mathbb{C}_\infty$ be the projection map of Γ_f defined by $(u, \lambda) \in \Gamma_f \mapsto u$. For any point P_0, the restriction of π to a neighborhood of P_0 will be called u-coordinates centered at P_0.

We now define finite subsets of $\pi(\Gamma_f)$ whose members must be avoided when performing certain mathematical tasks.

Definition 1.11 (Problem points)**.** Express f according to equation 1.5. Let

$$S_{-1} = \{u \in \mathbb{C}_\infty : \ u = \infty \text{ or a singular point of } f \text{ is rooted at } u\}, \tag{1.7}$$

and also

$$S_0 = \{u \in \mathbb{C}_\infty : \ u = \infty \text{ or } a_n(u) = 0 \text{ or } \ R\left(f, f_\lambda\right)(u) = 0\}. \tag{1.8}$$

An element of S_0 is called a problem point of f.

Proposition 1.3.2 (Properties of S_0 and its complement)**.** *Given S_{-1} and S_0, as defined in definition 1.11, we record the following basic facts.*

1. *$S_{-1} \subset S_0$ and S_0 is finite.*

2. *If $P_0 \in \Gamma_f \setminus \pi^{-1}(S_0)$, then there is a neighborhood U_0 of P_0 for which $\pi|_{U_0}$ is invertible, and the function $\lambda_0(u)$ on $\pi(U_0)$, defined implicitly by*

$$f(u, \lambda_0(u)) = 0 \text{ for } u \in \pi(U_0),$$

   is invertible. Moreover,

$$\left(\pi|_{U_0}\right)^{-1}(u) = (u, \lambda_0(u)).$$

3. *Suppose P_0 is defined according to (2). Let P_1 be defined similar to P_0: replace 0 with 1 in (2). If $U_0 \cap U_1$ is not empty, then the maps defined in (2) agree on the intersection of U_0 and U_1.*

Proof. We omit the proof as it is straightforward. We do note that item 2 is a consequence of the inverse function theorem. □

It is well known that for any point of f there exist local coordinates constructed using Puiseux series and that are compatible with the conformal structure of the surface. I utilize results in [3, 9, 28], and use the `puiseux` command to compute these coordinates explicitly.

Theorem 1.1 (Coordinates determined by Puiseux series)**.** *For any point $P_0 \in \Gamma_f$, there is a unique non-zero integer r_0 and integer N_0 such that, on a sufficiently small coordinate ball B_0 at 0, the map ϕ_0 defined by*

$$(u, \lambda) \mapsto (u - u_0)^{\frac{1}{r_0}} \quad \textit{for } (u, \lambda) \textit{ satisfying } (u - u_0)^{\frac{1}{r_0}} \in B_0$$

and its inverse ϕ_0^{-1} defined by

$$z \in B_0 \mapsto (z^{r_0} + u_0, \sum_{N_0=k}^{\infty} c_k z^k)$$

determine local coordinates in the neighborhood U_0 of P_0, where

$$U_0 = \{(u, \lambda) : \textit{ for some } z \in B_0, \ u = z^{r_0} + u_0 \textit{ and } \lambda = \sum_{N_0=k}^{\infty} c_k z^k\}.$$

Proof. We omit the proof. This is established in many books on Riemann surfaces. See [3, 16, 30]. □

Definition 1.12 (p-coordinates of a curve)**.** Let $P_0 \in \Gamma_f$. Then we call the coordinates of Theorem 1.1, defined by

$$P(z) = (u(z), \lambda(z)) = \left(z^{r_0} + u_0, \ \sum_{N_0=k}^{\infty} c_k z^k \right),$$

the p-coordinates of P_0.

Definition 1.13 (Riemann surface)**.** A Riemann surface is a tuple $(\Gamma, \{\psi_\alpha\}_{\alpha \in A})$ with the property that Γ is a 2-dimensional, connected topological manifold and for each $\alpha \in A$ the chart (U_α, ψ_α) defined by the local coordinate $\psi_\alpha : U_\alpha \to \psi_\alpha(U_\alpha)$ satisfies the following properties.

1. The collection $\{U_\alpha\}_{\alpha \in A}$ forms a cover of Γ by open sets

$$\Gamma \subset \bigcup_{\alpha \in A} U_\alpha.$$

2. For $\alpha,\ \beta \in A$ so that $U_\alpha \cap U_\beta$ is non-empty, the transition map $\psi_\alpha \circ \psi_\beta^{-1}$ that transitions from chart (U_β, ψ_β) to (U_α, ψ_α) is bi-holomorphic.

I note that this definition assumes that our topological manifolds are Hausdorff spaces and have countable bases.

Definition 1.14 (Conformal isomorphisms)**.** Let M_1 and M_2 be Riemann surfaces and suppose $\Phi : M_1 \to M_2$. Then Φ is a conformal isomorphism if it is a homeomorphism such that every local coordinate representation of Φ is bi-holomorphic. In this case, M_1 is conformally isomorphic to M_2.

Now that we have a conformal structure we can discuss how Γ_f can be viewed as a Riemann surface. If the points of f that do not admit a local inverse to π are removed, then a Riemann surface can be formed. This surface is the most natural surface to associate with f, and it covers, under π, the punctured u-plane. However, in light of Theorem 1.1 we can de-singularize points of f.

The formulation of the de-singularization of Γ_f can proceed in multiple ways. For instance, in [16, Section 9.3], a collection of charts $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ is formed and then inverted to form a collection of inverted charts $\{(\psi_\alpha(U_\alpha), \psi_\alpha^{-1})\}_{\alpha \in A}$. This collection is then patched together by forming a disjoint union. From this point, a quotient is formed using the compatibility of charts. At each stage of this construction, a map is naturally defined via the collection $\{\psi_\alpha^{-1}\}_{\alpha \in A}$. The final map constructed via this process is called the resolution of singularities of Γ_f.

Other approaches exist but what they all have in common is that the singular points of the curve are resolved in a manner compatible with solving the primary problem.

We follow the procedure used in the `algcurves` package to de-singularize f using the Puiseux series ([9]).

Definition 1.15 (The de-singularized Riemann surface of a curve)**.** Let the collection of charts $\{U_P, \psi_P\}_{P \in \Gamma_f}$ be defined so that, for $P \in \Gamma_f$, U_P and ψ_P are defined via the p-coordinates (Definition 1.12). Then $(\Gamma_f, \{U_P, \psi_P\}_{P \in \Gamma_f})$ satisfies Definition 1.13 and is a

Riemann surface. Each singular point P is now made distinct by using Puiseux series in the construction of the chart (U_P, ψ_P).

We call the Riemann surface constructed above the de-singularized Riemann surface associated with f.

The final statement of this section is the fact that every compact Riemann surface can be obtained as prescribed in this section.

Theorem 1.2. *Let Γ be a compact Riemann surface. Then Γ is conformally isomorphic to a Riemann surface of an irreducible curve.*

Proof. I recommend [30, Chapters 4 & 5] for a detailed proof. □

1.3.3 Essential topological considerations

Let f be an irreducible curve. Assume that all of the mathematical objects associated with f of the previous section are also defined. It is well known that compact, orientable topological manifolds are homeomorphic to a g-holed torus. In this section, I make clear the relevant topological structures, needed for the rest of the thesis.

Definition 1.16 (Genus of Γ_f)**.** The compact Riemann surface Γ_f is of genus g if it is homemorphic to torus with g holes.

Definition 1.17 (Symplectic Homology basis of Γ_f)**.** The symplectic homology basis of Γ_f is denoted by $\{\alpha_i,\ \beta_i\}_{i=1}^{g}$ and has the following intersection indicies:

$$\alpha_i \circ \alpha_j = \beta_i \circ \beta_j = 0, \quad \alpha_i \circ \beta_j = \delta_{ij}, \text{ for } i, j \in \{1, \ldots, g\}.$$

This basis can be used to represent Γ_f as a quotient of a polygonal region in the complex plane.

Definition 1.18 (The $4g$-gon)**.** Let $\widetilde{\Gamma}$ denote the regular $4g$-gon embedded in $\mathbb{R}^2$ with its center at 0. Its boundary $\partial\widetilde{\Gamma}$ is a closed loop composed of $4g$ edges. Choose any edge and

label it a_1. Then, by successive labeling and recording edges of the boundary in a counter clockwise fashion, we have encoded the names and order of the edges that form the boundary by the word

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\dots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}.$$

Recall that systematically cutting the Riemann surface along the loops in $\{\alpha_i,\ \beta_i\}_{i=1}^g$ homeomorphically transforms the surface into a quotient space on $\widetilde{\Gamma}$. If we retain the labels of the edges described in Definition 1.18 and the encoded word

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1}\dots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1},$$

then the standard topological gluing operations define the quotient space homeomorphic to Γ_f. As an example, if $(1-t)e_1+te_2$ and $(1-t)\ell_1+t\ell_2$ are parameterizations of edges named a_1 and a_1^{-1} respectively, then glue together $(1-t)e_1+te_2$ and $t\ell_1+(1-t)\ell_2$.

Definition 1.19 (Cohomology basis of Γ)**.** The co-homolgy basis consist of g differentials, so that the ith differential of the basis is denoted by ω_i, for $i\in\{1,\dots,g\}$. The vector of the cohomology basis is denoted by $\boldsymbol{\omega}$ with

$$\boldsymbol{\omega}[i]=\omega_i,\ \text{for each } i.$$

We express the basis in local coordinates (u-coordinates)

$$\omega_i = h_i\left(u,\lambda\right)du,\ \text{for each } i. \tag{1.9}$$

For each i, ω_i is holomorphic, which means that for any chart (U,z) the local representation is of the form

$$\omega_i = H_i(z)dz$$

for some holomorphic function H_i. These differentials form a basis of the holomorphic differentials of Γ_f: every holomorphic differential can be expressed as $\boldsymbol{c_0}^T\boldsymbol{\omega}$, for $\boldsymbol{c_0}\in\mathbb{C}^g$.

The holomorphic differentials are also called Abelian differentials of the first kind. I define the the Abelian differentials.

Definition 1.20 (Abelian differentials of Γ_f)**.** Let ω be a differential. The Abelian differentials of the first kind were defined in Definition 1.19, and ω is of the first kind if, for an arbitrary chart (U, z), the local coordinate representation satisfies

$$\omega = H(z)dz, \ H(z) \text{ is holomorphic.}$$

The differential ω is of the second kind if there is a unique $P_0 \in \Gamma_f$ and $N_0 > 0$ such that, for a chart (U_0, z) centered at P_0, ω is of the form

$$\omega = H(z)dz, \text{ and } H(z) \text{ has a pole of multiplicity } N_0 + 1 \text{ at } 0.$$

In this case we re-express $H(z)$ so that

$$H(z) = \frac{c_0}{z^{N_0+1}} + H'(z), \ H' \text{ is holomorphic and } c_0 \in \mathbb{C}^\times.$$

We call the meromorphic portion of ω, $c_0/z^{N_0+1}dz$, the principal part of ω.

Finally, ω is a differential of the third kind if there are two points P_1 and P_2 and some $r \in \mathbb{C}^\times$ so that P_1 is a simple pole with residue r and P_2 is a simple pole with residue $-r$.

We now begin integrating differentials along the homology basis.

Definition 1.21 (The a and b periods of a differential)**.** Let ω be any Abelian differential. The a-periods of ω are the g integrals $\{\oint_{\alpha_j} \omega\}_{j=1}^g$ and similarly the b-periods are the g integrals $\{\oint_{\beta_j} \omega\}_{j=1}^g$.

The $g \times g$ matrices Π_A and Π_B defined by

$$(\Pi_A)_{i,j} = \oint_{\alpha_j} \omega_i \ \text{ and } (\Pi_B)_{i,j} = \oint_{\beta_j} \omega_i, \text{ for each } i, j \tag{1.10}$$

are called the matrix of a-periods and matrix of b-periods, respectively.

The utility of $\widetilde{\Gamma}$ will be demonstrated in what follows. It is not hard to show that $\left(\widetilde{\Gamma_f}\right)^\circ$ is homeomorphic to

$$U = \{P \in \Gamma_f : \ P \text{ is not on the cycles } \alpha_i, \beta_i, \text{ for each } i\}.$$

If $P_0 \in U$ is identified with $z_0 \in \left(\widetilde{\Gamma_f}\right)^\circ$, then fix any $z_1 \in \widetilde{\Gamma_f}$ and let γ be a straight line from z_0 to z_1. This ensures that in the worst case $\gamma(1)$ is on a edge. Hence, for a closed differential ω on $\widetilde{\Gamma}$ the expression

$$f(z_1) = \int_{z_0}^{z_1} \omega = \int_0^1 \omega_{\gamma(t)}(\gamma'(t))dt$$

is well-defined. Moreover, the pull-back with respect to the homeomorphism ensures that we could have alternatively defined f on Γ_f. Integration on this space is used to simplify calculations and prove results, such as the ones below.

Theorem 1.3. *Let ω and ω' be closed differentials on Γ_f. Then*

1. *for the a-periods and b-periods of ω, $\{A_i,\ B_i\}_{i=1}^g$, and the a-periods and b-periods of ω', $\{A_i',\ B_i'\}_{i=1}^g$, we have*

$$\int_{\Gamma_f} \omega \wedge \omega' = \sum_{i=1}^{g} \left(A_i B_i' - A_i' B_i\right);$$

2. *if ω is holomorphic and each of the a-periods of ω are 0, then ω is identically zero.*

Proof. See [11]. □

Consider $\boldsymbol{\omega}$ as defined in Definition 1.19 and also the associated matrix of a-periods Π_A defined in Definition 1.21. It is well known that Π_A is invertible. Indeed, since

$$\left(\Pi_A^T \boldsymbol{v}\right)_j = \sum_{i=1}^{g} \boldsymbol{v}[i] \oint_{\alpha_j} \omega_i = \oint_{\alpha_j} \sum_{i=1}^{g} \boldsymbol{v}[i]\omega_i,$$

we see that, by Theorem 1.3, the fact that $\boldsymbol{\omega}$ forms a basis implies Π_A must be invertible.

Definition 1.22 (Normalization of Abelian differentials & the Riemann matrix)**.** The vector whose components consist of the normalized cohomology basis is

$$\boldsymbol{\omega}^\ell = \Pi_A^{-1}\boldsymbol{\omega}, \text{ with } \boldsymbol{\omega}^\ell[i] = \omega_i^\ell. \tag{1.11}$$

The a-periods of this basis are

$$\oint_{\alpha_j} \omega_i^\ell = \delta_{ij}, \text{ for each } i \text{ and } j.$$

Let ω be an Abelian differential of the either second kind or third kind. If $\boldsymbol{C} \in \mathbb{C}^g$ is given by

$$\boldsymbol{C}[i] = \int_{\alpha_i} \omega.$$

Then $\omega - \boldsymbol{C}^T\boldsymbol{\omega}^\ell$ and ω have the same principal part. However,

$$\oint_{\alpha_i} \omega - \boldsymbol{C}^T\boldsymbol{\omega}^\ell = \oint_{\alpha_i} \omega - \boldsymbol{C}_i = 0,$$

for each i.

Hence the normalization of an Abelian differential of the second or third kind is a differential ω with the same principal part and zeros for a-periods.

Definition 1.23 (Riemann matrix)**.** A Riemann matrix B is symmetric and has the property that $\mathrm{Im}B$ is positive definite. Moreover, if B can be considered the normalized b-periods of a cohomology basis with respect to some Riemann surface, then we say that B can be obtained from a curve.

Theorem 1.4 (Periods of differentials of 2nd & 3rd kind)**.** *With respect to Definition 1.22.*

1. *Let ω be the normalized differential of the second kind with a singularity at P_0 of multiplicity $N_0 + 1$, with $N_0 > 1$. Let (U_0, z) be a centered at P_0. Then for each i,*

$$\omega_j^\ell = H_j(z)dz$$

and

$$\oint_{\beta_i} \omega_{P_0}^{N_0} = \frac{1}{N_0!}\left(\frac{d^{N_0}}{dz^{N_0}} H_i\right)(0).$$

2. *Let ω be the normalized differential of the third kind with simple poles at P_1 and P_{-1} of residue 1 and -1 respectively. Then, for each i,*

$$\oint_{\beta_i} \omega = \int_{P_2}^{P_1} \omega_i^\ell.$$

Proof. Essentially, this is a restatement of [11, Lemma 2.1.2]. □

1.3.4 The Abel map, The Riemann Theta function, & Jacobi inversion

Let f and be a curve expressed as in equation 1.5 and fix all of the mathematical objects of the previous sections that are associated to f. Fix a point $P_\ell \in \Gamma_f$ to be used throughout this section.

Since B is a Riemann matrix of f it makes sense to define Riemann theta functions.

Definition 1.24 (The Riemann theta function)**.** Let B' be a Riemann matrix, as in Definition 1.23. The Riemann theta function, or just theta function, of B' is the map

$$\theta\left(\boldsymbol{z}|B'\right) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} e^{2\pi i\left(\boldsymbol{n}^T\frac{B'}{2}\boldsymbol{n}+\boldsymbol{n}^T\boldsymbol{z}\right)}. \tag{1.12}$$

When it is clear what matrix is being used as a parameter of the Riemann theta function we write $\theta\left(\boldsymbol{z}\right)$.

Proposition 1.3.3. *Let B' be a Riemann matrix and let θ be the Riemann theta function of B'. Then, for $\boldsymbol{z}\in\mathbb{C}^g$,*

(a) $\theta\left(\boldsymbol{z}+\boldsymbol{N}+B'\boldsymbol{M}\right) = e^{-2\pi i\left(\boldsymbol{M}^T\frac{B'}{2}\boldsymbol{M}+\boldsymbol{N}^T\boldsymbol{z}\right)}\theta\left(\boldsymbol{z}\right)$, *for* $\boldsymbol{M},\ \boldsymbol{N}\in\mathbb{Z}^g$, *and*

(b) $\theta\left(-\boldsymbol{z}\right) = \theta\left(\boldsymbol{z}\right)$.

We use the `algcurves` command `RiemannTheta` to work with theta functions numerically (see [7]).

Next, we consider the Abel map and properties.

Definition 1.25 (The Abel map)**.** The map $\mathcal{A}:\Gamma_f^2\to\mathbb{C}^g$ defined by

$$\left(\mathcal{A}\left(P_1,P_2\right)\right)_i = \int_{P_1}^{P_2}\omega_i^\ell,\ \text{for } i\in\{1,\ \ldots,\ g\} \tag{1.13}$$

is called the Abel map. The `algcurves` command `AbelMap` allows us to compute the Abel map using Puisuex series expansions via the `algcurves` command `puisuex`.

If, for any P, the path from P_ℓ to P has been fixed, then we write

$$\mathcal{A}_\ell^P = \mathcal{A}\left(P_\ell,P\right).$$

Definition 1.26 (Divisors of Γ_f)**.** A divisor, D, is a formal, finite sum of points in Γ_f:

$$D = \sum_{i=1}^{N} n_i D_i, \quad \text{for } \{D_i\}_{i=1}^{N} \subset \Gamma_f, \text{ and } \{n_i\}_{i=1}^{N} \subset \mathbb{Z}. \tag{1.14}$$

The degree of D is

$$\deg(D) = \sum_{i=1}^{N} n_i$$

and if L is a linear map on Γ_f it is naturaly extended to the divisor D:

$$\mathrm{L}(D) = \sum_{i=1}^{N} n_i \mathrm{L} D_i.$$

The theta function is defined on $\mathbb{C}^g$ and it is quasi-periodic, whereas $\mathcal{A}_\ell$ maps into $\mathbb{C}^g$ and a different choice in paths only adds integer multiples of the periods to the theta function. We now consider the Jacobian of Γ_f.

Definition 1.27 (Jacobian of Γ_f)**.** The Jacobian of Γ_f, denoted by $\mathrm{J}(\Gamma_f)$ is the quotient of $\mathbb{C}^g$ via the equivalence: $\boldsymbol{z}_1 \sim \boldsymbol{z}_2$ if

$$\boldsymbol{z}_1 - \boldsymbol{z}_2 = \boldsymbol{N} + B\boldsymbol{M} \ \ (\boldsymbol{N}, \boldsymbol{M} \in \mathbb{Z}^g).$$

Theorem 1.5 (Abel's Theorem)**.** *For $N > 0$ and points $\{P_i,\ Q_i\}_{i=1}^{N} \subset \Gamma_f$,*

$$\sum_{i=1}^{N} \mathcal{A}_\ell^{P_i} - \sum_{i=1}^{N} \mathcal{A}_\ell^{Q_i} \equiv \mathbf{0}$$

if and only if there is a meromorphic function on Γ_f whose N zeros are the points P_i and the N poles are the points Q_i.

Abel's theorem defines an equivalence on the set of divisors. Other results, in particular the Riemann-Roch theorem, can be used to analyze divisors by degree, among other properties. I define the most relevant terminology.

Definition 1.28 (More divisor terminology)**.** I define a divisor of a meromorphic function and differential, linear equivalence between divisors, and special types of divisors.

1. For any meromorphic function h on Γ_f, let (h) denote the divisor

$$(h) = (P_1 + P_2 + \cdots + P_N) - (Q_1 + Q_2 + \cdots + Q_N)$$

consisting of the N zeros $P_1,\ P_2,\ \ldots, P_N$ and N poles $Q_1,\ Q_2,\ \ldots, Q_N$ of h. For any Abelian differential ω, let (ω) be the zeros and poles defined similarly.

2. Linear equivalence is the equivalence relation on the divisors, given by

$$D_1 \sim D_2 \text{ if } \deg(D_1) = \deg(D_2) \text{ and } \mathcal{A}_\ell^{D_1} \equiv \mathcal{A}_\ell^{D_2}.$$

The canonical class $\mathcal{C}$ consists of divisors of Abelian differentials.

3. A divisor $\sum_i n_i D_i$ whose coefficients are positive integers will be called positive. A positive divisor D of degree g is special if there is a meromorphic function whose poles are the g points of D. Otherwise, it is called non-special. A point P is called a Weierstrass point if there is a meromorphic function with a single pole at P of multiplicity g or less.

Key to what remains of this section is the Vector of Riemann Constants $\boldsymbol{K}_\ell$ defined by

$$(\boldsymbol{K}_\ell)_i = \frac{1 + B_{ii}}{2} - \sum_{j \neq i} \oint_{\alpha_j^\ell} \left(\omega_j^\ell \int_{P_\ell}^{P} \omega_i^\ell \right), \quad \text{for each } i. \tag{1.15}$$

Theorem 1.6 (Jacobi inversion)**.** *Let $\boldsymbol{\zeta} \in \mathbb{C}^g$ be fixed and consider*

$$F(P) = \theta\left(\mathcal{A}_\ell^P - \boldsymbol{\zeta}\right).$$

There are only two possibilities: if $F \equiv \mathbf{0}$, then

$$\boldsymbol{\zeta} = \mathcal{A}_\ell^Q + \boldsymbol{K}_\ell$$

for some special divisor Q of degree g; otherwise, F has g zeros and

$$\boldsymbol{\zeta} = \mathcal{A}_\ell^D + \boldsymbol{K}_\ell$$

for a non-special divisor D of degree g. Moreover, the zeros of F are the g points of D.

1.3.5 Theta functions with characteristics, half-periods, & theta constants

We continue from the preceding section. Theta functions can be generalized to theta functions with characteristics.

Definition 1.29 (Theta functions with characteristics)**.** Let B' be a Riemann matrix. Then $\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]$ denotes a theta function with characteristics, with $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{R}^g$, and is defined as

$$\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right](\boldsymbol{z})=e^{2\pi i\left(\frac{\boldsymbol{\alpha}^T B'\boldsymbol{\alpha}}{2}+\boldsymbol{\alpha}^T\boldsymbol{z}+\boldsymbol{\beta}\right)}\theta\left(\boldsymbol{z}+\boldsymbol{\beta}+B'\boldsymbol{\alpha}\right).$$

Proposition 1.3.4. *Let B' be a Riemann matrix. And let $\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]$ denote a theta function with characteristics $\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]$. Then, for $\boldsymbol{z}\in\mathbb{C}^g$,*

$$\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right](\boldsymbol{z}+\boldsymbol{N}+B'\boldsymbol{M})=e^{-2\pi i\left(\frac{\boldsymbol{M}^T B'\boldsymbol{M}}{2}+\boldsymbol{M}^T(\boldsymbol{z}+\boldsymbol{\beta})-\boldsymbol{\alpha}^T\boldsymbol{N}\right)}\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right](\boldsymbol{z}),\ \textit{for}\ \boldsymbol{M},\ \boldsymbol{N}\in\mathbb{Z}^g.$$

We are primarily interested in $\theta\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]$ with $\boldsymbol{\alpha},\ \boldsymbol{\beta}\in\mathbb{Z}/\left(2\mathbb{Z}\right)$.

Definition 1.30 (Half-periods)**.** We call $\left[\boldsymbol{\alpha},\boldsymbol{\beta}\right]$ $\boldsymbol{\alpha},\ \boldsymbol{\beta}\in\mathbb{Z}/\left(2\mathbb{Z}\right)$ a half-period. This definition extends to the associated vector $\boldsymbol{\beta}+B'\boldsymbol{\alpha}$ under identification.

To simplify notation, we first introduce an enumeration of $\mathbb{Z}/\left(2\mathbb{Z}\right)$.

Definition 1.31 (Enumeration of half-periods)**.** For $\boldsymbol{h},\ \boldsymbol{h'}\in\mathbb{Z}/\left(2\mathbb{Z}\right)$, define

$$\mathrm{I}_{(\boldsymbol{h},\boldsymbol{h'})}=\min\{i:\boldsymbol{h}[i]\neq\boldsymbol{h'}[i]\},$$
$$\mathrm{Level}(\boldsymbol{h})=\sum_{i=1}^{g}\boldsymbol{h}[i].$$

Then, for $\boldsymbol{h},\boldsymbol{h'}\in\mathbb{Z}/\left(2\mathbb{Z}\right)$, we write $\boldsymbol{h}<\boldsymbol{h'}$ if

$$\mathrm{Level}(\boldsymbol{h})<\mathrm{Level}(\boldsymbol{h'})$$

or

$$\mathrm{Level}(\boldsymbol{h})=\mathrm{Level}(\boldsymbol{h'})\text{ and }\boldsymbol{h}[\mathrm{I}_{(\boldsymbol{h},\boldsymbol{h'})}]<\boldsymbol{h'}[\mathrm{I}_{(\boldsymbol{h},\boldsymbol{h'})}].$$

Using this order, let

$$\mathcal{H} = \{\boldsymbol{h}_n\}_{n=1}^{2^g}$$

denote the enumerated half-periods. We also identify $(\mathbb{Z}/2\mathbb{Z})^g$ with binary numbers under the mapping

$$\boldsymbol{h} \in (\mathbb{Z}/2\mathbb{Z})^g \mapsto b_{\boldsymbol{h}} = \boldsymbol{h}[g]\boldsymbol{h}[g-1]\ldots\boldsymbol{h}[1].$$

Notation 1.32 (Half-periods)**.** With respect to Definition 1.31. We let $\boldsymbol{h}_{(i,j)}$ denote

$$\boldsymbol{h}_{(i,j)} = \frac{B'\boldsymbol{h}_i}{2} + \frac{\boldsymbol{h}_j}{2}$$

for $i, j \in \{1, \ldots 2^g\}$. For $\boldsymbol{h}_{(i,0)}$ we write $\boldsymbol{h}_i$. This, at times, identifies $\boldsymbol{h}_{(i,0)}$ with $\mathcal{H}$ and we allow the context to resolve the ambiguity.

Now we define the theta constants.

Definition 1.33 (Theta constants)**.** Let $\boldsymbol{h}_s \in \mathcal{H}$. Then the theta constants associated with $\boldsymbol{h}_s$ are

$$\hat{\theta}^s = \theta\left[\boldsymbol{h}_s\right](\boldsymbol{0}|2B'), \tag{1.16}$$

$$\hat{\theta}^s_{ij} = \partial^2_{z_i z_j}\left(\theta\left[\boldsymbol{h}_s\right](\boldsymbol{z}|2B')\right)\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \tag{1.17}$$

$$\hat{\theta}^s_{ijkl} = \partial^4_{z_i z_j z_k z_l}\left(\theta\left[\boldsymbol{h}_s\right](\boldsymbol{z}|2B')\right)\Big|_{\boldsymbol{z}=\boldsymbol{0}}, \tag{1.18}$$

$$\hat{\theta}^s_{\boldsymbol{U}} = \sum_{ijkl} u_i u_j u_k u_l \hat{\theta}^s_{ijkl}. \tag{1.19}$$

The following proposition is of importance to the work in Chapter 3.

Proposition 1.3.5 (Non-singularity condition)**.** *Let*

$$M' = \begin{bmatrix} \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \ldots & \hat{\theta}^1_{1g} & \hat{\theta}^1_{22} & \ldots & \hat{\theta}^1_{gg} & \hat{\theta}^1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \hat{\theta}^{2^g}_{11} & \hat{\theta}^{2^g}_{12} & \ldots & \hat{\theta}^{2^g}_{1g} & \hat{\theta}^{2^g}_{22} & \ldots & \hat{\theta}^{2^g}_{gg} & \hat{\theta}^{2^g} \end{bmatrix}.$$

If B' is a matrix that comes from a surface, then M' is of full rank. On the other hand, if B' can be expressed as a block matrix, then M' is not of full rank.

Proof. This statement follows from [11, Lemma 4.3.1] and its proof. □

1.3.6 Construction of explicit representation of Γ_f

The previous sections build off of each other to construct a Riemann surface Γ_f. However, this is not the complete picture when computing and working numerically with a Riemann surface is desired. The key difference is that every step must be made explicit and we must start on the u-plane.

First, we must obtain the monodromy group so that we can obtain a homology basis. To do this, the u-plane must be punctured. Constructing S_0, from Definition 1.11, is easy to do, as it involves finding the zeros of specific polynomials in u. A set S_1 can be formed by adding a finite number of points to S_0, for instance by adding bounding points or even u_Q of proposition 2.2.1. Next, a starting point must be determined. We denote this point by P_ℓ. Using u_ℓ as the base point for all paths, π is used to compute the monodromy group. This requires generators of the fundamental group of $\mathbb{C} \setminus S_1$ to be formed. Enumeration of the points rooted at u_ℓ gives a label to sheets of the cover π, and numerical analytic continuation encodes the monodromy action of the fundamental group on the sheets as a permutation. The `monodromy` command of the `algcurves` package allows us to compute this group. For more details see [3, Chapter 2] as well as [9] and [28].

Once the monodromy behavior of the generators is cataloged, an explicit symplectic homology basis $\{\alpha_i^\ell,\ \beta_i^\ell\}_{i=1}^g$ can be formed (see [9]). The command `homology` of the `algcurves` package computes this for a given curve. It should be noted that there are other approaches to getting an explicit basis. For example, see [4]. In [4], the generators of the fundamental group are formed from a Voronoi diagram; however, it is not configured for use with `homology`. This can be done, but I do not detail this here.

Once the homology has a representation, the next task is to obtain a basis for the cohomology. The `differentials` command of the `algcurves` package obtains a basis for the cohomology ([9]), and this can be numerically integrated against the representations of the homology. This allows for an explicit representation of the normalized cohomology.

Provided local representations of the cohomology basis are known and used, integration

against the representatives of the homology becomes a quadrature problem. See [26] for an excellent resource. We use the `AbelMap` command of `algcurves` to integrate between paths. It should be noted that defining a function on Γ_f requires a consistent choice of paths. Moreover, some quantities are only well-defined once representatives for the symplectic basis have been fixed (see Section 2.2.4).

Once the data above is made explicit, computing the Riemann matrix presents no greater difficulty. Having access to a command, such as the `RiemannTheta` command of `algcurves`, we can then compute everything desired in this thesis explicitly.

The discussion above suggests that an appropriate mathematical framework for computing an explicit Riemann surface is the analytic continuation of a germ. It can be shown that constructing the representative of Γ_f and then choosing a starting point P_ℓ is equivalent to constructing the ramified Riemann surface of the germ defined by $(u_\ell, \lambda_\ell, f)$.

Definition 1.34 (Explicit Riemann surface)**.** An explicit representation, or realization, of the Riemann surface associated with f consists of a base point P_ℓ, a finite collection of points S_1 that contains S_0, a fixed collection of loops on $\mathbb{C} \setminus S_1$ that start at u_ℓ and generate the fundamental group, a cohomology basis $\boldsymbol{\omega}$, an integration scheme so that $\mathcal{A}_\ell$ can be defined, and local coordinates at the points rooted in S_0 that can be used to numerically integrate points near problem points.

1.4 Contributions of this thesis

This thesis details the implementation of two Maple procedures, *Krichever's Construction Method* (`KCM`) and *Extended Dubrovin's Method* (`EDM`), that compute finite-genus solutions of KP. The approaches contrast: `KCM` encapsulates Krichever's construction, as discussed in Chapter 2, using knowledge of the underlying Riemann surface, whereas `EDM` extends Dubrovin's work in [11, Chapter 4], as discussed in Chapter 3, using only a Riemann matrix and a few parameters.

A primary problem that is addressed in this thesis is the determination of the Vector of

Riemann Constants $\boldsymbol{K}_\ell$, defined by equation 1.15 and discussed in Section 2.2.4. I demonstrate that if the Riemann surface is constructed explicitly so that a base point P_ℓ as well as representatives for the homology and cohomolgy are fixed, then under reasonable assumptions the half-period $\boldsymbol{h}_\ell$ that relates $\boldsymbol{K}_\ell$ to the canonical class can be determined uniquely (Proposition 2.2.6). This result gives an affirmative answer to a problem posed by Patterson in his thesis [28, page 104].

As mentioned, in conjunction with this thesis, Maple procedures have been developed. These procedures are stored in a Maple package and can be found at the Git repository `https://github.com/MSBennett21/kp.git`.

Chapter 2

KRICHEVER'S METHOD FOR COMPUTING FINITE-GENUS SOLUTIONS OF THE KADOMTSEV-PETVIASHVILI EQUATION

Establishing a closed form expression for periodic and quasi-periodic solutions to KdV in terms of Riemann theta functions gave insight on the relationship between spectral theory of certain differential operators and algebraic geometry. From this insight, Krichever developed a method for constructing solutions of nonlinear partial differential equations that can be expressed in the form of equation 1.4 where L and A are linear differential operators in x, and each has coefficients that are matrix functions of x, y and t [22].

As one of the primary goals of this thesis is to compute finite-genus solutions to equation 1.3, it will be of great benefit to examine Krichever's scheme applied to KP, as described in [11]. From there, I detail how certain quantities are computed. In particular, a proof is given of a conjecture posited by Patterson ([28, Page 104]): there are conditions that can be enforced that uniquely determine the half-period associated to $\boldsymbol{K}_\ell$ of equation (Proposition 2.2.6). After detailing how I compute parameters, I present *Krichever's Construction Method* (KCM), along with numerical data from its application.

2.1 Krichever's construction

Before we analyze the construction, I define Baker-Akhiezer functions.

Definition 2.1 (Baker-Akhiezer function)**.** Fix any point P_Q on a compact Riemann surface Γ. Choose any coordinates $z = k^{-1}$ centered at P_Q and any g-divisor

$$D = D_1 + \cdots + D_g.$$

Then $\Psi : \Gamma \to \mathbb{C}_\infty$ is a **Baker-Akhiezer function** if

i. Ψ is meromorphic on $\Gamma \setminus \{P_Q\}$, with exactly g poles at the points D_i for $i \in \{1, \ldots, g\}$, and

ii. there is a constant c such that $\Psi(P) \sim ce^{xk+yk^2+tk^3}$ for P near P_Q.

It is easy to see that Baker-Akhiezer functions form a vector space. However, when D is non-special, as in Definition 1.28, even more can be said.

Proposition 2.1.1. *For a fixed z and non-special g divisor D, the space of functions that satisfy Definition 2.1 form a one-dimensional linear space.*

Proof. This follows from [11, Theorem 3.1.1]. □

Krichever's construction, as described in [11, Chapter 3], is essentially a constructive proof of the following result.

Theorem 2.1 (Krichever's construction)**.** *Let Γ be a compact Riemann surface of genus $g > 0$, and let f be an irreducible planar algebraic curve with the property that Γ can be realized as*

$$\Gamma_f = \{(u, \lambda) : f(u, \lambda) = 0\}.$$

Then, for any point

$$(u_Q, \lambda_Q) = P_Q \in \Gamma_f,$$

there is a family of finite-genus solutions to equation 1.3 parameterized by P_Q.

Proof. Fix Γ_f and $P_Q \in \Gamma_f$.

Begin by fixing an explicit representation of Γ_f (as described in Section 1.3.6). A base point P_ℓ and paths to points on the surface can always be chosen so that $P_\ell \neq P_Q$ and P_Q is not in the image of the paths to be traveled. By de-singularizing as in Definition 1.15, we can also assume that f is not singular (see [22, Page 17]). In particular, we have explicit representatives $\left\{\alpha_i^\ell,\ \beta_i^\ell\right\}_{i=1}^g$ of the symplectic homology basis, as well as a cohomology basis $\{\omega_i\}_{i=1}^g$, and the normalized basis $\left\{\omega_i^\ell\right\}_{i=1}^g$. The Abel map, given in Definition 1.25, is well

defined, and so is the linear extension of this map that arises by allowing P to be a g-divisor (definition 1.26).

Each family depends on a choice of local coordinates and a g divisor. Let z denote any local coordinate centered at P_Q, and choose any positive, non-special g divisor D. Write

$$k = z^{-1}, \quad q(k) = xk + yk^2 + tk^3, \quad \text{and } D = D_1 + \cdots + D_g.$$

Let Ω be the normalized Abelian differential of the second kind, as in Definition 1.22, determined by the principal part of

$$dq = xdk + ydk^2 + tdk^3.$$

A Baker-Akhiezer function Ψ_Q can be defined explicitly:

$$\Psi_Q(P) = e^{\int_{P_\ell}^{P}\Omega}\,\frac{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)}, \tag{2.1}$$

where

$$(\boldsymbol{U}_{dq})_i = \oint_{\beta_i^\ell}\Omega, \quad \text{for each } i,$$

$\boldsymbol{K}_\ell$ is the Vector of Riemann Constants, (equation 1.15), and B is the Riemann matrix whose entries are the normalized β^ℓ-periods of the cohomology basis.

It is straightforward to see that $\Psi_Q(P)$ is in the vector space of Baker-Akhiezer functions associated with the parameters given. It can be shown that the definition of $\Psi_Q(P)$ is independent of the choice made when determining all paths; see [11]. By construction, $\Psi_Q(P)$ is meromorphic on $\Gamma_f \setminus \{P_Q\}$. In fact, the non-special nature of D allows us to determine the zeros of the denominator. By the Jacobi inversion Theorem 1.6, $\theta\left(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell\right)$ has exactly g zeros and they are the points D_i, for each i. Similarly, for almost every x, y and z, the vector $\mathcal{A}_\ell^D - \boldsymbol{U}_{dq}$ corresponds to a non-special divisor and there are exactly g zeros of the numerator $\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})$, none of which are points of D. It is also worth noting that, by construction, P_Q cannot be P_ℓ, but we have not ruled out the possibility that P_Q could be a point of D. Since Ω has vanishing α^ℓ-periods,

$$\Omega = dq + \boldsymbol{c}^T\mathcal{A}_\ell^P, \quad \text{with } \boldsymbol{c}[i] = -\oint_{\alpha_i^\ell} dq,$$

and, for P close to P_Q,

$$\int_{P_\ell}^{P} \Omega = q(k(P)) - q(P_\ell) + \boldsymbol{c}^T \mathcal{A}_\ell^P,$$

where $q(P_\ell)$ is the expression obtained by analytically continuing $q(k)$ along the path from P_ℓ to P, in reverse order. This shows that

$$h_Q(P) = \Psi_Q(P) e^{-q(k)} \to c \text{ as } P \to P_Q, \tag{2.2}$$

where

$$c = \begin{cases} \infty & \text{whenever } P_Q = D_i \text{ for some } i \in \{1, \dots, g\}, \\ e^{-q(P_\ell) + \boldsymbol{\lambda} \cdot \mathcal{A}_\ell^{P_Q}} \frac{\theta(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)} & \text{otherwise.} \end{cases}$$

Now, consider the formal differential operators

$$L = \partial_x^2 + u, \tag{2.3}$$

$$A = \partial_x^3 + \frac{3}{2} u \partial_x + w. \tag{2.4}$$

We will show that u is a solution of equation 1.3 by showing that

$$\begin{cases} [L - \partial_y] \Psi = 0, \\ [A - \partial_t] \Psi = 0, \end{cases}$$

where

$$\Psi = r \Psi_Q(P) \quad \text{for some } r \in \mathbb{C}_\infty.$$

With c defined in equation 2.2, we can disregard the case $c = \infty$, since in this case we see that u and w are identically zero and there is nothing to show. Since we have no interest in constant functions, suppose that $c \neq \infty$. Then, we expand Ψ_Q in local coordinates,

$$\Psi_Q = e^q \left(c + \frac{\bar{\xi}_1}{k} + \frac{\bar{\xi}_2}{k^2} + \dots \right). \tag{2.5}$$

The normalized Baker-Akhiezer function Ψ is defined by

$$\Psi = e^q \left(1 + \frac{\xi_1}{k} + \frac{\xi_2}{k^2} + \dots \right); \tag{2.6}$$

however, the asymptotic behavior near P_Q of any non-constant $\Psi_0 \in \Lambda(D)$ gives us

$$\Psi = \frac{\Psi_0}{c_0}, \text{ for some } c_0 \in \mathbb{C}^\times.$$

Next, formally, we express Ψ as

$$\Psi = e^q \sum_{s=0}^{\infty} \xi_s k^{-s},$$

and apply the differential operators to this expansion. Applying $L - \partial_y$ yields

$$\begin{aligned}
[L-\partial_y]\Psi &= \sum_{s=0}^{\infty} [L-\partial_y](e^q\xi_s)\,k^{-s} \\
&= \sum_{s=0}^{\infty} \left((\xi_s k^2 + 2\xi_{sx}k + \xi_{sxx}) - (\xi_s k^2 + \xi_{sy}) + u\xi_s\right) e^q k^{-s} \\
&= \sum_{s=0}^{\infty} \left(2\xi_{sx}k^{1-s} + (\xi_{sxx} + u\xi_s - \xi_{sy})k^{-s}\right) e^q \\
&= e^q \left(\mathrm{Eqn}_0 + \mathrm{Eqn}_1 k^{-1} + O(k^{-2})\right),
\end{aligned}$$

where

$$\begin{cases} \mathrm{Eqn}_0 = u\xi_0 + 2\xi_{1x}, \\ \mathrm{Eqn}_1 = \xi_{1xx} + u\xi_1 - \xi_{1y} + 2\xi_{2x}. \end{cases}$$

Similarly, applying $A - \partial_t$ we obtain

$$\begin{aligned}
[A-\partial_t]\Psi &= \sum_{s=0}^{\infty} [A-\partial_t](e^q\xi_s)\,k^{-s} \\
&= \sum_{s=0}^{\infty} \left((\xi_s k^3 + 3(\xi_{sx}k^2 + \xi_{sxx}k) + \xi_{sxxx}) + \frac{3}{2}(u\xi_s k + u\xi_{sx}) - (\xi_s k^3 + \xi_{st}) + w\xi_s\right) e^q k^{-s} \\
&= \sum_{s=0}^{\infty} \left(3\xi_{sx}k^{2-s} + 3(\xi_{sxx} + \frac{u}{2}\xi_s)k^{1-s} + (\xi_{sxxx} + \frac{3}{2}u\xi_{sx} - \xi_{st} + w\xi_s)k^{-s}\right) e^q \\
&= e^q \left(\mathrm{Eqn}_2 k + \mathrm{Eqn}_3 + \mathrm{Eqn}_4 k^{-1} + O(k^{-2})\right),
\end{aligned}$$

where

$$\begin{cases} \mathrm{Eqn}_2 = \frac{3}{2}u\xi_0 + 3\xi_{1x}, \\ \mathrm{Eqn}_3 = w\xi_0 + 3\xi_{1xx} + \frac{3}{2}u\xi_1 + 3\xi_{2x}, \\ \mathrm{Eqn}_4 = \xi_{1xxx} + \frac{3}{2}u\xi_{1x} - \xi_{1t} + w\xi_1 + 3\xi_{2xx} + \frac{3}{2}u\xi_2 + 3\xi_{3x}. \end{cases}$$

A key assumption in the construction process is that Ψ behaves well at the essential singularity P_Q. This is achieved by enforcing the vanishing of coefficients (see [11, Lemma 3.1.1]). In this case, we enforce

$$\text{Eqn}_i = 0, \quad \text{for } i \in \{0, 2, 3\},$$

so that we have the expressions

$$\begin{aligned} u &= -2\xi_{1x}, \\ w &= -3\xi_{1xx} - \frac{3}{2}u\xi_1 - 3\xi_{2x}. \end{aligned}$$

This also implies

$$[L - \partial_y]\,\Psi = O(k^{-1})e^q \quad \text{and} \quad [A - \partial_t]\,\Psi = O(k^{-1})e^q,$$

which are actually global relations on the surface. To see this, note that $[L - \partial_y]\,\Psi$ and $[A - \partial_t]\,\Psi$ are Baker-Akhiezer functions that vanish as P_Q is approached. Since D is non-special, they both must be the zero function. The fact that these relations are global yields more equations in u, w, and their derivatives. These equations, together with the enforcement of the compatibility condition for the operators in equations 2.3 and 2.4,

$$[L - \partial_y, A - \partial_t] = 0,$$

can be used to derive equation 1.3 in u. This establishes u as a finite-genus solution of 1.3, parameterized by the choice of z and D.

Now, consider the Baker-Akhiezer function

$$\widetilde{\Psi} = e^q \frac{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)},$$

with an expansion in k of the form

$$\widetilde{\Psi} = e^q \left(\widetilde{\xi}_0 + \frac{\widetilde{\xi}_1}{k} + \frac{\widetilde{\xi}_2}{k^2} + \cdots \right).$$

Since $\Psi - \widetilde{\Psi} \in \Lambda\,(D)$, its behavior near P_Q allows us to assert

$$\widetilde{\xi}_1 = \widetilde{c}x + ay + bt + \xi_1, \text{ for some } \widetilde{c}, a, b \in \mathbb{C}. \tag{2.7}$$

Expand $\mathcal{A}_\ell$ locally, in k and apply Theorem 1.4: for P near P_Q,

$$\begin{aligned}\mathcal{A}_\ell^P &= \mathcal{A}_\ell^{P_Q} + \boldsymbol{\omega}^\ell(P_Q)k^{-1} + O(k^{-2}) \\ &= \mathcal{A}_\ell^{P_Q} - \boldsymbol{U}k^{-1} + O(k^{-2}),\end{aligned}$$

where, for each i,

$$(\boldsymbol{U})_i = \oint_{\beta_i^\ell} dk + \cdots,$$

and the dots indicate regular terms. Next, note that

$$\log\left(\widetilde{\Psi}\right) = \log e^q + \log\frac{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)},$$

and so the k^{-1} order behavior of $\widetilde{\Psi}$ can be obtained by computing the k^{-1} order behavior of $\log\frac{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)}$. This means

$$\begin{aligned}\widetilde{\xi}_1 &= \frac{d}{dz}\log\frac{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})}{\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)}\bigg|_{P=P_Q} \\ &= -\frac{\sum_{i=1}^g \theta_i(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})\boldsymbol{U}_i}{\theta(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})} + \frac{\sum_{i=1}^g \theta_i(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)\boldsymbol{U}_i}{\theta(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)} \\ &= -\partial_x \log\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})\bigg|_{P=P_Q} + \frac{\sum_{i=1}^g \theta_i(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)\boldsymbol{U}_i}{\theta(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell)}.\end{aligned}$$

If we differentiate with respect to x,

$$\widetilde{\xi}_{1x} = -\partial_x^2 \log\theta(\mathcal{A}_\ell^P - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq})\bigg|_{P=P_Q}, \tag{2.8}$$

the equation

$$\widetilde{\xi}_{1x} = \xi_{1x} + \widetilde{c} \tag{2.9}$$

is obtained. We have seen that

$$u = -2\xi_{1x};$$

hence it follows that

$$u(x,y,z) = 2\partial_x^2 \log\theta\left(\mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell + \boldsymbol{U}_{dq}\right) + 2\widetilde{c}. \tag{2.10}$$

□

2.2 Krichever's construction method

In this section, I describe *Krichever's Construction Method* (KCM): a `Maple` procedure whose primary function is to encapsulate the proof given for Theorem 2.1, within the scope of producing finite-genus solutions to equation 1.3. First, I present my model for KCM in the form of a proposition (Proposition 2.2.1). Then, I discuss computational considerations such as the computation of parameters that describe a solution u and local coordinates. This is followed by a results section in which I provide an algorithm that implements KCM and data that demonstrates the capabilities of the approach taken in KCM.

2.2.1 Modeling & encapsulation

Suppose that we are given Γ, P_Q, z, and D as defined in the previous section. In order to produce finite-genus solutions that agree with Krichever's construction in Section 2.1, we need only construct the parameters in equation 2.10. Thus, it suffices for KCM to be a black-box program that outputs these parameters, given user input. Both the input and output can be reduced to mathematical objects that are more favorable to our computational framework.

The model used as a framework for KCM is best described in a proposition.

Proposition 2.2.1 (KCM Framework)**.** *Let f be an irreducible planar algebraic curve,*

$$P_Q = (u_Q, \lambda_Q), \text{ with } f(P_Q) = 0,$$

local coordinates $z = k^{-1}$ centered at P_Q, and

$$D = D_1 + \cdots + D_g, \text{ with } f(D_i) = 0, \ D_i \neq P_Q, \quad \text{for each } i.$$

Suppose that Γ_f has been constructed explicitly, as in Definition 1.34, and that the base point

$$P_\ell \neq P_Q \text{ with } f(P_\ell) = 0,$$

is chosen to be a regular point on f. Then u is a finite-genus solution to equation 1.3 whenever u is of the form

$$u(x,y,z) = 2\partial_x^2 \log\theta\left(x\boldsymbol{U} + y\boldsymbol{V} + z\boldsymbol{W} + \boldsymbol{\Phi}|B\right) + C, \tag{2.11}$$

where B is the $g\times g$ matrix of β^ℓ-periods of the normalized cohomology basis; $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W} \in \mathbb{C}^g$ are vectors of β^ℓ-periods of normalized Abelian differentials with principal parts dk, dk^2 and dk^3 respectively; and both $\boldsymbol{\Phi} \in \mathbb{C}^g$ and $C \in \mathbb{C}$ can be arbitrary. Moreover, if it is the case that

$$\boldsymbol{\Phi} \equiv \mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^{D} - \boldsymbol{K}_\ell, \tag{2.12}$$

where equivalence is on the Jacobian (Definition 1.27) and

$$C = 2\widetilde{c}, \text{ with } \widetilde{c} \text{ defined by equation 2.7,}$$

then u can be viewed as a finite-genus solution of equation 1.3 obtained directly from Krichever's construction.

Proof. Consider Krichever's construction with the input Γ_f, P_Q, z, and D. This provides us the finite-genus solution u derived and expressed in equation 2.10. The vector of β^ℓ-periods of dq can be simplified: for each i,

$$\begin{aligned}(\boldsymbol{U}_{dq})_i &= \oint_{\beta_i^\ell}\left(xdk + ydk^2 + tdk^3 - \sum_j\left(\oint_{\alpha_j^\ell} dq\right)\omega_j^\ell\right),\\ &= x\,(\boldsymbol{U}_{dk})_i + y\,(\boldsymbol{U}_{dk^2})_i + z\,(\boldsymbol{U}_{dk^3})_i\,,\end{aligned}$$

where

$$(\boldsymbol{U}_{dk^m})_i = \oint_{\beta_i^\ell}\left(dk^m - \sum_j\left(\oint_{\alpha_j^\ell} dk^m\right)\omega_j^\ell\right) \text{ for } m\in\{1,2,3\}. \tag{2.13}$$

The rest of the proposition is immediate from equation 2.10 and elementary calculus. □

2.2.2 Computing Wave vectors $\boldsymbol{U}$, $\boldsymbol{V}$, & $\boldsymbol{W}$

In this section, I provide an algorithm for computing the *wave vectors* $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ in the context of Krichever's Construction. The dependence of the vectors on the choice of local coordinate z is made explicit. The use of fixed internal coordinates is discussed.

In the context of finite-genus solutions of KP, the wave vectors are

$$\boldsymbol{U} = \boldsymbol{U}_{dk}, \quad \boldsymbol{V} = \boldsymbol{U}_{dk^2}, \quad \text{and } \boldsymbol{W} = \boldsymbol{U}_{dk^3},$$

with $\boldsymbol{U}_{dk^m}$ for $m \in \{1,2,3\}$ defined by equation 2.13 and as seen in Proposition 2.2.1. Thus, considering u in equation 2.11 as an ansatz requires an additional non-trivial assumption on the parameters that cannot be ignored. There are special cases: for example, when $g < 4$, there are ways to work with wave vectors that seemingly have little to do with the construction. In fact, for small genus these vectors are often used to understand and model shallow water wave phenomena (see [12], [18],[19]). Methods of computing solutions outside of the framework presented here do exist, such as the method I discuss in the next chapter.

Fortunately, the calculation of wave vectors in regards to equation 2.11 is simple and efficient. In what follows, fix i. Express the ith differential of the cohomology basis ω_i in local coordinates. Specifically, write

$$\omega_i = h_i(u,\lambda)\,du$$

so that

$$\omega_i^\ell = H_i(u,\lambda)\,du = \sum_j (\Pi_A)^{-1}_{ij} h_i(u,\lambda)\,du.$$

Then, fixing $m \in \{1,2,3\}$, since

$$dk^m = \frac{-m}{z^{m+1}}dz,$$

Proposition 1.6 yields

$$\boldsymbol{U}_{dk^m} = -m \oint_{\beta_i^\ell} \left(\frac{dz}{z^{m+1}} - \sum_j \left(\oint_{\alpha_j^\ell} \frac{dz}{z^{m+1}} \right) \omega_j^\ell \right) = \frac{-1}{m-1!} \frac{d^{m-1}}{dz^{m-1}} \left(H_i(P(z)) \frac{du}{dz} \right) \Bigg|_{z=0}. \tag{2.14}$$

Suppose that z is the local coordinate derived from the Puiseux series expansion at P_Q, or an approximation of this local coordinate. Such is the case when the command `puiseux` of `algcurves` is called. Recall that z is called the p-coordinates of P_Q as in Definition 1.12. Then the relationship between the u and z coordinates is given by

$$P(z) = (u(z), \lambda(z)) = \left(z^{r_Q} + u_Q, \sum_{m=N_Q} a_m z^m \right), \tag{2.15}$$

for some $r_Q, N_Q \in \mathbb{Z}$ with $r_Q \neq 0$. Furthermore, the change of coordinates comes with the associated changes in the local representation of the differentials and derivatives

$$\frac{d}{dz} = \frac{du}{dz}\frac{d}{du} = r_Q t^{r_Q - 1}\frac{d}{du} = r_Q \left(u - u_Q\right)^{\frac{r_Q - 1}{r_Q}}\frac{d}{du}, \tag{2.16}$$

$$du = \frac{du}{dz}dz = r_Q t^{r_Q - 1}dz, \tag{2.17}$$

where

$$\frac{d}{du} = \partial_u + \frac{d\lambda}{du}\partial_\lambda \quad \text{and} \quad \frac{d\lambda}{du} = -\frac{f_u}{f_\lambda}.$$

It is well known that differentials in the cohomology basis can be chosen to be of the form

$$\omega_i = \frac{p_i\left(u, \lambda(u)\right)}{f_\lambda\left(u, \lambda(u)\right)}du, \tag{2.18}$$

where each p_i is a polynomial. For example this is what is done using the command `differential` of `algcurves` [9].

Proposition 2.2.2. *Suppose the cohomology basis is locally expressed as in equation 2.18 and consider the p-coordinates of equation 2.15. Then*

$$\boldsymbol{U} = \Pi_A^{-1}\widetilde{\boldsymbol{U}}, \ \boldsymbol{V} = \Pi_A^{-1}\widetilde{\boldsymbol{V}}, \ \textit{and} \ \boldsymbol{W} = \Pi_A^{-1}\widetilde{\boldsymbol{W}},$$

with

$$\left(\widetilde{\boldsymbol{U}}\right)_i = -\left(\frac{du}{dz}h_i\right)(0) = -\left(\frac{du}{dz}\frac{p_i}{f_\lambda}\right)(0), \tag{2.19}$$

$$\left(\widetilde{\boldsymbol{V}}\right)_i = -\left(\frac{d^2u}{dz^2}h_i + \frac{du}{dz}^2\frac{dh_i}{du}\right)(0), \tag{2.20}$$

and

$$\left(\widetilde{\boldsymbol{W}}\right)_i = -\left(\frac{1}{2}\frac{d^3u}{dz^3}h_i + \frac{3}{2}\left(\frac{d^2u}{dz^2}\frac{du}{dz}\right)\frac{dh_i}{du} + \frac{1}{2}\left(\frac{du}{dz}\right)^3\frac{d^2h_i}{du^2}\right)(0). \tag{2.21}$$

If P_Q is regular, these formulas reduce to equations 2.25.

Furthermore, the only errors that occur are in expressing λ as a function of z, in constructing Π_A, and in inverting Π_A.

Proof. The non-normalized cohomology basis $\widetilde{\boldsymbol{\omega}}$ is normalized using matrix-vector multiplication:

$$\boldsymbol{\omega} = \Pi_A^{-1}\widetilde{\boldsymbol{\omega}}, \quad \text{where, for each } i,j,\ (\Pi_A)_{ij} = \oint_{\alpha_j^\ell} \omega_i.$$

Therefore, for a fixed i,

$$H_i(z) = \frac{du}{dz}h_i = \sum_j (\Pi_A)^{-1}_{ij} \frac{du}{dz} \frac{\widetilde{p}_j\left(u(z),\lambda(z)\right)}{f_\lambda\left(u(z),\lambda(z)\right)}.$$

Since equation 2.14 relates the wave vectors to h_i, we see that

$$\boldsymbol{U} = \Pi_A^{-1}\widetilde{\boldsymbol{U}},\ \boldsymbol{V} = \Pi_A^{-1}\widetilde{\boldsymbol{V}},\ \text{ and } \boldsymbol{W} = \Pi_A^{-1}\widetilde{\boldsymbol{W}},$$

for some $\widetilde{\boldsymbol{U}}$, $\widetilde{\boldsymbol{V}}$, and $\widetilde{\boldsymbol{W}}$. For fixed z near 0, let us define

$$\widetilde{\boldsymbol{U}}_z = \Pi_A \begin{bmatrix} -H_1(z) \\ \vdots \\ -H_g(z) \end{bmatrix},\ \widetilde{\boldsymbol{V}}_z = \Pi_A \begin{bmatrix} -\frac{d}{dz}(H_1)(z) \\ \vdots \\ -\frac{d}{dz}(H_g)(z) \end{bmatrix},\ \text{and } \widetilde{\boldsymbol{W}}_z = \Pi_A \begin{bmatrix} -\frac{d^2}{dz^2}\left(\frac{H_1}{2}\right)(z) \\ \vdots \\ -\frac{d^2}{dz^2}\left(\frac{H_g}{2}\right)(z) \end{bmatrix}.$$

Now fix i. We have

$$\frac{du}{dz}h_i = \frac{du}{dz}\frac{p_i}{f_\lambda}, \tag{2.22}$$

$$\frac{d}{dz}\left(\frac{du}{dz}h_i\right) = \frac{d^2u}{dz^2}h_i + \frac{du}{dz}^2\frac{d}{du}(h_i), \tag{2.23}$$

$$\frac{d^2}{dz^2}\left(\frac{du}{dz}h_i\right) = \frac{d^3u}{dz^3}h_i + 3\left(\frac{d^2u}{dz^2}\frac{du}{dz}\right)\frac{d}{du}(h_i) + \left(\frac{du}{dz}\right)^3\frac{d^2}{du^2}(h_i). \tag{2.24}$$

Since

$$h_i\frac{du}{dz}dz = \frac{p_i}{f_\lambda}du = \omega_i,$$

$h_i\frac{du}{dz}$ is holomorphic at 0. Moreover,

$$\Pi_A^{-1}\begin{bmatrix} -\frac{du}{dz}\frac{\widetilde{p}_1}{f_\lambda}(z) \\ \vdots \\ -\frac{du}{dz}\frac{\widetilde{p}_g}{f_\lambda}(z) \end{bmatrix} = \boldsymbol{U},\ \text{which implies } \widetilde{\boldsymbol{U}} = \widetilde{\boldsymbol{U}}_0 = \begin{bmatrix} -\left(\frac{du}{dz}\frac{\widetilde{p}_1}{f_\lambda}\right)(0) \\ \vdots \\ -\left(\frac{du}{dz}\frac{\widetilde{p}_g}{f_\lambda}\right)(0) \end{bmatrix}.$$

Equation 2.22 gives us

$$\widetilde{\boldsymbol{V}} = \begin{bmatrix} -\left(\frac{du}{dz^2}h_1 + \frac{du}{dz}^2\frac{d}{du}\left(h_1\right)\right)(0) \\ \vdots \\ -\left(\frac{du}{dz^2}h_g + \frac{du}{dz}^2\frac{d}{du}\left(h_g\right)\right)(0) \end{bmatrix}, \text{ and } \widetilde{\widetilde{\boldsymbol{W}}} = \begin{bmatrix} -\left(\frac{1}{2}\frac{d^3u}{dz^3}h_1 + \frac{3}{2}\left(\frac{d^2u}{dz^2}\frac{du}{dz}\right)\frac{dh_1}{du} + \frac{1}{2}\left(\frac{du}{dz}\right)^3\frac{d^2h_1}{du^2}\right)(0) \\ \vdots \\ -\left(\frac{1}{2}\frac{d^3u}{dz^3}h_g + \frac{3}{2}\left(\frac{d^2u}{dz^2}\frac{du}{dz}\right)\frac{dh_g}{du} + \frac{1}{2}\left(\frac{du}{dz}\right)^3\frac{d^2h_g}{du^2}\right)(0) \end{bmatrix}.$$

Now, $u(z)$ defined in equation 2.15 and $h_i(u,\lambda)$ do not include any errors as they are exact. The same is true of the z derivatives of u and the u derivative of h_i. If P_Q is regular, then $\dfrac{du}{dz} = 1$, so that

$$\widetilde{\boldsymbol{U}} = \begin{bmatrix} -\left(\frac{p_1}{f_\lambda}\right)(u_Q,\lambda_Q) \\ \vdots \\ -\left(\frac{p_g}{f_\lambda}\right)(u_Q,\lambda_Q) \end{bmatrix}, \widetilde{\boldsymbol{V}} = \begin{bmatrix} -\frac{dh_1}{du}(u_Q,\lambda_Q) \\ \vdots \\ -\frac{h_g}{du}(u_Q,\lambda_Q) \end{bmatrix}, \text{ and } \widetilde{\widetilde{\boldsymbol{W}}} = \begin{bmatrix} -\frac{1}{2}\frac{d^2h_1}{du^2}(u_Q,\lambda_Q) \\ \vdots \\ -\frac{1}{2}\frac{d^2h_g}{du^2}(u_Q,\lambda_Q) \end{bmatrix}. \quad (2.25)$$

The only source of numerical error will be in expressing λ as a function of z.

To compute $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ we just need to multiply by Π_A^{-1}, which may introduce numerical error. □

What follows is an algorithm that encapsulates the approach I have outlined in this section.

getUVW (2.1)
INPUT: $f(u,\lambda) = 0$, $P_Q = (u_Q,\lambda_Q)$ such that $f(u_Q,\lambda_Q) = 0$.
1: Call procedures to obtain Π_A, $\widetilde{\boldsymbol{\omega}}$, and z coordinates (eqns 2.15) for P_Q.
2: Store $\{h_i\}_{i=1}^g$, defined in equation 2.18, in a vector $\boldsymbol{H}_{u0}$, and symbolically compute and store the vectors of derivatives $\boldsymbol{H}_{u1}$ and $\boldsymbol{H}_{u2}$:

Continuation of getUVW (2.1)

for $m \in \{1, 2\}$,

$$(\boldsymbol{H}_{um})_i = \frac{d^{m-1}}{du^{m-1}} h_i = \partial_u \left(\boldsymbol{H}_{u(m-1)}\right)_i - \frac{f_u}{f_\lambda} \partial_\lambda \left(\boldsymbol{H}_{u(m-1)}\right)_i, \quad \text{for } i \text{ from 1 to } g.$$

3: Symbolically, compute the analogous vectors

$$\boldsymbol{H_{zm}}, \text{ for } m \in \{0, 1, 2\},$$

given by either equations 2.19-2.21 or equation 2.25. Do not evaluate.

4: Let F be a command that takes as input a rational function $\frac{p}{q}$ of z, and is defined by

$$F\left(\frac{p}{q}\right) = [\bar{p}(0), \bar{q}(0)], \text{ with } \frac{p}{q} = \frac{\bar{p}}{\bar{q}} \text{ and } \mathrm{GCD}(\bar{p}, \bar{q}) \in \mathbb{C}.$$

Then, for i from 1 to g, numerically compute

$$\widetilde{\boldsymbol{U}}_i = F\left((\boldsymbol{H}_{z0})_i\right), \quad \widetilde{\boldsymbol{V}}_i = F\left((\boldsymbol{H}_{z1})_i\right), \text{ and } \widetilde{\boldsymbol{W}}_i = F\left((\boldsymbol{H}_{z2})_i\right).$$

6: Set

$$\widetilde{\boldsymbol{U}}_i = -\frac{\widetilde{\boldsymbol{U}}_i[1]}{\widetilde{\boldsymbol{U}}_i[2]}, \quad \widetilde{\boldsymbol{V}}_i = -\frac{\widetilde{\boldsymbol{V}}_i[1]}{\widetilde{\boldsymbol{V}}_i[2]}, \quad \widetilde{\boldsymbol{W}}_i = -\frac{1}{2}\frac{\widetilde{\boldsymbol{W}}_i[1]}{\widetilde{\boldsymbol{W}}_i[2]}, \text{ for } i \text{ from 1 to } g.$$

7: Numerically compute Π_A^{-1} and

$$\boldsymbol{U} = \Pi_A^{-1}\widetilde{\boldsymbol{U}}, \quad \boldsymbol{V} = \Pi_A^{-1}\widetilde{\boldsymbol{V}}, \quad \boldsymbol{W} = \Pi_A^{-1}\widetilde{\boldsymbol{W}}, \text{ for } i \text{ from 1 to } g.$$

OUTPUT: The wave vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ of Krichever's construction associated local coordinates given by Puisuex expansions (equations 2.15).

2.2.3 Local coordinate considerations

Let f and P_Q be fixed. In this section, we consider how a local coordinate $\tilde{z}$ relates to the p-coordinates (equations 2.15) z.

Proposition 2.2.3. *Consider the proof of Theorem 2.1 with $\Gamma = \Gamma_f$, P_Q, any non-special g divisor D, and coordinates $\tilde{z}$. Write*

$$\tilde{z} = \widetilde{k}^{-1} \quad \text{and} \quad z = k^{-1},$$

and expand $\widetilde{k}^{-1}$ in z,

$$\widetilde{k}^{-1} = \lambda z^{-1} + \alpha + bt + \sum_{j=2}^{\infty} c_j z^j.$$

Then

$$\widetilde{\boldsymbol{U}} = \lambda \boldsymbol{U}, \tag{2.26}$$

$$\widetilde{\boldsymbol{V}} = \lambda^2 \boldsymbol{V} + 2\lambda\alpha \boldsymbol{U}, \tag{2.27}$$

$$\widetilde{\boldsymbol{W}} = \lambda^3 \boldsymbol{W} + 3\lambda^2\alpha \boldsymbol{V} + 3(\lambda\alpha^2 + \lambda^2 b)\boldsymbol{U}, \tag{2.28}$$

where $\widetilde{\boldsymbol{U}}$, $\widetilde{\boldsymbol{V}}$, $\widetilde{\boldsymbol{W}}$ are the wave vectors obtained from Krichever's construction using local coordinates $\tilde{z}$, and $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ are defined similarly but with respect to local coordinates given by Puisuex expansions (equations 2.15) and calculated in Proposition 2.2.2.

Proof. In what follows regular terms of the differential are denoted by $\cdots$. Recall that the wave vectors $\widetilde{\boldsymbol{U}}$, $\widetilde{\boldsymbol{V}}$, and $\widetilde{\boldsymbol{W}}$ are defined to be the β^ℓ periods of the differentials $d\widetilde{k}$, $d\widetilde{k}^2$, and $d\widetilde{k}^3$, respectively. Applying d to the expansion of $\widetilde{k}$ in k, we have

$$d\widetilde{k} = d\left(\lambda k + \alpha + bt + O\left(z^2\right)\right) = \lambda dk + \cdots.$$

This implies equation 2.26 holds. Similarly,

$$d\widetilde{k}^2 = d\left(\lambda k + \alpha + bt + O\left(z^2\right)\right)^2 = \lambda^2 dk^2 + 2\lambda\alpha dk + \cdots,$$

as well as

$$d\widetilde{k}^3 = d\left(\lambda^3 k^3 + 3\lambda^2\alpha k^2 + 3\left(\lambda\alpha^2 + \lambda^2 b\right)k + O\left(1\right)\right) = \lambda^3 dk^3 + 3\lambda^2\alpha dk^2 + 3\left(\lambda\alpha^2 + \lambda^2 b\right)dk + \cdots,$$

so that equations 2.27 and 2.28 hold, respectively. The claim follows. □

Proposition 2.2.2 makes computing wave vectors associated to p-coordinates explicit whereas Proposition 2.2.3 demonstrates that all other wave vectors can be obtained from these ones. Thus, using p- coordinates internally is justified. Another benefit is that other quantities can be made explicit. For example, for P near P_Q,

$$dq = -x\frac{dz}{z^2} - 2y\frac{dz}{z^3} - 3t\frac{dz}{z^4} = \left(\frac{-x}{(u-u_Q)^{\frac{1+r_Q}{r_Q}}} + \frac{-2y}{(u-u_Q)^{\frac{2+r_Q}{r_Q}}} + \frac{-3t}{(u-u_Q)^{\frac{3+r_Q}{r_Q}}}\right)\frac{du}{r_Q}.$$

The importance of this calculation is the following: given a numerical analytic continuation scheme we can compute q at any point on the surface and, therefore, numerically integrate dq. If the numerical analytic continuation is sufficiently accurate and efficient, the asymptotic behavior of Baker-Akhiezer functions in equation 2.2 can be used to compute quantities, such as the value of C in equation 2.11, with less dependence on the evaluation of theta functions. This could result in a very efficient method for calculating C.

2.2.4 The phase shift $\mathbf{\Phi}$ and the Vector of Riemann Constants $\boldsymbol{K}_\ell$

Accurate calculation of the quantity $\mathbf{\Phi}$, defined in Proposition 2.2.1 and expressed here as

$$\mathbf{\Phi} = \mathcal{A}_\ell^{P_Q} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell,$$

depends on both the accuracy of the Abel map $\mathcal{A}_\ell$ and on the accuracy of the Vector of Riemann Constants $\boldsymbol{K}_\ell$ in equation 1.15. However, computing $\boldsymbol{K}_\ell$ requires analysis. For example, applying Theorem 1.6 to the components of $\boldsymbol{K}_\ell$,

$$(\boldsymbol{K}_\ell)_i = \frac{1+B_{ii}}{2} - \sum_{j\neq i}\oint_{\alpha_j^\ell}\omega_j^\ell \int_{P_\ell}^{P}\omega_i^\ell, \quad P\in\alpha_j^\ell,$$

we see that $(\boldsymbol{K}_\ell)_i$ can be expressed as

$$(\boldsymbol{K}_\ell)_i = \frac{1+B_{ii}}{2} - \sum_{j\neq i}\oint_{\alpha_j^\ell}\omega_j^\ell \oint_{\beta_i^\ell}\omega_{PP_\ell}, \quad P\in\alpha_j^\ell,$$

where $\omega_{\omega_{PP_\ell}}$ is the normalized differential of the third kind characterized by the simple poles P_ℓ (with residue -1) and P (with residue 1) (as in Definition 1.22). For a fixed P, the expression $\oint_{\beta_i} \omega_{PP_\ell}$ is ambiguous without an explicit representation of a homology basis (see [30, Proplem 7.5]). Thus, not only must I compute $\boldsymbol{K}_\ell$ accurately, I must ensure that it is computed without ambiguity.

Since the framework includes a realized Riemann surface Γ_f, this is less of a concern.

Proposition 2.2.4. *Let $P_0 \in \Gamma_f$. Then the vector of Riemann constants with base point P_0 is*

$$\boldsymbol{K}_{P_0} \equiv (g-1)\mathcal{A}_\ell^{P_0} + \boldsymbol{K}_\ell,$$

where $\equiv$ is with respect to the Jacobian 1.27. Thus, the phase shift, defined in equation 2.11, is invariant with respect to the base point:

$$\boldsymbol{\Phi} \equiv \mathcal{A}(P_0, P_Q) - \mathcal{A}(P_0, D) + \boldsymbol{K}_{P_0}.$$

Proof. This relies on the following calculation:

$$\begin{aligned}
(\boldsymbol{K}_{P_0})_i &= \frac{1+B_{ii}}{2} - \sum_{j\neq i} \oint_{\alpha_j^\ell} \omega_j^\ell \int_{P_0}^{P} \omega_i^\ell, \\
&= \frac{1+B_{ii}}{2} - \sum_{j\neq i} \oint_{\alpha_j^\ell} \omega_j^\ell \left(\int_{P_0}^{P_\ell} \omega_i^\ell + \int_{P_\ell}^{P} \omega_i^\ell \right), \\
&= \left(\frac{1+B_{ii}}{2} - \sum_{j\neq i} \oint_{\alpha_j^\ell} \omega_j^\ell \int_{P_\ell}^{P} \omega_i^\ell \right) + (g-1) \int_{P_\ell}^{P_0} \omega_i^\ell.
\end{aligned}$$

Now that we have the relation involving $\boldsymbol{K}_\ell$,

$$\begin{aligned}
\boldsymbol{\Phi} &\equiv \left(\mathcal{A}(P_0, P_Q) - \mathcal{A}_\ell^{P_0}\right) - \left(\mathcal{A}(P_0, D) - g\mathcal{A}_\ell^{P_0}\right) + \left(\boldsymbol{K}_{P_0} - (g-1)\mathcal{A}_\ell^{P_0}\right) \\
&= \mathcal{A}(P_0, P_Q) - \mathcal{A}(P_0, D) + \boldsymbol{K}_{P_0}.
\end{aligned}$$

□

For $g = 1$, computing $\boldsymbol{K}_\ell$ is straightforward:

$$\boldsymbol{K}_\ell = \left(\frac{1+B_{11}}{2} \right).$$

Therefore, in what follows I make the assumption that $g > 1$. In this case, I make use of the well known fact that, for any divisor, $\mathcal{C}$, in the canonical class (see Definition 1.28),

$$2\boldsymbol{K}_\ell \equiv -\mathcal{A}_\ell^{\mathcal{C}}.$$

To use this relation I need to fix $\mathcal{C}$. This will lead to a relation on $\mathbb{C}^g$:

$$2\boldsymbol{K}_\ell = -\mathcal{A}_\ell^{\mathcal{C}} + B\boldsymbol{M}_\ell + \boldsymbol{N}_\ell, \quad \boldsymbol{M}_\ell, \boldsymbol{N}_\ell \in \mathbb{Z}^g;$$

and so $\boldsymbol{K}_\ell$ and $\mathcal{A}_\ell^{\mathcal{C}}$ are related by a theta characteristic $\boldsymbol{h}_\ell$:

$$\boldsymbol{K}_\ell = -\frac{\mathcal{A}_\ell^{\mathcal{C}}}{2} + B\frac{\boldsymbol{M}_\ell}{2} + \frac{\boldsymbol{N}_\ell}{2} = -\frac{\mathcal{A}_\ell^{\mathcal{C}}}{2} + \boldsymbol{h}_\ell. \tag{2.29}$$

Since our primary interest in computing $\boldsymbol{K}_\ell$ is the roll it plays in calculating the zeros of the theta function (as in Theorem 1.6), we can restrict $\boldsymbol{h}_\ell$ to restrict to half-periods; see Section 1.3.5. There are 4^g possible choices for $\boldsymbol{h}_\ell$ and I do not know if this number can be reduced.

Any choice of Abelian differential will work to obtain $\mathcal{C}$. I believe that there are better choices than others. In the KCM framework, the zeros and poles of a member of the non-normalized cohomology basis $\boldsymbol{\omega}$ is chosen. Specifically

$$\omega_j = h_j du = \frac{p_j}{f_\lambda} du \tag{2.30}$$

is chosen among the components of $\boldsymbol{\omega}$ with minimal $\deg(p_j, \lambda)$ in λ and among this sub collection the one with minimal $\deg(p_j, u)$ is chosen (lexicographic order with respect to λ, u).

The following proposition is the basis for the method I use to obtain the divisor of ω_j.

Proposition 2.2.5. *Let*

$$\omega = h\, du = \frac{p(u,\lambda)}{f_\lambda(u,\lambda)} du$$

be a holomorphic differential that is given and also suppose that P_0 and the associated p-coordinates are given:

$$P(z) = (u(z),\ \lambda(z)) = \left(z^{r_0} + u_0,\ \sum_{m=N_0} a_m z^m \right),\ \textit{with } P_0 = (u_0, \lambda_0) = P(0).$$

If P_0 is a point of the divisor (ω), as in Definition 1.28, then it can only be a zero. Moreover either (a) P_0 is a zero of p, (b) u_0 is infinite, or (c) there is a branch point or singularity at u_0.

Proof. Suppose that P_0 is a point of the divisor $h\,du$. Since the differential is holomorphic, the local representation $h\,\dfrac{du}{dz}$ is holomorphic and therefore vanishes at 0. Now, if $r_0 = 1$, then the holomorphic local expansion takes on the form

$$h\frac{du}{dz} = \frac{p\left(z+u_0, \lambda\left(z+u_0\right)\right)}{f_\lambda\left(z+u_0\right)},$$

so that, for a holomorphic function g and $s \in \mathbb{Z}$,

$$h(z) = z^s g(z), \text{ with } g(0) \neq 0, \ s > 0.$$

This can only occur if p vanishes at u_0. The exponent r_0 is negative if and only if u_0 is infinite. Finally, if $r_0 > 1$, then u_0 cannot be infinite. This implies that u_0 corresponds to a branch point. □

Below is an algorithm that obtains the divisor of ω_j.

`getDivisor (2.2)`
`INPUT:` $f\left(u,\lambda\right) = 0$, $\quad h_j = \frac{p}{f_\lambda}$ as in equation 2.30.

Continuation of getDivisor (2.2)

1: Call a procedure to obtain the problem points S_0 (Section 1.3.6) of f:

$$S_0 = \{u_0 : u_0 = \infty,\ f_y(u_0) = 0\}.$$

2: Let $\mathrm{R}(p, f)$ denote the resolvent of p and f with respect to λ (Definition 1.6) and let ''solve'' be a command that computes the zeros of a polynomial of one variable. Compute and store the zeros Z_0 of p:

$$Z_0 = \begin{cases} \emptyset, & \text{if } p \in \mathbb{C}, \\ \mathtt{solve}\left(p(u) = 0\right), & \text{if } \operatorname{degree}(p, y) = 0, \\ \mathtt{solve}\left(\mathrm{R}\left(p, f\right) = 0\right), & \text{if } \operatorname{degree}(p, y) > 0. \end{cases}$$

3: Initialize

$$T_{\deg} = 0 \text{ and } \mathcal{C} = [\,].$$

For every $u_0 \in S_0 \bigcup Z_0$,

3.i: Obtain p-coordinates for each point rooted at u_0: X_{u_0}.

3.ii: For each pair of z-coordinates, $P_{u_0} \in X_{u_0}$,

$$P_{u_0}(z) = \left(u = z^{r_0} + u_0, \lambda = \sum_{m=N_0} a_m t^m\right),$$

express p and f_λ in these coordinates and expand.

3.iii: Let $\mathtt{ldeg}(g, z)$ be as in Notation 1.4.

Continuation of getDivisor (2.2)

Then compute

$$crt_{\deg} = \mathrm{ldeg}(p, z) - \mathrm{ldeg}(f_\lambda, z) + r_0 - 1.$$

3.iv: If $crt_{\deg} < 0$, terminate with an error. If $crt_{\deg} > 0$, update $\mathcal{C}$:

Add $[P_{u_0}, crt_{\deg}]$ to $\mathcal{C}$ and set $T_{\deg} = T_{\deg} + crt_{\deg}$.

4: If

$$T_{\deg} \neq 2g - 2,$$

then terminate with an error.

OUTPUT:

An explicit representation of a member of the canonical class

$$\mathcal{C} = [\,[P_{u_1}, crt_{\deg_1}], \ldots, [P_{u_n}, crt_{\deg_n}]\,].$$

Now that we have $\mathcal{C}$, we have reduced the problem of computing $\boldsymbol{K}_\ell$ to the sub-problem of determining $\boldsymbol{h}_\ell$. For a half-period $\boldsymbol{h}$ and any non-special g divisor D, define the vector

$$\boldsymbol{K}_{\boldsymbol{h}} = \boldsymbol{h} - \frac{\mathcal{A}_\ell^{\mathcal{C}}}{2},$$

as well as the function

$$F_{\boldsymbol{h}}^{D}(P) = \theta\left(\mathcal{A}_\ell^{P} - \mathcal{A}_\ell^{D} - \boldsymbol{K}_{\boldsymbol{h}}\right).$$

Proposition 2.2.6. *Suppose P_ℓ is not a Weierstrass point, and further suppose*

$$D = D_1 + \cdots + D_g$$

is a non-special g divisor with the property that

$$D_i \neq D_j, \text{ whenever } i \neq j, \text{ and } D_i \neq P_\ell \text{ for each } i.$$

If the half period $\boldsymbol{h}$ has the properties that

1. $F^D_{\boldsymbol{h}}(P) \neq 0$ *for some* $P \in \Gamma$, *and*

2. $\boldsymbol{h}$ *minimizes the map*

$$\boldsymbol{h}_0 \mapsto \sum_{i=1}^{g} |F^D_{\boldsymbol{h}_0}(D_i)| + |\theta(\boldsymbol{K}_{\boldsymbol{h}_0})|,$$

then

$$\boldsymbol{h} = \boldsymbol{h}_\ell,$$

and $\boldsymbol{K}_{\boldsymbol{h}}$ *is the Vector of Riemann Constants.*

Proof. First note that such a D exists. For example, choose any g distinct points in $\Gamma_f \setminus \{P_\ell\}$ and form the g divisor $\widetilde{D}$. If $\widetilde{D}$ is special, there are non-special g divisors in every one of its neighborhoods. By choosing a non-special divisor sufficiently close to $\widetilde{D}$, we can ensure that it exhibits the desired properties.

In what follows, I make use of Theorem 1.6 multiple times. Since P_ℓ is not a Weierstrass point gP_ℓ is not special. This means

$$F^{gP_\ell}_{\boldsymbol{h}_\ell} = \theta(\mathcal{A}^P_\ell - \boldsymbol{K}_\ell)$$

is not identically zero. Suppose $\boldsymbol{h}$ satisfies the claim in the hypothesis. Then $\boldsymbol{h}$ minimizes

$$\boldsymbol{h}_0 \mapsto \sum_{i=1}^{g} |F^D_{\boldsymbol{h}_0}(D_i)| + |\theta(\boldsymbol{K}_{\boldsymbol{h}_0})|. \tag{2.31}$$

Then since, $\boldsymbol{K}_\ell$ satisfies

$$\sum_{i=1}^{g} |F^D_{\boldsymbol{h}_\ell}(D_i)| + |\theta(\boldsymbol{K}_\ell)| = 0,$$

it must be the case that

$$F^D_{\boldsymbol{h}}(D_1) = \cdots = F^D_{\boldsymbol{h}}(D_g) = 0 \text{ and } \theta(\boldsymbol{K}_{\boldsymbol{h}}) = 0.$$

Now, since

$$\boldsymbol{K}_{\boldsymbol{h}} = (\boldsymbol{h} - \boldsymbol{h}_\ell) + \boldsymbol{K}_\ell$$

is a zero of the theta function, $\boldsymbol{K}_{\boldsymbol{h}}$ can be expressed as

$$\boldsymbol{K}_{\boldsymbol{h}} \equiv \mathcal{A}_\ell^{D^0} + \boldsymbol{K}_\ell$$

for some non-special g divisor D^0, and thus

$$(\boldsymbol{h} - \boldsymbol{h}_\ell) \equiv \mathcal{A}_\ell^{D^0}.$$

By hypothesis $F_{\boldsymbol{h}}^D$ is not the zero map and has exactly g zeros. Express $F_{\boldsymbol{h}}^D$ in terms of $\boldsymbol{K}_\ell$ and observe that, for each i, both

$$F_{\boldsymbol{h}}^D(D_i) = \theta\left(\mathcal{A}_\ell^{D_i} - (\boldsymbol{h} - \boldsymbol{h}_\ell) - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell\right) \quad \text{and} \quad F_{\boldsymbol{h}_\ell}^D(D_i) = \theta\left(\mathcal{A}_\ell^{D_i} - \mathcal{A}_\ell^D - \boldsymbol{K}_\ell\right)$$

vanish. Since, $F_{\boldsymbol{h}}^D$ is not identically zero, the vanishing of $F_{\boldsymbol{h}}^D$ is equivalent to the Jacobi inversion problem with respect to $(\boldsymbol{h} - \boldsymbol{h}_\ell) + \mathcal{A}_\ell^D$: there is a non-special g divisor, $\widetilde{D}$, such that

$$\mathcal{A}_\ell^{\widetilde{D}} \equiv (\boldsymbol{h} - \boldsymbol{h}_\ell) + \mathcal{A}_\ell^D \equiv \mathcal{A}_\ell^{D^0} + \mathcal{A}_\ell^D. \tag{2.32}$$

Thus $(\boldsymbol{h} - \boldsymbol{h}_\ell) + \mathcal{A}_\ell^D$ and $\widetilde{D}$ are determined uniquely by the zeros of $F_{\boldsymbol{h}}^D$, which are precisely the points that make up D. It follows that

$$\widetilde{D} = D \quad \text{and} \quad \mathbf{0} \equiv (\boldsymbol{h} - \boldsymbol{h}_\ell) \equiv \mathcal{A}_\ell^{D^0}.$$

The non-speciality of gP_ℓ then ensures $D^0 = gP_\ell$ and thus the claim is valid. □

The algorithm below encapsulates this idea.

getVRC (2.3)
INPUT: $f(u,\lambda) = 0$, $\mathcal{C}$ (obtained from Algorithm getDivisor (2.2)), D, and P_ℓ (that satisfy the hypotheses of Proposition 2.2.6).

Continuation of getVRC (2.3)

1: Compute the Abel map, with respect to P_ℓ, of

$$\mathcal{A}_\ell^{\mathcal{C}} = \mathcal{A}_\ell^{\mathcal{C}_1} + \cdots + \mathcal{A}_\ell^{\mathcal{C}_j} + \cdots + \mathcal{A}_\ell^{\mathcal{C}_{2g-2}},$$

$$\mathcal{A}_\ell^{\hat{D}_i} = \sum_{j \neq i} \mathcal{A}_\ell^{D_j}, \text{ for each } i \text{ from } 1 \text{ to } g.$$

2: Populate a list H containing all 4^g half periods; store the vector

$$\boldsymbol{AC} = \frac{\mathcal{A}_\ell^{\mathcal{C}}}{2}.$$

3: For each $\boldsymbol{h} \in H$, replace $\boldsymbol{h}$ with

$$H_{\boldsymbol{h}} = \left[h, \theta\left(\boldsymbol{h} - \boldsymbol{AC}\right), \theta\left(-\mathcal{A}_\ell^{\hat{D}_1} - \left(\boldsymbol{h} - \boldsymbol{AC}\right)\right), \ldots, \theta\left(-\mathcal{A}_\ell^{\hat{D}_g} - \left(\boldsymbol{h} - \boldsymbol{AC}\right)\right)\right].$$

4: Select among $H_{\boldsymbol{h}}$ the list that minimizes

$$|H_{\boldsymbol{h}}[2]| + |H_{\boldsymbol{h}}[3]| + \cdots + |H_{\boldsymbol{h}}[g+2]|.$$

OUTPUT:

An explicit representation of $\boldsymbol{K}_\ell$ as in equation 1.15,

$$\boldsymbol{K}_\ell = \boldsymbol{h} - \boldsymbol{AC}.$$

It should be noted that the terms in the sum 2.31 can be replaced. For example, the term $|\theta(\boldsymbol{K}_{\boldsymbol{h}_0})|$ can be replaced with $K|\theta(\boldsymbol{K}_{\boldsymbol{h}_0})|$, for any $K > 0$.

2.2.5 The constant C

The constant C of Proposition 2.2.1 can be computed in a variety of ways. For example, at the end of Section 2.2.3, we discussed a method that may avoid the use of theta functions.

The method for computing C that I utilize involves the use of Hirota's Bilinear Form:

Theorem 2.2. *Let u be given by equation 2.11 and let $\theta(\boldsymbol{z})$, with*

$$\boldsymbol{z} = x\boldsymbol{U} + y\boldsymbol{V} + t\boldsymbol{W} + \boldsymbol{\Phi},$$

be the associated theta function. Then, there is some $d \in \mathbb{C}$, such that for all $x, y, t \in \mathbb{R}$,

$$\theta_{4x}\theta - 4\theta_{3x}\theta_x + 3\theta_{xx}^2 - 4\theta_{xt}\theta + 4\theta_x\theta_t + 3\theta_{yy}\theta - 3\theta_y^2 + 8d\theta^2 + 6C\theta_{xx}\theta - 6C\theta_x^2 = 0. \quad (2.33)$$

if and only if u solves equation 1.3.

Proof. See [11]. □

Assuming u is a solution, evaluating the left hand side of equation 2.33 and enforcing vanishing will give approximations to both C and d. A better approach would be to choose a phase shift $\boldsymbol{\Phi}$ so that

$$\theta(\boldsymbol{\Phi}) = 0, \quad \sum_{i=1}^{g} \boldsymbol{U}_i \theta_i(\boldsymbol{\Phi}) \neq 0. \quad (2.34)$$

Then,

$$C = -\frac{2}{3}\frac{\theta_{3x}}{\theta_x} + \frac{1}{2}\left(\frac{\theta_{xx}}{\theta_x}\right)^2 + \frac{2}{3}\frac{\theta_t}{\theta_x} - \frac{1}{2}\left(\frac{\theta_y}{\theta_x}\right)^2, \text{ with } \boldsymbol{z} = \boldsymbol{\Phi},$$

and C can be calculated explicitly. In practice, candidates for such a $\boldsymbol{\Phi}$ are numerous. For example, $\boldsymbol{K}_\ell$ is a good candidate as well as any odd half period that satisfies equation 2.34. If the search takes too long, we can employ methods from Chapter 3 to obtain an accurate d, so that C can be computed.

2.3 Results

In the previous sections, we reduced various portions of Krichever's construction to modular components. This compartmentalization can be brought together to produce an encapsulation of the construction.

KCM (2.4)

INPUT: $f(u,\lambda)=0$, $[u_Q,\lambda_Q]$, $[D_1,\ \ldots,\ D_g]$ such that

$$P_Q=(u_Q,\lambda_Q) \quad \text{and} \quad D=D_1+\cdots+D_g,$$

as specified in Proposition 2.2.1, and $\lambda,\alpha,b\in\mathbb{C}$ with $\lambda\neq 0$, as specified in Proposition 2.2.3.

1: Construct Γ_f explicitly as 1.34. This includes procedures to obtain Π_A, Π_B, B, $\widetilde{\boldsymbol{\omega}}$, and p-coordinates for P_ℓ, P_Q, and D_i for i from 1 to g.

2: Using ω_j, as defined in the discussion of equation 2.30, call Algorithm getDivisor (2.2) to obtain an explicit representative of the canonical class $\mathcal{C}$.

3: Find a g-divisor $\widetilde{D}$ that satisfies the condition in Proposition 2.2.6 and compute $\mathcal{A}^{\mathcal{C}}_{\ell}$; then call Algorithm getVRC (2.3) to obtain $\boldsymbol{K_h}$.

4: Using the computed data, call Algorithm getUVW (2.1) to obtain the wave vectors corresponding to $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ with respect to p-coordinates at P_Q.

5: Use λ, α and b in the formulas of Proposition 2.2.3 to transform the wave vectors $\boldsymbol{U}$, $\boldsymbol{V}$ and $\boldsymbol{W}$ into the desired wave vectors $\widetilde{\boldsymbol{U}}$, $\widetilde{\boldsymbol{V}}$ and $\widetilde{\boldsymbol{W}}$.

6: Compute C using the ideas in Section 2.2.5, and compute $\boldsymbol{\Phi}$ according to the formula in equation 2.12.

OUTPUT: The parameters $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{\Phi}$, and C that parameterize a solution of equation 1.3 derived from Krichever's construction in Section 2.1.

2.3.1 Genus 3 example

Consider the surface Γ_f defined by

$$f(u,\lambda)=144(u^4+\lambda^4)-225(u^2+\lambda^2)+350u^2\lambda^2+810=0.$$

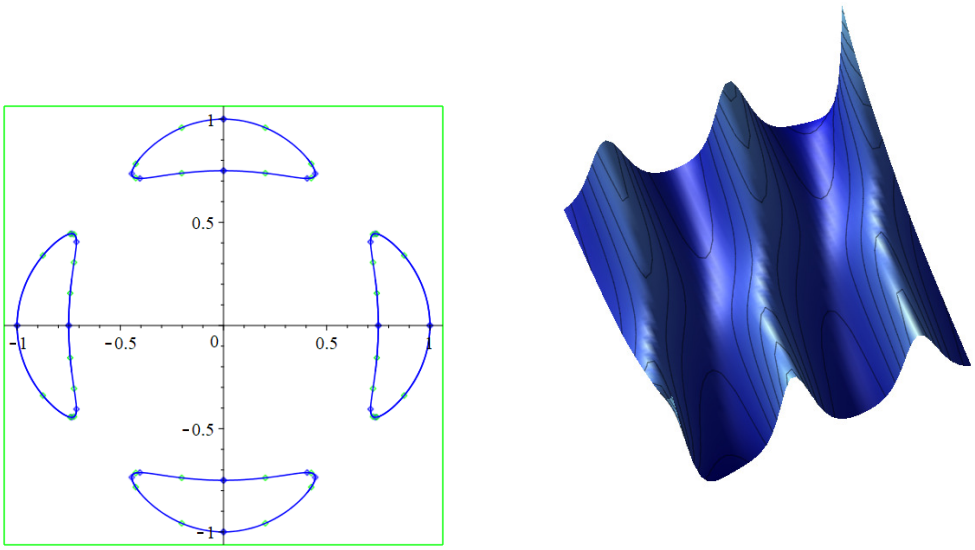

Figure 2.1: *Left:* The intersection of $f(u,\lambda)=0$ with the real plane. *Right:* The constructed solution at $t=0,\ -1\le x\le 1$, and $0\le y\le 2$.

For input we use

$$P_Q=(0,1)\,,\ \ (\lambda,\alpha,b)=(1,0,0)\,,\text{ and } D=[D_1,\ D_2,\ D_3],$$

where

$$D_1=\left(-\frac{1}{2},\frac{\sqrt{-\sqrt{2135}i+275}}{24}\right),\ D_2=\left(\frac{1}{2},\frac{\sqrt{-\sqrt{2135}i+275}}{24}\right),$$

and

$$D_3=\left(\frac{i}{2},\frac{\sqrt{-\sqrt{53665}+625}}{24}\right).$$

I obtain the following parameters as output. The Riemann matrix

$$B=\begin{bmatrix}1+1.80507i & -1-0.90254i & 1+0.90254i\\ -1-0.90254i & 0.19129+1.22883i & -0.80871-1.016883i\\ 1+0.90254i & -0.80871-1.016883i & 0.19129+1.22883i\end{bmatrix}$$

and the base point

$$P_\ell=(-1.1,-1.12762i)$$

are obtained from the `monodromy` and `periodmatrix` commands from the `algcurves` package. Using the `differential` and `puiseux` commands within Algorithm 2.1, the wave vectors are approximately

$$\boldsymbol{U} = \begin{bmatrix} 0.57344 \\ 0.12154 - 0.55728i \\ 0.12154 + 0.69194i \end{bmatrix}, \boldsymbol{V} = \begin{bmatrix} 0.57344 \\ -0.12154 - 0.35405i \\ -0.12154 + 0.21939i \end{bmatrix},$$

and

$$\boldsymbol{W} = \begin{bmatrix} 2.01549i \\ 0.42718 - 2.29389i \\ 0.42718 + 2.76720i \end{bmatrix}.$$

The phase

$$\boldsymbol{\Phi} = \mathcal{A}_\ell^Q - \mathcal{A}_\ell^D - \boldsymbol{K_h}$$

is

$$\boldsymbol{\Phi} = \begin{bmatrix} -0.49069 - 1.68604i \\ 0.98160 + 0.42829i \\ -0.83214 + 0.08543i \end{bmatrix},$$

with

$$\mathcal{A}_\ell^Q = \begin{bmatrix} 0.17580 - 0.19777i \\ 0.21225 + 0.24358i \\ -0.36568 + 0.04581i \end{bmatrix}, \quad \mathcal{A}_\ell^D = \begin{bmatrix} 0.51808 + 1.30296i \\ -0.27197 + 0.29650i \\ -0.19202 - 0.41530i \end{bmatrix},$$

and

$$\boldsymbol{K_h} = \begin{bmatrix} 0.14841 + 0.18531i \\ -0.49737 - 0.48121i \\ 0.65849 + 0.37569i \end{bmatrix}.$$

Finally,

$$C = -1.52741 - 0.84712i,$$

and the constant d that appears in equation 2.33 is

$$d = 4.00856 + 4.43536i.$$

The absolute error in equation 2.33 at $(x, y, t) = (0, 0, 0)$ is on the order of 10^{-45} (if all the digits obtained in the calculation are used) and this scales with the global precision. As the wave vectors are computed via formulas, the error here is important in measuring the error in C and d.

As for the error in $\boldsymbol{K}_{\boldsymbol{h}}$, I computed each term of the sum

$$|F^D_{\boldsymbol{h}}(D_1)| + |F^D_{\boldsymbol{h}}(D_2)| + |F^D_{\boldsymbol{h}}(D_3)| + 3|\theta(\boldsymbol{K}_{\boldsymbol{h}})|,$$

where $F^D_{\boldsymbol{h}}$ was defined in Section 2.2.4. Each term is on the order of 10^{-24}, and the sum was also of this order. Moreover, for any other choice of half period, the sum was significantly larger: on the order of 10^{-1}. This suggests that this is indeed the correct choice. Recall the order on half-periods, as defined in Notation 1.32 in Section 1.3.5. The $\boldsymbol{h}$ obtained was

$$b_{\boldsymbol{h}} = 100\ 000,$$

when represented in binary. This corresponds to

$$\boldsymbol{h} = B\frac{\boldsymbol{e_1}}{2}, \text{ with } \boldsymbol{e}_{\boldsymbol{1}}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Moreover, this half-period was obtained in each test, as the number of digits were increased.

2.3.2 Genus 4 example

Consider the surface Γ_f defined by

$$f(u, \lambda) = (u^2 - 1)(u^2 - 4)(u^2 - 16)(u^2 - 25)(u^2 - 36) + \lambda^2 = 0.$$

With

$$P_Q = \left(-11, \sqrt{12029472000}\right), \ (\lambda, \alpha, b) = (1, 0, 0),$$

and

$$D = [D_1, D_2, D_3, D_4],$$

with

$$D_1 = \left(-12, \sqrt{32934021120}\right), \ D_2 = \left(-13, \sqrt{81226696320}\right),$$

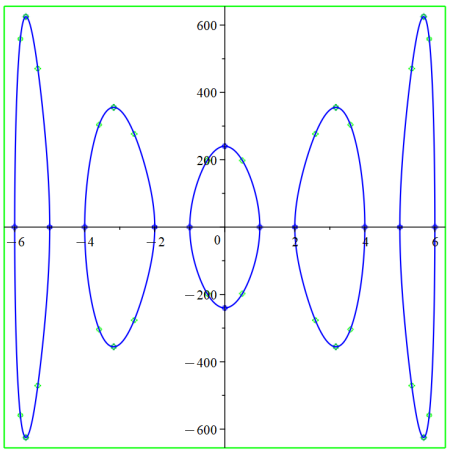

Figure 2.2: The intersection of $f(u, \lambda) = 0$ with the real plane.

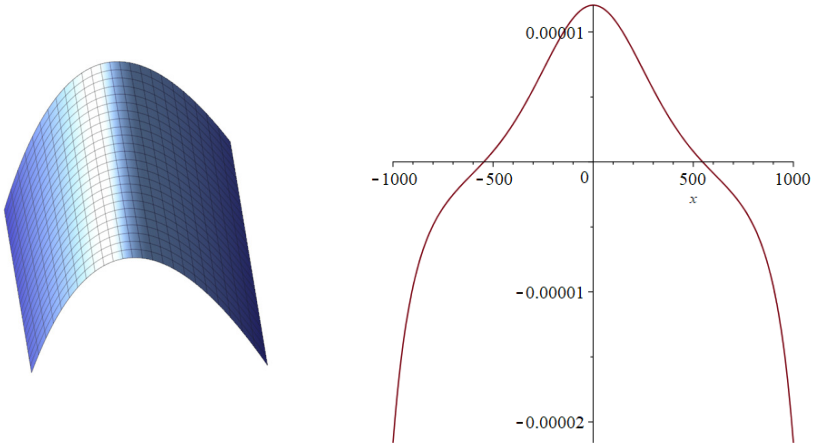

Figure 2.3: (*Left*): The constructed solution at $t = 0$ and $-10 \leq x, y \leq 10$. (*Right*):A slice of the constructed solution with $y = 0$ and $-1000 \leq x \leq 1000$.

$$D_3 = \left(-12, -\sqrt{32934021120}\right), \; D_4 = \left(-13, -\sqrt{81226696320}\right).$$

Of concern is the accuracy of the Riemann Constant Vector $\boldsymbol{K}_{\boldsymbol{h}}$. Let T_{di} be

$$T_{\text{di}} = |F^D_{\boldsymbol{h}}(D_1)| + |F^D_{\boldsymbol{h}}(D_2)| + |F^D_{\boldsymbol{h}}(D_3)| + |F^D_{\boldsymbol{h}}(D_4)| + 4|\theta(\boldsymbol{K}_{\boldsymbol{h}})|,$$

at di digits of precision.

di=	$b_{\boldsymbol{h}}$	T_{di}	$\lvert F^D_{\boldsymbol{h}}(D_1)\rvert$	$\lvert F^D_{\boldsymbol{h}}(D_2)\rvert$	$\lvert F^D_{\boldsymbol{h}}(D_3)\rvert$	$\lvert F^D_{\boldsymbol{h}}(D_4)\rvert$	$4\lvert\theta(\boldsymbol{K}_{\boldsymbol{h}})\rvert$
10	1110 1010	0.24095	0.08663	0.05982	0.046287	0.04822	1.3755×10^{-24}
20	1110 1010	0.22281	0.06795	0.04733	0.044123	0.06341	1.4876×10^{-44}
100	1110 1010	0.15145	0.04860	0.03895	0.040696	0.02317	1.6134×10^{-205}
20	1110 1011	0.41334	0.12328	0.08971	0.08416	0.11619	1.850×10^{-44}

Table 2.5: Recorded values

The final column of Table 2.5 demonstrates that the obtained Riemann Constant Vector $\boldsymbol{K}_{\boldsymbol{h}}$ for the half-period $\boldsymbol{h} = 1110\ 1010$ is a zero of the theta function. As Table 2.5 also shows, values of the absolute error T_{di} are not as close to zero as in the genus 3 example above. Moreover, $\boldsymbol{h} = 1110\ 1010$ is not the only half-period for which $|\theta(\boldsymbol{K}_{\boldsymbol{h}})|$ is small, as is pointed out by the final row. This suggests that the remaining error is not coming from the $\mathcal{A}^{\mathcal{C}}_{\ell}$ term; otherwise, I would expect to obtain nonzero values for $|\theta(\boldsymbol{K}_{\boldsymbol{h}})|$. I also see that the sum, however, does indeed decrease as the number of digits increases. Moreover, the same half-period $\boldsymbol{h} = 1110\ 1010$ is obtained as the number of digits increases. At 20 digits, the 2nd smallest value of the sum is achieved by half-period 1110 1011 and has the values 0.4. Also observed is the fact that any half-period, not including $\boldsymbol{h}$, has the property that one of the terms, $|F^D_{\boldsymbol{h}}(D_i)|$, is larger than the corresponding term in row 2 of table 2.5, by a whole order of magnitude.

2.3.3 Conclusion

Initial testing suggests that the procedure works when there is an $\boldsymbol{h}_\ell$ that can be obtained. However, there are times for which the sum of Proposition 2.2.6 is not as small as we would like. In this case, one hopes that using more digits can resolve the discrepancy. To be conclusive, however, more analysis would be needed.

There are a number of places where improvements could be made to the code (including programs the code calls). The current implementation could be ported to the most recent version of `Maple`. Moreover, the code is not integrated within the `algcurves` package. This lack of integration makes it harder to control precision at each stage of the computation, and can result in run-time errors due to incompatibility of different parts of the whole package.

Once these issues are resolved, an error analysis of the program could be pursued, with the goal of being able to guarantee upper bounds on the errors in the output of the code. Such an analysis would require knowledge of several items as inputs to the analysis, including

1. the error in the input values,

2. the error that results from the choice of integration scheme,

3. the error in the computation of the Riemann theta function and its derivatives.

Chapter 3

EXTENDING DUBROVIN'S EFFECTIVIZATION OF FINITE-GENUS SOLUTIONS

Krichever's method for constructing finite-genus solutions of KP establishes the fact that, given any irreducible planar curve and any point on the curve, there is an associated family of finite-genus solutions to KP. To use the closed-form expression of such a solution requires a Riemann matrix B that can be derived from the periods of a Riemann surface. Now, even if it were known that a Riemann matrix B comes from a surface, the remaining parameters must still be computed and they are non-trivially related to B through the underlying surface. Thus, quantifying the parameters in the closed-form expression in equation 2.11 correctly is a non-trivial endeavor.

In this chapter, we treat the closed-form expression for u as an ansatz, with the goal of computing the different parameters. Certain choices are made to "normalize" this expression. This expression can be used to derive a system of 2^g quartics, in the components of $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$, that must simultaneously vanish. First, we essentially follow Dubrovin's approach in [11, Chapter 4] to compute solutions of genus $g \leq 3$. Then, a scheme, the *Extended Dubrovin's Method* (EDM), is described that provides a mathematical framework for computing parameters for higher genus solutions. This is followed by a discussion on implementation considerations and results. We wrap up the chapter with a discussion on how this relates to the constructive Schottky problem.

It is worth noting that an alternate approach to this problem has already been developed. In [2], an algorithm for computing solutions of KP using the system of quartics was proposed. This algorithm was developed in [1] and has been used to numerically reconstruct the curve.

3.1 Normalization of solutions & Dubrovin quartics

Using the Lie-point symmetries of the KP equation, Dubrovin [11] shows that it suffices to consider the ansatz

$$u\,(x,y,t) = 2\partial_x^2 \log\theta\,(x\boldsymbol{U} + y\boldsymbol{V} + t\boldsymbol{W})\,, \tag{3.1}$$

where the phase vector $\boldsymbol{\Phi}$ can be reintroduced if so desired, without any change in the other parameters. The components of the wave vectors are unknown, and we use u_i, v_i, and w_i in place of $\boldsymbol{U}[i]$, $\boldsymbol{V}[i]$, and $\boldsymbol{W}[i]$.

The ansatz (3.1) is used in equation 1.3. Elementary manipulations lead to

$$\theta_{4x}\theta - 4\theta_{3x}\theta_x + 3\theta_{xx}^2 - 4\theta_{xt}\theta + 4\theta_x\theta_t + 3\theta_{yy}\theta - 3\theta_y^2 + 8d\,\theta^2 = 0, \tag{3.2}$$

for $x, y, t \in \mathbb{R}$ and some $d \in \mathbb{C}$. Here I drop the argument $\boldsymbol{z}$ of the theta functions,

$$\boldsymbol{z} = x\boldsymbol{U} + y\boldsymbol{V} + t\boldsymbol{W}\,.$$

This expression can be exploited further, leading to the Dubrovin quartics. First, enumerate the half-periods according to Definition 1.31:

$$\mathcal{H} = \{\boldsymbol{h}_n\}_{n=1}^{2^g}.$$

Using theta constants, as in Definition 1.33, the system of Dubrovin quartics is a 2^g-dimensional system of polynomial equations in the components of the wave vectors, given by

$$\hat{\theta}^n_{\boldsymbol{U}} + \sum_{1\le i,j\le g}\left(-u_iw_j + \frac{3}{4}v_iv_j\right)\hat{\theta}^n_{(i,j)} + d\,\hat{\theta}^n = 0, \quad 1 \le n \le 2^g. \tag{3.3}$$

The following useful statements are given without proof (for details see [11]).

Proposition 3.1.1 (Properties of the ansatz u)**.** *Let B be a Riemann matrix that comes from a surface. Then*

1. *u, defined by equation 3.1, is a solution of equation 1.3 if and only if equation 3.3 is valid for some $d \in \mathbb{C}$.*

2. The Dubrovin quartics are invariant with respect to transformations of the form

$$\begin{aligned}\boldsymbol{U} &\mapsto \lambda \boldsymbol{U},\\ \boldsymbol{V} &\mapsto \pm\left(\lambda^2 \boldsymbol{V} + 2\lambda\alpha \boldsymbol{U}\right),\\ \boldsymbol{W} &\mapsto \lambda^3 \boldsymbol{W} + 3\lambda^2\alpha \boldsymbol{V} + 3\lambda\alpha^2 \boldsymbol{U},\\ d &\mapsto \lambda^4 d.\end{aligned}$$

Proof. See [11] for (1). For (2), apply the transformation directly to a quartic equation in the system of quartic equations. □

3.2 Dubrovin's effectivization

Suppose that B comes from a surface. We are interested in computing finite-genus solutions of equation 1.3 that are of the form given in equation 3.1 for $g \in \{1,\ 2,\ 3\}$ in a manner that encapsulates the primary ideas used in [11, Chapter 4]. We justify transformations using Proposition 3.1.1.

The $g = 1$ case is trivial. Any choice of parameters,

$$U, V, \text{ and } W \in \mathbb{C}, \text{ with } U \neq 0,$$

gives rise to a solution. To see this, note that the Lie-point symmetry transformation with

$$\lambda = \frac{1}{U} \quad \text{and } \alpha = -\frac{V}{2U^2}$$

demonstrates that u is equivalent to

$$u\left(x, y, t\right) = 2\partial_x^2 \log\theta\left(x + y\left(\frac{V}{U^2} + 2\alpha\right) + t\left(\frac{W}{U^3} + 3\frac{\alpha V}{U^2} + 3\alpha^2\right)\right) = 2\partial_x^2 \log\theta\left(x + t\xi\right)$$

for some ξ.

Next, consider the $g = 2$ case. Then, as in Definition 1.31, the half-periods are enumerated as

$$\mathcal{H} = \{h_n\}_{n=1}^{2^g} = \left[\mathbf{0}, \frac{\boldsymbol{e_1}}{\mathbf{2}}, \frac{\boldsymbol{e_2}}{\mathbf{2}}, \frac{\boldsymbol{e_1} + \boldsymbol{e_2}}{\mathbf{2}}\right].$$

Start with an arbitrary choice for $\boldsymbol{U}$,

$$\boldsymbol{U}^T = \begin{bmatrix} z_1 & z_2 \end{bmatrix}$$

with $z_1,\ z_2 \in \mathbb{C}$ and, without loss of generality assume that $z_2 \neq 0$. Scale the parameters, using

$$\lambda = z_2^{-1},\ \alpha = 0,$$

and relabel so that,

$$\boldsymbol{U} = \begin{bmatrix} \hat{z}_1 \\ 1 \end{bmatrix},\ \boldsymbol{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},\ \text{and } \boldsymbol{W} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

with

$$\hat{z}_1 = \frac{z_1}{z_2}.$$

Since

$$u_2 = 1,$$

we can transform again, using

$$\lambda = 1 \text{ and } \alpha = -\frac{v_2}{2},$$

so that

$$\boldsymbol{U} = \begin{bmatrix} \hat{z}_1 \\ 1 \end{bmatrix},\ \boldsymbol{V} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix},\ \text{and } \boldsymbol{W} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Using the notation of Definition 1.33, compute the matrix

$$M_{\boldsymbol{U}} = \begin{bmatrix} \hat{\theta}^1_{\boldsymbol{U}} & \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \hat{\theta}^1_{22} & \hat{\theta}^1 \\ \hat{\theta}^2_{\boldsymbol{U}} & \hat{\theta}^2_{11} & \hat{\theta}^2_{12} & \hat{\theta}^2_{22} & \hat{\theta}^2 \\ \hat{\theta}^3_{\boldsymbol{U}} & \hat{\theta}^3_{11} & \hat{\theta}^3_{12} & \hat{\theta}^3_{22} & \hat{\theta}^3 \\ \hat{\theta}^4_{\boldsymbol{U}} & \hat{\theta}^4_{11} & \hat{\theta}^4_{12} & \hat{\theta}^4_{22} & \hat{\theta}^4 \end{bmatrix}.$$

Since B is from a surface, the submatrix

$$M' = \begin{bmatrix} \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \hat{\theta}^1_{22} & \hat{\theta}^1 \\ \hat{\theta}^2_{11} & \hat{\theta}^2_{12} & \hat{\theta}^2_{22} & \hat{\theta}^2 \\ \hat{\theta}^3_{11} & \hat{\theta}^3_{12} & \hat{\theta}^3_{22} & \hat{\theta}^3 \\ \hat{\theta}^4_{11} & \hat{\theta}^4_{12} & \hat{\theta}^4_{22} & \hat{\theta}^4 \end{bmatrix}$$

is invertible (Proposition 1.3.5). Compute

$$\boldsymbol{X'}(\boldsymbol{U}) = -(M')^{-1}\boldsymbol{v}(\boldsymbol{U}),$$

where

$$(\boldsymbol{v}(\boldsymbol{U}))^T = \begin{bmatrix} \hat{\theta}^1_{\boldsymbol{U}} & \hat{\theta}^2_{\boldsymbol{U}} & \hat{\theta}^3_{\boldsymbol{U}} & \hat{\theta}^4_{\boldsymbol{U}} \end{bmatrix}$$

is the first column of M_U. It follows that the linear system,

$$M_{\boldsymbol{U}}\boldsymbol{X}(\boldsymbol{U}) = \boldsymbol{0} \text{ with } \boldsymbol{X}(\boldsymbol{U}) = \begin{bmatrix} 1 \\ \boldsymbol{X'}(\boldsymbol{U}) \end{bmatrix},$$

can be used to solve equation 3.3, as I now make clear. Drop the argument of $\boldsymbol{X}$ and relate the computed components of $\boldsymbol{X}$ to the appropriate relationships amongst the components of the wave vectors:

$$\begin{aligned}
\boldsymbol{X}[2] &= -u_1w_1 + \frac{3}{4}v_1^2 = -\hat{z}_1w_1 + \frac{3}{4}v_1^2, \\
\boldsymbol{X}[3] &= -u_1w_2 - u_2w_1 + \frac{3}{2}v_1v_2 = -\hat{z}_1w_2 - w_1, \\
\boldsymbol{X}[4] &= -u_2w_2 + \frac{3}{4}v_2^2 = -w_2, \\
\boldsymbol{X}[5] &= d.
\end{aligned}$$

This system can be solved completely: d and w_2 are determined immediately, then w_1 is determined, and then finally v_1 is determined. It should be noted that the vector $\boldsymbol{V}$ can only be determined up to a sign. After choosing the sign for v_1, a solution to the linear system is obtained. Moreover, by replacing $\boldsymbol{X}$ with these relations we can recover the Dubrovin quartics, thereby ensuring that we have a solution.[1]

Next, suppose that $g = 3$. The half-periods are enumerated according to

$$\mathcal{H} = \{h_n\}_{n=1}^8 = \left[\boldsymbol{0}, \frac{\boldsymbol{e_1}}{\boldsymbol{2}}, \frac{\boldsymbol{e_2}}{\boldsymbol{2}}, \frac{\boldsymbol{e_3}}{\boldsymbol{2}}, \frac{\boldsymbol{e_1}+\boldsymbol{e_2}}{\boldsymbol{2}}, \frac{\boldsymbol{e_1}+\boldsymbol{e_3}}{\boldsymbol{2}}, \frac{\boldsymbol{e_2}+\boldsymbol{e_3}}{\boldsymbol{2}}, \frac{\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}}{\boldsymbol{2}}\right].$$

[1] This procedure also works for $g = 1$: Choosing $U = z_1$ and $V = z_2$, compute the 2×3 matrix M_U and use M' to solve for W and d.

The matrices, $M_{\boldsymbol{U}}$ and M', are easy to express in symbols:

$$M_{\boldsymbol{U}} = \begin{bmatrix} \hat{\theta}^1_U & \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \hat{\theta}^1_{13} & \hat{\theta}^1_{22} & \hat{\theta}^1_{23} & \hat{\theta}^1_{33} & \hat{\theta}^1 \\ \hat{\theta}^2_U & \hat{\theta}^2_{11} & \hat{\theta}^2_{12} & \hat{\theta}^2_{13} & \hat{\theta}^2_{22} & \hat{\theta}^2_{23} & \hat{\theta}^2_{33} & \hat{\theta}^2 \\ \hat{\theta}^3_U & \hat{\theta}^3_{11} & \hat{\theta}^3_{12} & \hat{\theta}^3_{13} & \hat{\theta}^3_{22} & \hat{\theta}^3_{23} & \hat{\theta}^3_{33} & \hat{\theta}^3 \\ \hat{\theta}^4_U & \hat{\theta}^4_{11} & \hat{\theta}^4_{12} & \hat{\theta}^4_{13} & \hat{\theta}^4_{22} & \hat{\theta}^4_{23} & \hat{\theta}^4_{33} & \hat{\theta}^4 \\ \hat{\theta}^5_U & \hat{\theta}^5_{11} & \hat{\theta}^5_{12} & \hat{\theta}^5_{13} & \hat{\theta}^5_{22} & \hat{\theta}^5_{23} & \hat{\theta}^5_{33} & \hat{\theta}^5 \\ \hat{\theta}^6_U & \hat{\theta}^6_{11} & \hat{\theta}^6_{12} & \hat{\theta}^6_{13} & \hat{\theta}^6_{22} & \hat{\theta}^6_{23} & \hat{\theta}^6_{33} & \hat{\theta}^6 \\ \hat{\theta}^7_U & \hat{\theta}^7_{11} & \hat{\theta}^7_{12} & \hat{\theta}^7_{13} & \hat{\theta}^7_{22} & \hat{\theta}^7_{23} & \hat{\theta}^7_{33} & \hat{\theta}^7 \\ \hat{\theta}^8_U & \hat{\theta}^8_{11} & \hat{\theta}^8_{12} & \hat{\theta}^8_{13} & \hat{\theta}^8_{22} & \hat{\theta}^8_{23} & \hat{\theta}^8_{33} & \hat{\theta}^8 \end{bmatrix}, \; M' = \begin{bmatrix} \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \hat{\theta}^1_{13} & \hat{\theta}^1_{22} & \hat{\theta}^1_{23} & \hat{\theta}^1_{33} & \hat{\theta}^1 \\ \hat{\theta}^2_{11} & \hat{\theta}^2_{12} & \hat{\theta}^2_{13} & \hat{\theta}^2_{22} & \hat{\theta}^2_{23} & \hat{\theta}^2_{33} & \hat{\theta}^2 \\ \hat{\theta}^3_{11} & \hat{\theta}^3_{12} & \hat{\theta}^3_{13} & \hat{\theta}^3_{22} & \hat{\theta}^3_{23} & \hat{\theta}^3_{33} & \hat{\theta}^3 \\ \hat{\theta}^4_{11} & \hat{\theta}^4_{12} & \hat{\theta}^4_{13} & \hat{\theta}^4_{22} & \hat{\theta}^4_{23} & \hat{\theta}^4_{33} & \hat{\theta}^4 \\ \hat{\theta}^5_{11} & \hat{\theta}^5_{12} & \hat{\theta}^5_{13} & \hat{\theta}^5_{22} & \hat{\theta}^5_{23} & \hat{\theta}^5_{33} & \hat{\theta}^5 \\ \hat{\theta}^6_{11} & \hat{\theta}^6_{12} & \hat{\theta}^6_{13} & \hat{\theta}^6_{22} & \hat{\theta}^6_{23} & \hat{\theta}^6_{33} & \hat{\theta}^6 \\ \hat{\theta}^7_{11} & \hat{\theta}^7_{12} & \hat{\theta}^7_{13} & \hat{\theta}^7_{22} & \hat{\theta}^7_{23} & \hat{\theta}^7_{33} & \hat{\theta}^7 \\ \hat{\theta}^8_{11} & \hat{\theta}^8_{12} & \hat{\theta}^8_{13} & \hat{\theta}^8_{22} & \hat{\theta}^8_{23} & \hat{\theta}^8_{33} & \hat{\theta}^8 \end{bmatrix}.$$

The matrix M' is full rank, by Proposition 1.3.5. However, this time M' is not square. Since $M_{\boldsymbol{U}}$ is square, we exploit the fact that $M_{\boldsymbol{U}}$ must have a non-trivial null space, which implies that

$$0 = \det(M_{\boldsymbol{U}}) = \sum_{i=1}^{8} (-1)^{i-1} \hat{\theta}^i_U \det M_{1i}, \tag{3.4}$$

where M_{1i} is the submatrix of $M_{\boldsymbol{U}}$ obtained by removing its first column and ith row. The determinant is quartic in the components of $\boldsymbol{U}$. It follows from the full rank of M' that at least one of the terms $\det M_{1i}$ is non-trivial.

Start with an arbitrary choice $z_1,\ z_2 \in \mathbb{C}$ for the first two components of $\boldsymbol{U}$, with the third component u_3 to be determined,

$$\boldsymbol{U}^T = \begin{bmatrix} z_1 & z_2 & u_3 \end{bmatrix},$$

and without loss of generality assume that $z_2 \neq 0$. Apply the transformations analogously to what was done in the previous case and relabel so that

$$\boldsymbol{U} = \begin{bmatrix} \hat{z}_1 \\ 1 \\ u_3 \end{bmatrix}, \; \boldsymbol{V} = \begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix}, \text{ and } \; \boldsymbol{W} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Equation 3.4 gives four choices for u_3. In what follows, let $\boldsymbol{U}$ correspond to any one of these choices.

Our next task is to compute $\boldsymbol{X}$, given $\boldsymbol{U}$. A natural way to do this would be to find an invertible 7×7 submatrix A of the 8×7 matrix M'. We present this computation in a manner more amenable to generalization. Since M' is full rank, search the 8 rows for a collection of 7 rows that form an linearly independent set. There are only 8 possible candidates for A. For each $i \in \{1,\ldots,8\}$, let A_i be the submatrix of M' with rows associated with the half-periods

$$\mathcal{F}_i = \{h_{n_k} : \ n_k \neq i, \ k = 1,\ldots,7\}; \tag{3.5}$$

A_i here is the same as M_{1i} in equation 3.4. Choose i so that A_i is invertible. With

$$\boldsymbol{v}^T = \begin{bmatrix} \hat{\theta}^1_{\boldsymbol{U}} & \cdots & \hat{\theta}^8_{\boldsymbol{U}} \end{bmatrix}, \text{ and } \ \boldsymbol{v}^T_{\mathcal{F}_i} = \begin{bmatrix} \hat{\theta}^{n_1}_{\boldsymbol{U}} & \cdots & \hat{\theta}^{n_7}_{\boldsymbol{U}} \end{bmatrix},$$

compute

$$\boldsymbol{X'} = -\left(A_i\right)^{-1} \boldsymbol{v}_{\mathcal{F}_i}.$$

Since $\boldsymbol{U}$ was chosen so that $M_{\boldsymbol{U}}$ has a non-trivial null space and M' is of full rank, it follows that, using $\boldsymbol{X}$ defined as before,

$$M_{\boldsymbol{U}}\boldsymbol{X} = \boldsymbol{v} + M'\boldsymbol{X'} = \boldsymbol{0}.$$

The final step is to solve the appropriate system of equations:

$$\begin{aligned}
\boldsymbol{X}[2] &= -u_1w_1 + \frac{3}{4}v_1^2 = -\hat{z}_1w_1 + \frac{3}{4}v_1^2,\\
\boldsymbol{X}[3] &= -u_1w_2 - u_2w_1 + \frac{3}{2}v_1v_2 = -\hat{z}_1w_2 - w_1,\\
\boldsymbol{X}[4] &= -u_1w_3 - u_3w_1 + \frac{3}{2}v_1v_3 = -\hat{z}_1w_3 - u_3w_1 + \frac{3}{2}v_1v_3,\\
\boldsymbol{X}[5] &= -u_2w_2 + \frac{3}{4}v_2^2 = -w_2,\\
\boldsymbol{X}[6] &= -u_2w_3 - u_3w_2 + \frac{3}{2}v_2v_3 = -w_3 - u_3w_2,\\
\boldsymbol{X}[7] &= -u_3w_3 + \frac{3}{4}v_3^2,\\
\boldsymbol{X}[8] &= d.
\end{aligned}$$

Here the unknowns quantities are $w_1,\ w_2,\ w_3,\ v_1,\ v_3$ and d. Both d and w_2 are determined immediately. Then w_1 and w_3 follow. To obtain the components of $\boldsymbol{V}$, we must choose a

sign. Solve for v_1 and choose a sign. Then v_3 can be computed, and it must satisfy the equation whose left-hand side is $\boldsymbol{X}[4]$. This solves the system, and we now have a list

$$L_1 = [\boldsymbol{U},\ \boldsymbol{V},\ \boldsymbol{W},\ d]$$

for the choice $\boldsymbol{U}$. If we replace $\boldsymbol{X}$ with the relations, we can again recover the quartic equations, thereby ensuring we have a solution. We can repeat this for the remaining three possibilities for u_3. This gives four lists.

We have shown that given z_1, z_2 with $z_2 \neq 0$ we can obtain finite-genus solutions for $g = 2$ and $g = 3$. The same holds if we instead require $z_1 \neq 0$. In the case that $g = 3$ and $z_1 = z_2 = 0$, we can set v_3 to zero and choose a sign for either v_1 or v_2. Then $\boldsymbol{V}$, d, and w_3 are known immediately. This gives the remaining components of $\boldsymbol{W}$.

3.3 Extending Dubrovin's approach

We now generalize this approach for use in higher genus settings. For the remainder of this section, let B be a fixed Riemann matrix of genus $g > 3$ that can be derived from a Riemann surface. Further assume that B together with the g-dimensional vectors

$$\boldsymbol{U}^T = \begin{bmatrix} u_1 & u_2 & \dots & u_g \end{bmatrix}, \quad \boldsymbol{V}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_g \end{bmatrix},$$

and

$$\boldsymbol{W}^T = \begin{bmatrix} w_1 & w_2 & \dots & w_g \end{bmatrix},$$

solve the quartic equations 3.3. The goal of this section is to develop a scheme that can determine $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$ from an explicit B and a few parameters.

3.3.1 Chapter-specific conventions & notation

Before proceeding, we give labels to a few quantities that will appear throughout the remainder of this chapter.

Let

$$r = \frac{g(g+1)}{2} + 1,$$

and equip

$$\mathrm{IJ} = \{(i,j) : \ 1 \leq i \leq j \leq g\}$$

with the following order:

$$(i_1, j_1) \leq (i_2, j_2), \text{ if } i_1 < i_2 \text{ or } i_1 = i_2 \text{ and } j_1 \leq j_2.$$

I will call $(i,j) \in \mathrm{IJ}$ a pair. For $i \in \{1, \ldots, g\}$, define the integers A_i inductively:

$$\begin{cases} A_i = 1, & \text{provided } i = 1, \\ A_i = A_{i-1} + g - (i-1), & \text{whenever } 1 < i \leq g. \end{cases} \tag{3.6}$$

Proposition 3.3.1. *The map ind on IJ defined by*

$$(i,j) \mapsto A_i + j,$$

for A_i defined in equation 3.6, maps IJ onto the integers in the interval from 2 to r. Moreover, ind is bijective and order preserving. In particular, if

$$I = [c_0, \ (1,1), \ \cdots, \ (i,j), \ \cdots, \ (g,g), \ c_{-1}]$$

for some $c_0, c_{-1} \in \mathbb{C}$, then I is a list of size $r+1$ with the pairs ordered in I accordingly, and

$$I_{ind((i,j))} = (i,j).$$

Proof. Fix $i \in \{1, \ldots, g\}$ and consider any pair (i,j) with $j \in \{i, \ldots, g\}$. If $i = 1$, then it is straightforward to see that

$$\mathrm{ind}((1,j)) = A_1 + j = 1 + j,$$

and

$$I_{\mathrm{ind}((1,j))} = (1,j).$$

Suppose $i > 1$. Further, suppose that

$$I_{\mathrm{ind}((m,j_0))} = (m, j_0), \text{ whenever } 1 \leq m < i \text{ and } j_0 \in \{m, \ldots, g\}.$$

Then, by hypothesis,

$$I_{\mathrm{ind}(((i-1),g))} = ((i-1),g),$$

and thus

$$I_{\mathrm{ind}(((i-1),g))+1} = (i,i).$$

By definition of A_i and of ind,

$$\begin{aligned}\mathrm{ind}((i,i)) &= A_i + i,\\ &= A_{i-1} + g + 1,\\ &= \mathrm{ind}\left(((i-1),g)\right) + 1.\end{aligned}$$

Hence, we have shown that

$$I_{\mathrm{ind}((i,i))} = (i,i),$$

and it follows that

$$I_{\mathrm{ind}((i,j))} = (i,j).$$

By establishing the fact that a pair (i,j) is mapped to the index $\mathrm{ind}((i,j))$ of I, the rest of the claim is immediate. □

Instead of explicitly using ind we use the shorthand

$$\mathrm{I}_{(i,j)} = \mathrm{ind}((i,j)). \tag{3.7}$$

To illustrate the ordering we have defined, let $g = 6$. Then

$$\begin{aligned}A_2 &= g = 6,\\ A_3 &= 2g - 2 = 10,\\ A_4 &= 3g - 5 = 13,\\ A_5 &= 4g - 9 = 15,\\ A_6 &= 5g - 14 = 16.\end{aligned}$$

Moreover, if we construct the 23 dimensional array I and compare the indices and entries of elements of I, we see that

$$\begin{aligned}
\mathrm{I}_{(1,1)} &= A_1 + 1 = 2, \quad \text{and } \mathrm{I}_2 = (1,1),\\
\mathrm{I}_{(2,2)} &= A_2 + 2 = 8, \quad \text{and } \mathrm{I}_8 = (2,2),\\
\mathrm{I}_{(3,3)} &= A_3 + 3 = 13, \quad \text{and } \mathrm{I}_{13} = (3,3),\\
\mathrm{I}_{(4,4)} &= A_4 + 4 = 17, \quad \text{and } \mathrm{I}_{17} = (4,4),\\
\mathrm{I}_{(5,5)} &= A_5 + 5 = 20, \quad \text{and } \mathrm{I}_{20} = (5,5),\\
\mathrm{I}_{(6,6)} &= A_6 + 6 = 22, \quad \text{and } \mathrm{I}_{22} = (6,6).
\end{aligned}$$

3.3.2 Expressing quantities as functions of $\boldsymbol{U}$

In this section, we compute certain quantities required for determining the solution u using the undetermined vectors $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ and undetermined d, which appears in the quartic equation 3.3. We use these quantities to express $\boldsymbol{V}$, $\boldsymbol{W}$, and d as functions of the components of $\boldsymbol{U}$. Once these goals are achieved, determining a solution will be entirely dependent on computing the components of $\boldsymbol{U}$, at least theoretically.

Since B can be obtained from a Riemann surface, there is a curve f that can be used in Krichever's construction to obtain a solution. Thus, the system of Dubrovin quartics 3.3 is satisfied for some choice of wave vectors and some d. We can assume that the unknowns satisfy the Dubrovin quartics. Let $\boldsymbol{h}_\alpha \in \mathcal{H}$. Then the quartic equation associated with $\boldsymbol{h}_\alpha$,

$$\hat{\theta}^\alpha_{\boldsymbol{U}} + \sum_{1\le i,j\le g} \left(-u_i w_j + \frac{3}{4} v_i v_j\right) \hat{\theta}^\alpha_{ij} + d\,\hat{\theta}^\alpha = 0,$$

can be re-expressed as

$$\begin{aligned}
0 &= \hat{\theta}^\alpha_{\boldsymbol{U}} + \sum_{1\le i\le j\le g} \left(-u_i w_j + \frac{3}{4} v_i v_j + (1-\delta_{ij}) \left(-u_j w_i + \frac{3}{4} v_j v_i\right)\right) \hat{\theta}^\alpha_{ij} + d\,\hat{\theta}^\alpha\\
&= \hat{\theta}^\alpha_{\boldsymbol{U}} + m_\alpha \boldsymbol{X'},
\end{aligned}$$

where m_α denotes

$$m_\alpha = \begin{bmatrix} \hat{\theta}^\alpha_{11} & \hat{\theta}^\alpha_{12} & \dots & \hat{\theta}^\alpha_{1g} & \hat{\theta}^\alpha_{22} & \dots & \hat{\theta}^\alpha_{gg} & \hat{\theta}^\alpha \end{bmatrix}, \tag{3.8}$$

and $\boldsymbol{X'}$ is the r-vector of all of the coefficients (but the first, which is 1) of the theta constants; note that $\boldsymbol{X'}$ is independent of the index α. For $\boldsymbol{U}$, we have shown that the quartics are equivalent to

$$M_{\boldsymbol{U}} \boldsymbol{X} = \boldsymbol{0}, \text{ with } \boldsymbol{X}^T = \begin{bmatrix} 1 & \boldsymbol{X'}^T \end{bmatrix}, \tag{3.9}$$

and

$$M_{\boldsymbol{U}} = \begin{bmatrix} \hat{\theta}^1_U & \hat{\theta}^1_{11} & \hat{\theta}^1_{12} & \dots & \hat{\theta}^1_{1g} & \hat{\theta}^1_{22} & \dots & \hat{\theta}^1_{gg} & \hat{\theta}^1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \hat{\theta}^{2^g}_U & \hat{\theta}^{2^g}_{11} & \hat{\theta}^{2^g}_{12} & \dots & \hat{\theta}^{2^g}_{1g} & \hat{\theta}^{2^g}_{22} & \dots & \hat{\theta}^{2^g}_{gg} & \hat{\theta}^{2^g} \end{bmatrix}. \tag{3.10}$$

Let $\boldsymbol{v}$ be the first column of $M_{\boldsymbol{U}}$ and define the $2^g \times r$ sub-matrix M' according to the partition

$$M_{\boldsymbol{U}} = \begin{bmatrix} \boldsymbol{v} & M' \end{bmatrix}. \tag{3.11}$$

Proposition 3.3.2. *Suppose that B comes from a surface and consider $M_{\boldsymbol{U}}$, $\boldsymbol{v}$ and M' as defined in equations 3.10-3.11. Then the linear system*

$$\boldsymbol{M_U X} = \boldsymbol{0}$$

(where we have fixed $\boldsymbol{X}[1] = 1$) can be re-expressed as

$$\boldsymbol{M'X'} = -\boldsymbol{v}.$$

Moreover, $\boldsymbol{X}$ and $\boldsymbol{X'}$ are uniquely determined by $\boldsymbol{U}$, and as a function of the components of the wave vectors and the unknown d, $\boldsymbol{X}$ depends only on the components of $\boldsymbol{U}$.

Proof. By definition of $\boldsymbol{X}$,

$$\boldsymbol{0} = M_{\boldsymbol{U}} \boldsymbol{X} = \boldsymbol{v} + M' \boldsymbol{X'}.$$

By the non-singularity condition, Proposition 1.3.5, M' is full rank and thus injective. The only claim that needs consideration is the independence of $\boldsymbol{X}$ with respect to d and the components of $\boldsymbol{V}$ and $\boldsymbol{W}$. However, this is immediate since M' is a constant matrix and so the only dependence on parameters appears in the vector $\boldsymbol{v}$, and $\boldsymbol{v}$ depends only on $\boldsymbol{U}$. □

Notation 3.1 (Invertible submatricies)**.** Suppose that $\mathcal{A} \subset \mathcal{H}$ has r elements and that the associated $r \times r$ submatrix A of M', consisting of the r rows of M' corresponding to the elements of $\mathcal{A}$, is invertible. Choose an $\boldsymbol{h}_\beta \in \mathcal{H} \setminus \mathcal{A}$, let

$$\mathcal{A}_\beta = \mathcal{A} \cup \{\boldsymbol{h}_\beta\},$$

and let M_A be the $(r+1) \times (r+1)$ submatrix of M consisting of the $r+1$ rows of M_U corresponding to the elements of $\mathcal{A}_\beta$. We will suppress the dependence of M_A on β, assuming that β is chosen and fixed.

In addition, for any subset $\mathcal{S} \subset \mathcal{H}$, let $\boldsymbol{v}_\mathcal{S}$ denote the subvector of $\boldsymbol{v}$ consisting of the entries of $\boldsymbol{v}$ corresponding to the elements of $\mathcal{S}$.

Proposition 3.3.3. *Suppose $\mathcal{A}$ and $\mathcal{G}$ are both subcollections of the enumerated half-periods $\mathcal{H}$, consisting of r elements each, such that both associated submatrices A and G of M' are invertible, as in Notation 3.1. Assume also that $\mathcal{A} \cup \mathcal{G}$ is not all of $\mathcal{H}$. Choose $\boldsymbol{h}_\beta \in \mathcal{H} \setminus (\mathcal{A} \cup \mathcal{G})$ and consider $\mathcal{A}_\beta$, $\mathcal{G}_\beta$ (note: with the same $\boldsymbol{h}_\beta$), M_A, and M_G as in Notation 3.1. Let C be the $r \times r$ invertible matrix $C = G\,A^{-1}$; so we have $G = CA$. Then,*

1. *the linear system given in equation 3.9 can be solved using either A or G:*

$$\boldsymbol{X'} = -A^{-1}\boldsymbol{v}_\mathcal{A}; \quad \textit{and} \quad \boldsymbol{X'} = -G^{-1}\boldsymbol{v}_\mathcal{G}.$$

2. $\boldsymbol{v}_\mathcal{G} = C\boldsymbol{v}_\mathcal{A}$.

3. *Let P_A and P_G be the $(r+1) \times (r+1)$ permutation matrices that move the $\boldsymbol{h}_\beta$ row of M_A to the bottom and move the $\boldsymbol{h}_\beta$ row of M_G to the bottom, respectively, keeping the other rows in order. Then*

$$P_G M_G = \begin{bmatrix} C & \mathbf{0} \\ 0 & 1 \end{bmatrix} P_A M_A. \tag{3.12}$$

4. *The null space of M_G is the same as the null space of M_A.*

Proof. Since equation 3.9 holds,

$$A\boldsymbol{X'} = -\boldsymbol{v}_{\mathcal{A}} \quad \text{and} \quad G\boldsymbol{X'} = -\boldsymbol{v}_{\mathcal{G}},$$

and (1) follows; (2) follows immediately from (1).

By the definition of P_A and P_G,

$$P_G M_G = \begin{bmatrix} \boldsymbol{v}_{\mathcal{G}} & G \\ \boldsymbol{v}[\beta] & m_\beta \end{bmatrix} = \begin{bmatrix} C\boldsymbol{v}_{\mathcal{A}} & C\,A \\ \boldsymbol{v}[\beta] & m_\beta \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_{\mathcal{A}} & A \\ \boldsymbol{v}[\beta] & m_\beta \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 1 \end{bmatrix} P_A M_A,$$

giving (3); (4) follows because the other three $(r+1)\times(r+1)$ matrices in equation 3.12 are all invertible. □

Let $\mathcal{A}$ and M_A be as in Proposition 3.3.3. The importance of M_A in extending Dubrovin's approach can be seen by noting that, for $g = 3$,

$$M_A = M_{\boldsymbol{U}}.$$

In this previous case, the unknown component of $\boldsymbol{U}$ was determined by requiring it to be a zero of $\det M_{\boldsymbol{U}}$. Here, instead of using the matrix $M_{\boldsymbol{U}}$ which is rectangular for genus $g > 3$, we use the determinant of the $(r+1)\times(r+1)$ square submatrix M_A, setting it equal to zero, to get an equation involving $\boldsymbol{U}$. Let

$$\mathfrak{p}_A(\boldsymbol{U}) = \det M_A = \sum_{\ell=1}^{r+1} \left((-1)^{(\ell-1)} \det A_\ell\right) \hat{\theta}_{\boldsymbol{U}}^{n_\ell}, \tag{3.13}$$

where A_ℓ is the $r \times r$ submatrix of M_A obtained by removing its first column and ℓth row.

It follows from Proposition 3.3.3 that if $\boldsymbol{U}$ is any vector in $\mathbb{C}^g$ for which the linear system $M_U\boldsymbol{X} = 0$ has a non-trivial solution $\boldsymbol{X}$, and $\mathcal{A}$, $\mathcal{A}_\beta$, A, and M_A are as in Notation 3.1, then M_A is singular, the null space of M_A is the same as the null space of M_U, and

$$\mathfrak{p}_A(\boldsymbol{U}) = 0. \tag{3.14}$$

Equation 3.14 is thus a *necessary* condition for a vector $\boldsymbol{U} \in \mathbb{C}^g$ to be a wave vector.

Next, we use the notation introduced in Section 3.3.1 to define some vectors. Let $\boldsymbol{X}_{\mathrm{IJ}}$ and $\boldsymbol{X}_{\mathrm{JI}}$ be $r-1$ dimensional vectors defined by

$$\left(\boldsymbol{X}_{\mathrm{IJ}}\right)_{\mathrm{I}_{(i,j)}-1} = -u_i w_j + \frac{3}{4} v_i v_j, \text{ and } \left(\boldsymbol{X}_{\mathrm{JI}}\right)_{\mathrm{I}_{(i,j)}-1} = -u_j w_i + \frac{3}{4} v_j v_i, \tag{3.15}$$

for each pair (i,j). The vector of relations $\boldsymbol{X}_{\mathrm{rel}}$ is defined by

$$\left(\boldsymbol{X}_{\mathrm{rel}}\right)_{\mathrm{I}} = \begin{cases} \left(\boldsymbol{X}_{\mathrm{IJ}}\right)_{\mathrm{I}_{(i,j)}-1} + (1-\delta_{ij})\left(\boldsymbol{X}_{\mathrm{JI}}\right)_{\mathrm{I}_{(i,j)}-1}, & \text{if } \mathrm{I} = \mathrm{I}_{(i,j)} - 1, \\ d, & \text{if } \mathrm{I} = r. \end{cases} \tag{3.16}$$

Proposition 3.3.4. *Suppose that $\boldsymbol{U}$ is computed and it is known that*

$$M_{\boldsymbol{U}} \boldsymbol{X} = \boldsymbol{0}.$$

Further suppose that $\boldsymbol{V}$, $\boldsymbol{W}$, and d are found and satisfy

$$\boldsymbol{X'} = \boldsymbol{X}_{rel}$$

when, for each i, u_i, v_i, and w_i are replaced with $\boldsymbol{U}[i], \boldsymbol{V}[i]$, and $\boldsymbol{W}[i]$, and d is replaced with $\boldsymbol{X'}[r]$. Then u, as defined in equation 3.1, is a solution of equation 1.3 for the parameters B, $\boldsymbol{U}$, $\boldsymbol{V}$, and $\boldsymbol{W}$, and d.

Proof. Fix $\boldsymbol{h}_\alpha \in \mathcal{H}$. Then under the assumptions of the claim we have

$$\begin{aligned} 0 &= \boldsymbol{v}[\alpha] - m_\alpha \boldsymbol{X'} \\ &= \hat{\theta}^\alpha_{\boldsymbol{U}} + \sum_{1 \le i \le j \le g} \boldsymbol{X'}[I_{(i,j)} - 1] \hat{\theta}^\alpha_{(i,j)} + \boldsymbol{X'}[r] \hat{\theta}^\alpha \\ &= \hat{\theta}^\alpha_{\boldsymbol{U}} + \sum_{1 \le i \le j \le g} \left(-u_i w_j + \frac{3}{4} v_i v_j + \left(1 - \delta_{(i,j)}\right) \left(-u_j w_i + \frac{3}{4} v_j v_i \right) \right) \hat{\theta}^\alpha_{(i,j)} + d\hat{\theta}^\alpha \\ &= \hat{\theta}^\alpha_{\boldsymbol{U}} + \sum_{1 \le i,j \le g} \left(-u_i w_j + \frac{3}{4} v_i v_j \right) \hat{\theta}^\alpha_{(i,j)} + d\hat{\theta}^\alpha. \end{aligned}$$

Comparing with equation 3.3, we see that the claim follows. □

To wrap up this section, $\boldsymbol{V}$, $\boldsymbol{W}$, and d are expressed in terms of $\boldsymbol{U}$.

Proposition 3.3.5. *Let $z_1 \in \mathbb{C}$ and σ be a choice of sign. By enforcing*

$$\boldsymbol{X'} = \boldsymbol{X}_{rel},$$

we can express $\boldsymbol{V}$, $\boldsymbol{W}$, and d in terms of $\boldsymbol{U}$.

Proof. In this proof, let $I_{(j,i)}$ denote $I_{(i,j)}$ for $i < j$. Note that by Proposition 3.3.3

$$d = \boldsymbol{X'}[r] = -\sum_{k=1}^{r} \left(A_{r+1}^{-1}\right)_{rk} \hat{\theta}_{\boldsymbol{U}}^{n_k}.$$

Since an explicit expression for d in terms of the components of $\boldsymbol{U}$ has been found, denote d by $d(\boldsymbol{U})$. We use this convention also for the components of $\boldsymbol{V}$ and $\boldsymbol{W}$.

Let $\psi \in \{1, \ldots, g\}$ be such that the ψ component of $\boldsymbol{U}$ is non-zero. Then set this component to z_1, using the transformation

$$\lambda_0 = \frac{z_1}{u_\psi},\ \alpha_0 = 0.$$

Next, apply the transformation

$$\lambda_1 = 1,\ \alpha_1 = -\frac{v_\psi}{2z_1}.$$

This transforms the ψ component of $\boldsymbol{V}$ to 0.

These choices allow $\boldsymbol{W}$ to be computed right away. First,

$$w_\psi(\boldsymbol{U}) = -\frac{\boldsymbol{X'}_{I_{(\psi,\psi)}-1}}{z_1}.$$

For $j \neq \psi$,

$$\boldsymbol{X'}_{I_{(\psi,j)}-1} = -u_j w_\psi(\boldsymbol{U}) - z_1 w_j,$$

and thus

$$w_j(\boldsymbol{U}) = -\frac{\boldsymbol{X'}_{I_{(\psi,j)}-1} + u_j w_\psi(\boldsymbol{U})}{z_1}.$$

Next, we need formulas for $\boldsymbol{V}$. Let $\xi \neq \psi$. Then

$$\boldsymbol{X'}_{I_{(\xi,\xi)}-1} = -u_\xi w_\xi(\boldsymbol{U}) + \frac{3}{4} v_\xi^2,$$

and, using the sign σ,

$$v_\xi(\boldsymbol{U}) = \sigma\sqrt{\frac{4}{3}\left(\boldsymbol{X}'_{I_{(\xi,\xi)}-1} + u_\xi w_\xi(\boldsymbol{U})\right)}.$$

Let j be such that $j \neq \psi$ and $j \neq \xi$. Then

$$\boldsymbol{X}'_{I_{(\xi,j)}-1} = -u_\xi w_j(\boldsymbol{U}) - u_j w_\xi(\boldsymbol{U}) + \frac{3}{2} v_\xi(\boldsymbol{U})\, v_j,$$

and it is clear that v_j is also expressible only in terms of $\boldsymbol{U}$. If when choosing ξ there was no component that satisifed

$$\boldsymbol{X}'_{I_{(\xi,\xi)}-1} + u_\xi w_\xi(\boldsymbol{U}) \neq 0,$$

then

$$\boldsymbol{V} = \boldsymbol{0}.$$

Otherwise, with ξ such that

$$\boldsymbol{X}'_{I_{(\xi,\xi)}-1} + u_\xi w_\xi(\boldsymbol{U}) \neq 0,$$

we can express the components of $\boldsymbol{V}$:

$$v_j(\boldsymbol{U}) = \frac{2}{3}\frac{\boldsymbol{X}'_{I_{(\xi,j)}-1} + u_\xi w_j(\boldsymbol{U}) + u_j w_\xi(\boldsymbol{U})}{v_\xi(\boldsymbol{U})}.$$

□

3.4 Implementation

Let B be a Riemann matrix that comes from a surface. The scheme can be compartmentalized into four parts.

3.4.1 Procedure 1: Construct $M_{\boldsymbol{U}}$

Construct $M_{\boldsymbol{U}}$, M', $\boldsymbol{v}$ according to equation 3.10 and equation 3.11.

It must be noted that the computation of $M_{\boldsymbol{U}}$ requires the evaluation of a number of theta functions, which are g-fold sums over $\mathbb{Z}^g$. As g increases, the complexity of numerically calculating a theta function grows (according to [7]) and the number of theta functions that

must be evaluated grows exponentially. One way to reduce computation is to exploit the structure of the theta constants. For example, let $\boldsymbol{h}_\alpha \in \mathcal{H}$ and consider the theta function with double period

$$\theta[\boldsymbol{h}_\alpha](\boldsymbol{z}|2B) = e^{2\pi i\left(\boldsymbol{h}_\alpha^T B \boldsymbol{h}_\alpha + \boldsymbol{h}_\alpha^T \boldsymbol{z}\right)} \theta(\boldsymbol{z} + 2B\boldsymbol{h}_\alpha|2B).$$

By Proposition 1.3.4, this is an even function. Express this function as

$$g_0 = e^K f_0,$$

where g_0, K and f_0 depend on

$$\boldsymbol{z} = x\boldsymbol{U} + y\boldsymbol{V} + t\boldsymbol{W}.$$

For a function $h(\boldsymbol{z})$, let $\hat{h}$ denote $h(\boldsymbol{0})$, and if h is h_0, then let h_i denote $\partial_x^i h_0$. For K note that

$$\partial_x K = \pi i \boldsymbol{U}^T (2\boldsymbol{h}_\alpha) = K_x.$$

Then

$$g_1 = e^K \left(K_x f_0 + f_1\right),$$

and

$$g_2 = e^K \left(K_x^2 f_0 + K_x 2 f_1 + f_2\right).$$

Since g_1 is a odd function,

$$\hat{g}_1 = 0, \quad \text{which implies that } -K_x \hat{f}_0 = \hat{f}_1.$$

Hence

$$\hat{g}_2 = e^{\hat{K}} \left(-K_x^2 \hat{f}_0 + \hat{f}_2\right).$$

In a similar way, the formula for $\hat{g}_4$ is obtained:

$$\hat{g}_4 = e^{\hat{K}} \left(f_4 - 6K_x^2 f_2 + 5K_x^4 f_0\right).$$

For $i \in \{0, 2, 4\}$, note that $\hat{g}_i$ is the theta constant we desire and $\hat{f}_i$ is the ith x-derivative of $\theta(\boldsymbol{z} + 2B\boldsymbol{h}_\alpha|2B)$ evaluated at 0. The same reasoning applies for partial derivatives. Let the subscript n denote application of ∂_{z_n}, and let

$$K_n = \pi i \boldsymbol{U}[n](2\boldsymbol{h}_\alpha)_n,$$

so that

$$K_{i_1 \ldots i_m} = \prod_{j=1}^{m} K_{i_j}.$$

The formulas are now

$$\hat{g}_{ij} = e^{\hat{K}} \left(-K_{ij}\hat{f}_0 + \hat{f}_{ij} \right)$$

and

$$\hat{g}_{ijkl} = e^{\hat{K}} \left(\hat{f}_{ijkl} - 5K_{ijkl}\hat{f}_0 - K_{ij}\hat{f}_{kl} - K_{ik}\hat{f}_{jl} - K_{il}\hat{f}_{jk} - K_{jk}\hat{f}_{il} - K_{jl}\hat{f}_{il} - K_{kl}\hat{f}_{ij} \right).$$

Although many computations are still required, these formulas reduce the number of computations needed.

3.4.2 Procedure 2: Search for A

An invertible $r \times r$ submatrix of M' is needed. It may be preferable to try to find rows of M' that give the best condition number. Once A is computed, construct M_A by adding a row $\boldsymbol{h}_\beta$. The determinant of M_A, $\mathfrak{p}_A$, will be needed. In practice, coefficients of $\mathfrak{p}_A$ can take on extremely small values (relative to the number of digits of precision used).

3.4.3 Procedure 3: Compute $\boldsymbol{U}$

Next $\boldsymbol{U}$ must be determined. One approach would be to choose $\boldsymbol{U}$ that satisfies equation 3.14, $\mathfrak{p}_A(\boldsymbol{U}) = 0$, which is a necessary condition on $\boldsymbol{U}$.

Proposition 3.4.1. *Suppose $\boldsymbol{U} \in \mathbb{C}^g$ satisfies equation 3.14. Then $\boldsymbol{U}$ satisfies*

$$M_A \boldsymbol{X} = \boldsymbol{0}$$

for some $\boldsymbol{X} \in \mathbb{C}^{r+1}$ (with $\boldsymbol{X}[1] = 1$).

Proof. Equation 3.14 immediately implies the existence of a nonzero $\boldsymbol{X} \in \mathbb{C}^{r+1}$ satisfying $M_A\boldsymbol{X} = \mathbf{0}$. Since the submatrix A of M_A (as in Notation 3.1) is invertible, it follows that there is such an $\boldsymbol{X}$ with $\boldsymbol{X}[1] \neq 0$, and the result follows by scaling $\boldsymbol{X}$. The invertibility of A also implies that such an $\boldsymbol{X}$ with $\boldsymbol{X}[1] = 1$ is unique. □

Another approach for determining $\boldsymbol{U}$ is to require a subset of the components of $\boldsymbol{U}$ to be predetermined.

3.4.4 Procedure 4: Compute $\boldsymbol{V}$ and $\boldsymbol{W}$

After $\boldsymbol{U}$ is known, $\boldsymbol{X'}$ can be computed and the vectors $\boldsymbol{V}$, $\boldsymbol{W}$, and the number d must be determined. The procedure outlined in Proposition 3.3.5 gives formulas and a general procedure for computing $\boldsymbol{V}$, $\boldsymbol{W}$, and d. If these parameters are chosen so that

$$\boldsymbol{X'} = \boldsymbol{X}_{\text{rels}},$$

where $\boldsymbol{X}_{\text{rels}}$ is defined in equation 3.16, then the formulas of equation 3.15 can be used to recover the Dubrovin quartics.

3.5 Results

The current implementation takes a genus g matrix as input along with $g-1$ parameters for the vector $\boldsymbol{U}$. An invertible matrix A is found and $\mathfrak{p}_A$ is computed. The $g-1$ parameters are used in $\mathfrak{p}_A$ to determine 4 possible wave vectors $\boldsymbol{U}$. This extends Dubrovin's approach and in higher genus cases reduces computational complexity; however $\boldsymbol{U}$ need not result in a solution. Errors are measured in two ways. First, equation 3.2 is used to measure if a given u is indeed a solution with the parameters computed. Second, to obtain $\boldsymbol{W}$ and $\boldsymbol{V}$, only a few of the relations in $\boldsymbol{X}_{\text{rels}}$ are used. Since $\boldsymbol{X'}$ is explicit, evaluation of all of the components of

$$\boldsymbol{X'} - \boldsymbol{X}_{\text{rels}}$$

allows for a measure of error. Since the formulas for $\boldsymbol{V}$ and $\boldsymbol{W}$ introduce small, controllable errors, this is a better measure of $\boldsymbol{X'}$.

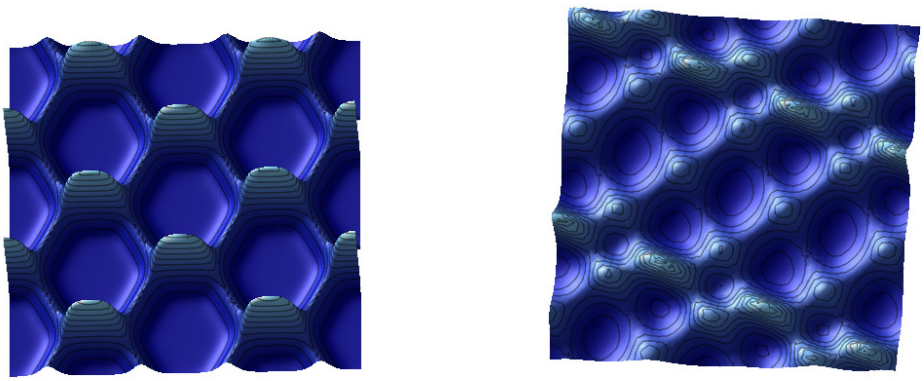

Figure 3.1: Waves recovered from [12] at $t = 0$ and on $-10 \leq x, y \leq 10$. (*Left*): A genus 2 solution that is symmetric as defined in [12, Page 143]. (*Right*): A genus 3 solution.

3.5.1 Recovery of wave parameters found in literature

The current implementation has been used to recover data from [12] and from [2]. In [12], wave vectors are given with respect to the form of the KP equation used in the paper and are not given completely. We have recovered the wave vectors from this paper with the same level of accuracy. Figure 3.1 shows two solutions obtained this way.

In [2], a genus 3 curve is given and a solution is plotted. After comparing the differentials and periods used in the paper with the differentials and periods obtained from `differentials`, `periodmatrix`, and `Siegel` commands, I was able to determine input for EDM that would give the solution depicted.

Besides correctness, these examples demonstrate that EDM can be used to study shallow water wave phenomena according to patterns in wave vectors. See [12] for more details on

this approach.

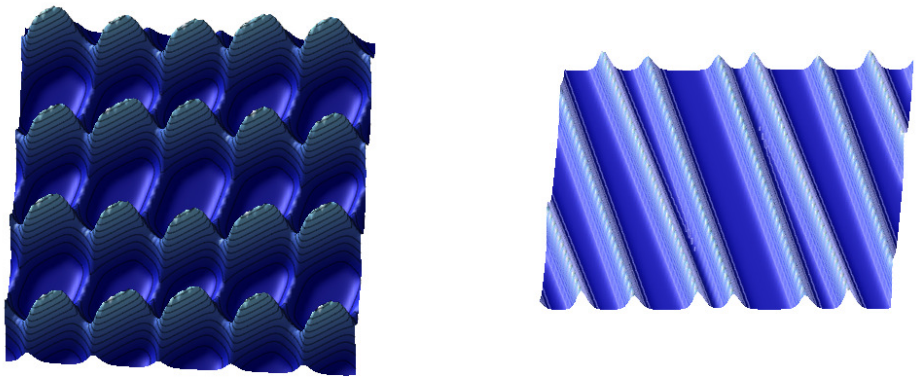

Figure 3.2: First two solutions recovered from the genus 4 M-curve at $t = 0$ and on $-10 \leq x, y \leq 10$.

3.5.2 Confirmation of data from genus 4 M-curve

To test whether or not EDM extends the method that Dubrovin uses, data from higher genus cases need to be tested. In particular, EDM should be able to recover a full set of data (the exact values of wave vectors and of d) using only a partial set. This is exactly what has taken place. I was given data from a M-curve of genus 4: a matrix B and 4 sets of parameters for the wave vectors. Using partial data from each of the four sets, I recovered the set of parameters with the same level of precision. Figure 3.2 and Figure 3.3 are snapshots of the solutions obtained.

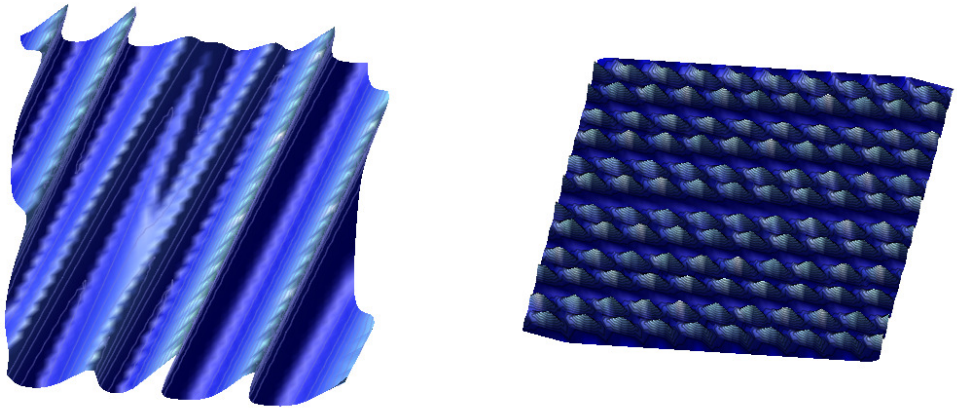

Figure 3.3: Final two solutions recovered from the genus 4 M-curve at $t = 0$. (*Left*): Solution from third parameter set given with $0 \leq x, y \leq 50$. (*Right*): Solution from fourth parameter set given with $-10 \leq x, y \leq 10$.

3.6 Final remarks: towards a constructive approach to the Schottky problem

I conclude this thesis with a brief discussion on how the results of this chapter relate to the Schottky problem.

So far, I have avoided an important point: a Riemann matrix need not come from a surface Γ_f. For example, a Riemann matrix B that is of the form

$$B = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$

cannot be from a Riemann surface. The problem of determining what Riemann matrices B come from a Riemann surface is known as the Schottky problem.

Krichever's construction, as detailed in Chapter 2, establishes the fact that a Riemann

matrix of a Riemann surface can be used to construct finite-genus solutions to the KP equation 1.3. Novikov conjectured that this property characterizes Riemann matrices that can be derived from a Riemann surface: if B can be used to construct a finite-genus solution to KP, it must be derivable from a Riemann surface. Shiota proved this conjecture to be true and in doing so solved the Schottky problem.

Although the problem has a resolution there are aspects that still deserve investigation. The constructive Schottky problem is the problem of determining relations that can be used to distinguish matrices B that come from a surface from those that do not.

In this chapter we have considered only Riemann matrices B that come from a surface. We can recast some of the results in this chapter to consider the case when it is not known whether B comes from a surface. Thus, the importance of EDM can be encapsulated by the following proposition:

Proposition 3.6.1. *Let B be a Riemann matrix of genus g. Suppose that the submatrix M' of $M_{\boldsymbol{U}}$ is constructed as in equations 3.10 and 3.11. (Recall that M' is a constant matrix, independent of $\boldsymbol{U}$.) Then there are three possibilities:*

1. *M' is not of full rank, and then B is decomposable;*

2. *M' is of full rank, but there is no $\boldsymbol{U} \in \mathbb{C}^g \setminus \{\mathbf{0}\}$ that satisfies*

$$M_{\boldsymbol{U}} \boldsymbol{X}(\boldsymbol{U}) = \mathbf{0},$$

   and thus B is not a Riemann matrix from a Riemann surface; or,

3. *M' is of full rank, and there is a vector $\boldsymbol{U} \in \mathbb{C}^g \setminus \{\mathbf{0}\}$ that satisfies*

$$M_{\boldsymbol{U}} \boldsymbol{X}(\boldsymbol{U}) = \mathbf{0};$$

   in this case, B is a Riemann matrix from a Riemann surface.

Proof. This is an application of the non-singularity condition of Proposition 1.3.5 and the results of Chapter 3. □

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