Nonlinear Stability in Integrable Hamiltonian Systems

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A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

2009

Program Authorized to Offer Degree: Applied Mathematics

University of Washington
Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

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Abstract

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The stability of periodic solutions of partial differential equations has been an area of increasing interest in the last decade. In this thesis, a new method for investigating the (nonlinear) orbital stability of periodic solutions of integrable Hamiltonian systems is presented. The method is demonstrated on the KdV equation, proving that all periodic finite-genus solutions are orbitally stable with respect to subharmonic perturbations (perturbations that have period equal to an integer multiple of the period of the amplitude of the solution). Also, a reduced form of the method is applied to the NLS and mKdV equations, establishing the orbital stability of elliptic solutions of the defocusing NLS equation and traveling wave solutions of the defocusing mKdV equation, both with respect to subharmonic perturbations.

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ACKNOWLEDGMENTS

I wish to express sincere appreciation to the Department of Applied Mathematics, especially to Professor Bernard Deconinck for his patience, guidance, knowledge, and enthusiasm. I would also like to thank my family for their unconditional support. Special thanks goes to my wife Melissa, the only person who “knows what I’m saying.”

DEDICATION

to Kurt Cobain

Chapter 1

INTRODUCTION

Integrable Hamiltonian systems are ubiquitous in nature; their range of applicability includes fiber optic communications, protein folding dynamics, and Bose-Einstein condensates [37, 53]. Characterized by their trademark soliton solutions (spatially localized waveforms persisting in time), examples include the Korteweg-deVries (KdV) equation, the nonlinear Schrödinger (NLS) equation, and the sine-Gordon equation. Here we are concerned with the stability of the periodic and quasi-periodic analogs of the soliton solutions, the finite-genus solutions.

Simply put, a solution is considered stable if nearby initial conditions lead to solutions that remain nearby. Stability is crucial to applications in science and engineering since one cannot expect the physical realization of solutions if they are not stable. For example, stability determines the feasibility of magnetohydrodynamic fusion devices and the emergence of ocean wave tsunamis [50]. Depending on one's definition of nearby, various levels of stability can be examined: spectral, linear, and nonlinear. More often than not, each successive level is increasingly difficult to prove.

The most general of the above stability types is nonlinear stability. Classically, a solution $u^*(x,t)$ is considered nonlinearly stable if for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$||u(x,0) - u^*(x,0)|| < \delta \Rightarrow ||u(x,t) - u^*(x,t)|| < \epsilon \tag{1.1}$$

for all $t > 0$.[1] However, this definition is too restrictive for Hamiltonian PDEs, as traveling waves of permanent form often exist. In such cases, two traveling waves can have initial profiles which are arbitrarily close at $t = 0$, but any difference in the individual velocities manifests itself as a non-arbitrary phase shift in some finite amount of time. Benjamin accounted for this in his study of the stability of solitary wave solutions of the KdV equation [10], coining the phrase *stability of*

[1] Of course, before one can consider stability, a global well-posedness result must exist for the initial-value problem in question. Since such results exist for the equations studied in this thesis (see [38, 56, 60] and the references therein), we assume general well-posedness in what follows.

shape. Essentially, this allows one to "re-align" the phases before measuring distance, minimizing the second norm in the above definition over all possible phase shifts:

$$||u(x,0) - u^*(x,0)|| < \delta \Rightarrow \inf_{x_0 \in \mathbb{R}} ||u(x,t) - u^*(x + x_0, t)|| < \epsilon. \tag{1.2}$$

This is an example of *orbital* stability. Orbital stability can also be defined with respect to more general symmetries (see [42, 68] and the definitions in the main text), which is necessary for our study of finite-genus solutions.

The objective of this thesis is to present a new method for establishing the (nonlinear) orbital stability of periodic and quasi-periodic solutions of integrable Hamiltonian systems with respect to *subharmonic perturbations* (periodic perturbations that have period equal to an integer multiple of the period of the amplitude of the solution). We restrict ourselves to periodic finite-genus solutions. It is often true that the finite-genus solutions of a given integrable Hamiltonian system are dense in the class of periodic functions. In such cases they completely solve the periodic initial value problem. Therefore, the stability of periodic finite-genus initial conditions in a suitable function space suggests the stability of general periodic initial conditions in that function space.

The vast majority of stability results for periodic solutions of PDEs are restricted to periodic perturbations of the same period as the solution. However, this is mostly due to convenience rather than pertinence. For example, periodic perturbations of the same period are traditionally easier to study numerically than subharmonic perturbations. Extension beyond periodic perturbations of the same period to subharmonic perturbations is important in that: (i) They are a significantly larger class of perturbations than the periodic ones of the same period, while retaining our ability to discuss completeness and separability of a suitable function space. For example, this would not be the case for quasi-periodic or almost periodic perturbations [12]. (ii) There are nontrivial examples of solutions which are stable with respect to periodic perturbations of the same period, but unstable with respect to subharmonic perturbations, such as the cn solutions of the focusing NLS equation [45]. (iii) They have a greater physical relevance than periodic perturbations of the same period, since in applications one usually considers domains which are larger than the period of the solution (ocean wave dynamics, for example).

The basis of our procedure is the Lyapunov method, which was first extended to infinite-dimensional systems (partial differential equations) by V.I. Arnold [6, 7] in his study of incompressible ideal fluid flows. Since its introduction, the Lyapunov method has formed the crux of subsequent nonlinear stability techniques, see [47, 50, 86], for instance. A more thorough analysis of past stability results is given in the main text.

In order to construct a Lyapunov function, we make extensive use of the nonlinear hierarchy associated with any integrable system:

- Since a finite-genus solution is not stationary with respect to the original PDE, we cannot define spectral stability in the conventional sense. However, each finite-genus solution is stationary with respect to one of the higher-order equations in the nonlinear hierarchy.

- We prove spectral stability with respect to the higher-order time variables by generalizing the squared eigenfunction method developed in [14].

- We use the ideas in [24, 68] to construct a candidate Lyapunov function. We show that it is indeed a Lyapunov function using the squared eigenfunction connection and the generalized spectral stability result. This establishes orbital stability from [42, 43].

Though presented for the KdV equation, the method is quite general and widely applicable to other integrable systems (the general steps and principles still apply, but the details will be different).

The outline of the thesis is as follows. In Chapters 2 and 3 we adapt the methods of [14, 24] to the defocusing NLS and modified KdV (mKdV) equations, establishing the spectral and orbital stability of their genus one solutions. This introduces the basic stability concepts involved, and eases one into the more abstract arguments and notions of stability required for the main result of this thesis: the (nonlinear) orbital stability of *all* periodic finite-genus solutions of the KdV equation. This is done in Chapter 4. In Chapter 5 we discuss how the method can be extended to any integrable Hamiltonian system, and draw some general conclusions. Chapters 2-4 are self contained and can be read independently of each other. However, it is recommended that at least one of Chapters 2 or 3 be read before reading Chapter 4. Chapter 4 should be read before reading Chapter 5.

Many of the explicit results in this thesis were obtained using MAPLE. The Weierstrass normal form algorithm in [85] was used in the computation of exact solutions of the mKdV equation.

The conservation law and integration algorithms from [27, 28, 29, 48] simplified many lengthy calculations arising in the study of the higher-order KdV, NLS, and mKdV equations.

Chapter 2

STABILITY OF ELLIPTIC SOLUTIONS OF THE NLS EQUATION

The defocusing one-dimensional nonlinear Schrödinger (NLS) equation with cubic nonlinearity is given by

$$i\Psi_t = -\frac{1}{2}\Psi_{xx} + \Psi\,|\Psi|^2\,. \tag{2.1}$$

Here $\Psi(x,t)$ is a complex-valued function, describing the slow modulation of a carrier wave in a dispersive medium. Due to both its physical relevance and its mathematical properties, (2.1) is one of the canonical equations of nonlinear dynamics. The equation has been used extensively to model, among other applications, waves in deep water [2, 90], propagation in nonlinear optics with normal dispersion [46, 61], Bose-Einstein condensates with repulsive self-interaction [44, 78] and electron plasma waves [21]. Equation (2.1) is completely integrable [1, 89]. This will be used extensively later on.

The equation has a large class of stationary solutions. These are solutions that are written as

$$\Psi = e^{-i\omega t}\phi(x), \tag{2.2}$$

where ω is a real constant. Among this class of solutions are the dark and grey solitons, for which $\phi(x)$ is expressed in terms of hyperbolic functions. These solutions may be regarded as limit cases of the so-called elliptic solutions studied in this paper. These solutions are either periodic or quasi-periodic as functions in x. The amplitude of $\phi(x)$ of the elliptic solutions is expressed in terms of Jacobi elliptic functions. A thorough discussion of the stationary solutions is found in, for instance, [18]. The details relevant to our investigations are presented in Section 2.1.

The stability analysis of the stationary solutions was begun in [90], where the now classical calculation for the modulational stability of the plane-wave solution ($\phi(x)$ constant) is given. The literature discussing the stability of the soliton solutions is extensive, see [62], and references therein.

Rowlands [79] may have been the first to consider the stability of the elliptic solutions directly. He studied the spectral stability problem for these solutions using regular perturbation theory with the Floquet parameter as a small expansion parameter. At the origin in the spectral plane, this parameter is zero, thus Rowlands was able to obtain expressions for the different branches of the continuous spectrum near the origin. For the focusing NLS equation these calculations demonstrate that the spectrum lies partially in the right-half plane, which leads to the conclusion of instability. For the defocusing NLS equation (2.1), the first approximation to these branches lies on the imaginary axis, and Rowlands' method is inconclusive with regard to stability or instability of the elliptic solutions. More recently, the stability of the elliptic solutions has been examined by Gallay and Hărăguş [38, 39]. In [39], they established the spectral stability of small-amplitude solutions of the form (2.2) of (2.1), as well as their (nonlinear) orbital stability with respect to perturbations that are of the same period as $|\phi(x)|$. In [38], the restriction on the amplitude for the orbital stability result is removed. Hărăguş and Kapitula [45] put some of these results in a more general framework valid for spectral problems with periodic coefficients originating from Hamiltonian systems. They establish that the small-amplitude elliptic solutions investigated in [39] are not only spectrally but also *linearly* stable. Lastly, we should mention a recent paper by Ivey and Lafortune [55]. They undertake a spectral stability analysis of the cnoidal wave solution of the focusing NLS equation, by exploiting the squared-eigenfunction connection, like we do in [14] for the cnoidal wave solutions of the Korteweg-de Vries equation and here, see below. Their calculations use Floquet theory for the spatial Lax operator to construct an Evans function for the spectral stability problem, whose zeros give the point spectrum corresponding to periodic perturbations. They also obtain a description of the continuous spectrum (which contains this point spectrum) using a Floquet discriminant. Their description of the spectrum is explicit in the sense that no differential equations remain to be solved. By computing level curves of this Floquet discriminant numerically, they obtain a numerical description of the spectrum.

In this chapter, we confirm the recent findings on spectral and orbital stability of the elliptic solutions of the defocusing equation and we extend their validity to solutions of arbitrary amplitude. In addition, we extend the stability results to the class of so-called subharmonic perturbations, *i.e.* perturbations that are periodic with period equal to an integer multiple of the period of the amplitude $|\phi(x)|$. Further, exploiting the integrability of (2.1), we are able to provide an explicit analytic description of the spectrum and the eigenfunctions associated with the linear stability problem of *all* elliptic solutions. We follow the same method as in [14], using the algebraic connection between the eigenfunctions of the Lax pair of (2.1) and those of the spectral stability problem. This explicit characterization of the spectrum is new. It appears that the methods of Ivey and Lafortune [55] allow for an equally explicit description when applied to the defocusing case. They rely on the general theory of hyperelliptic Riemann surfaces and theta functions, which are restricted to the elliptic case, through a nontrivial reduction process. We never leave the realm of elliptic functions, resulting in a significantly more straightforward approach. The explicit characterization of the spectrum is an obvious starting point for the stability analysis of more general solutions to non-integrable generalizations of the NLS equations, such as the two-dimensional NLS equation [19, 20] or one-dimensional perturbations of the NLS equation which might include such effects as dissipation or external potentials, see *e.g.*, [15, 55]. As in [38, 39, 45], we prove the spectral stability of the elliptic solutions of (2.1), without imposing a restriction on the amplitude. The results of [45] allow us to prove the completeness of the eigenfunctions of the linear stability problem, resulting in a conclusion of linear stability. Similarly to the last section of [24], we use the conserved quantities of the NLS equation to construct a candidate Lyapunov function. We then use the spectral stability result to prove that it is indeed a Lyapunov function. This allows us to invoke the classical results of Grillakis, Shatah and Strauss [42], from which (nonlinear) orbital stability follows.

It should be emphasized that our results are equally valid for elliptic solutions that have trivial phase ($\phi(x)$ real) as for solutions with a non-trivial phase profile ($\phi(x)$ not purely real). Similar calculations to the ones presented here apply to the focusing NLS equation, without the conclusion of stability, of course. That case is more complicated, due to the Lax operator associated with that integrable equation not being self adjoint.

The results of this chapter were obtained jointly with Nate Bottman and Bernard Deconinck.

2.1 Elliptic solutions of the defocusing NLS equation

The results of this section are presented in more detail in [18]. We restrict our considerations to the bare necessities for what follows.

Stationary solutions (2.2) of (2.1) satisfy the ordinary differential equation

$$\omega\phi = -\frac{1}{2}\phi_{xx} + \phi|\phi|^2. \tag{2.3}$$

Substituting an amplitude-phase decomposition

$$\phi(x) = R(x)e^{i\theta(x)} \tag{2.4}$$

in (2.3), we find ordinary differential equations satisfied by the amplitude $R(x)$ and the phase $\theta(x)$ by separating real and imaginary parts, after factoring out the overall exponential factor. Here we explicitly use that both amplitude and phase are real-valued functions. The equation for the phase $\theta(x)$ is easily solved in terms of the amplitude. One finds

$$\theta(x) = c\int_0^x \frac{1}{R^2(y)}\,dy. \tag{2.5}$$

Here c is a constant of integration. Using standard methods for elliptic differential equations (see for instance [17, 66]), one shows that the amplitude $R(x)$ is given by

$$R^2(x) = k^2\,\mathrm{sn}^2\,(x,k) + b, \tag{2.6}$$

where $\mathrm{sn}(x,k)$ is the Jacobi elliptic sine function, and $k \in [0,1)$ is the elliptic modulus [17, 66]. The amplitude $R(x)$ is periodic with period $2T(k) = 2K(k)$, where $K(k)$ is the complete elliptic integral of the first kind [17, 66]:

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2 y}}dy. \tag{2.7}$$

The form of the solution (2.6) leads to

$$\omega = \frac{1}{2}(1+k^2) + \frac{3}{2}b, \tag{2.8}$$

and

$$c^2 = b(b+1)(b+k^2). \tag{2.9}$$

Conditions on the reality of the amplitude and phase lead to the constraints on the offset parameter: $b \in \mathbb{R}^+$ (including zero). The class of solutions constructed here is not the most general class of stationary solutions of (2.1). We did not specify the full class of parameters allowed by the Lie point symmetries of (2.1), which allow for a scaling in x, multiplying by a unitary constant, *etc*. The methods introduced in the remainder of this chapter apply equally well and with similar results to the full class of stationary elliptic solutions.

If the constant c is zero, the solution is referred to as a trivial-phase solution. Otherwise it is called a nontrivial-phase solution. It is clear from the above that the only trivial-phase solutions are (up to symmetry transformations)

$$\Psi(x,t) = k\,\mathrm{sn}(x,k)e^{-\frac{i}{2}(1+k^2)t}. \tag{2.10}$$

This one-parameter family of solutions is found from the two-parameter family of stationary solutions by equating $b = 0$. The trivial-phase solutions are periodic in x. Their period is $4K(k)$. In contrast, the nontrivial-phase solutions are typically not periodic in x. The period of their amplitude is $2T(k) = 2K(k)$, whereas the period $\tau(k)$ of their phase is determined by $\theta(\tau(k)) = 2\pi$. Unless $\tau(k)$ and $2T(k)$ are rationally related, the nontrivial-phase solution is quasi-periodic instead of periodic.

This quasi-periodicity is more immediately obvious using a different form of the elliptic solutions, which will prove useful in Section 2.5. We split the integrand of (2.5) as

$$\frac{c}{R^2(x)} = \kappa(k,b) + \mathcal{K}(x;k,b), \tag{2.11}$$

where $\kappa(k,b)$ is the average value of $c/R^2(x)$ over an interval of length $2T(k)$. Thus the average value of $\mathcal{K}(x;k,b)$ is zero. Then the elliptic solutions may be written as

$$\Psi(x,t) = e^{-i\omega t + i\kappa x}\hat{R}(x), \tag{2.12}$$

where $\hat{R}(x + 2T(k)) = \hat{R}(x)$ is typically not real. It is clear from this formulation of the elliptic solutions that they are generically quasiperiodic with two incommensurate spatial periods $2T(k)$ and $2\pi/\kappa(k,b)$.

2.2 The linear stability problem

Before we study the orbital stability of the elliptic solutions, we examine their spectral and linear stability. To this end, we transform (2.1) so that the elliptic solutions are time-independent solutions of this new equation. Let

$$\Psi(x,t) = e^{-i\omega t}\psi(x,t). \tag{2.13}$$

Then

$$i\psi_t = -\omega\psi - \frac{1}{2}\psi_{xx} + \psi|\psi|^2. \tag{2.14}$$

As stated, the elliptic solutions are those solutions for which $\psi_t \equiv 0$. Next, we consider perturbations of such an elliptic solution. Let

$$\psi\left(x,t\right) = e^{i\theta(x)}\left(R(x) + \epsilon u(x,t) + i\epsilon v(x,t)\right) + \mathcal{O}\left(\epsilon^2\right), \tag{2.15}$$

where ϵ is a small parameter and $u(x,t)$ and $v(x,t)$ are real-valued functions. Since their dependence on both x and t is unrestricted, there is no loss of generality from factoring out the temporal and spatial phase factors. Substituting (2.15) into (2.1) and separating real and imaginary parts, the terms of zero order in ϵ vanish, since $R(x)e^{i\theta(x)}$ solves (2.1). Next, we equate terms of order ϵ to zero and separate real and imaginary parts, resulting in

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = J\mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} = J\begin{pmatrix} L_+ & S \\ -S & L_- \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{2.16}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.17}$$

and the linear operators L_-, L_+ and S are defined by

$$L_- = -\frac{1}{2}\partial_x^2 + R^2(x) - \omega + \frac{c^2}{2R^4(x)}, \tag{2.18}$$

$$L_+ = -\frac{1}{2}\partial_x^2 + 3R^2(x) - \omega + \frac{c^2}{2R^4(x)}, \tag{2.19}$$

$$S = \frac{c}{R^2(x)}\partial_x - \frac{cR'(x)}{R^3(x)} = \frac{c}{R(x)}\partial_x \frac{1}{R(x)}. \tag{2.20}$$

We wish to show that perturbations u and v that are initially bounded remain so for all times. By ignoring terms of order ϵ^2 and higher we are restricting ourselves to linear stability. The elliptic solution $\phi(x) = R(x)e^{i\theta(x)}$ is by definition *linearly stable* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $||u(x,0) + iv(x,0)|| < \delta$ then $||u(x,t) + iv(x,t)|| < \varepsilon$ for all $t > 0$. It should be noted that this definition depends on the choice of the norm $||\cdot||$ of the perturbations. In the next section this norm will be specified. The linear stability problem (2.16) is written in its standard form to allow for a straightforward comparison with the results of other authors, see for instance [38, 39, 45, 79], and many references where only the soliton case is considered. Some of our calculations are more conveniently done using a different form of the linear stability problem (2.16) or the spectral stability problem (2.22, below). These forms will be introduced as necessary.

Since (2.16) is autonomous in t, we may separate variables and consider solutions of the form

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} U(x,\lambda) \\ V(x,\lambda) \end{pmatrix}, \tag{2.21}$$

so that the eigenfunction vector $(U(x,\lambda), V(x,\lambda))^T$ satisfies the spectral problem

$$\lambda \begin{pmatrix} U \\ V \end{pmatrix} = J\mathcal{L} \begin{pmatrix} U \\ V \end{pmatrix} = J \begin{pmatrix} L_+ & S \\ -S & L_- \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{2.22}$$

In what follows, we suppress the λ-dependence of U and V. In order to show that the solution $\phi(x) = R(x)e^{i\theta(x)}$ is *spectrally stable*, we need to verify that the spectrum $\sigma(\mathcal{L})$ does not intersect the open right-half of the complex λ plane. To avoid confusion with other spectra defined below, we refer to $\sigma(\mathcal{L})$ as the *stability spectrum* of the elliptic solution $\phi(x)$. Since the nonlinear Schrödinger equation (2.1) is Hamiltonian [2], the spectrum of its linearization is symmetric with respect to both the real and the imaginary axis [87], so proving the spectral stability of an elliptic solution is equivalent to proving the inclusion $\sigma(\mathcal{L}) \subset i\mathbb{R}$.

Spectral stability of an elliptic solution implies its linear stability if the eigenfunctions corresponding to the stability spectrum $\sigma(\mathcal{L})$ are complete in the space defined by the norm $||\cdot||$. In that case all solutions of (2.16) may be obtained as linear combinations of solutions of (2.22).

The first goal of this chapter is to prove the spectral and linear stability of all solutions (2.2) by analytically determining the stability spectrum $\sigma(\mathcal{L})$, as well as its associated eigenfunctions. It is already known from [39] and [45] that the inclusion $\sigma(\mathcal{L}) \subset i\mathbb{R}$ holds for solutions of small amplitude, or, equivalently, solutions with small elliptic modulus, leading to spectral stability. We strengthen these results by providing a completely explicit description of $\sigma(\mathcal{L})$ and its eigenfunctions, without requiring any restriction on the elliptic modulus. To conclude the completeness of the eigenfunctions associated with $\sigma(\mathcal{L})$, and thus the linear stability of the elliptic solutions, we rely on the SCS lemma, see Hărăguş and Kapitula [45].

2.3 Numerical Results

In the next few sections, we determine the spectrum of (2.22) analytically. Before we do so, we compute it numerically, using Hill's method [26]. Hill's method is ideally suited to a periodic-coefficient problem such as (2.22). It should be emphasized that almost none of the elliptic solutions are periodic in x, as discussed in Section 2.1. Nevertheless, since we have factored out the exponential phase factor $e^{i\theta(x)}$ and the remaining coefficients are all expressed in terms of $R(x)$, the spectral problem (2.22) is a problem with periodic coefficients, even for elliptic solutions that are quasi-periodic.

Using Hill's method, we compute all eigenfunctions by using the Floquet-Bloch decomposition

$$\begin{pmatrix} U(x) \\ V(x) \end{pmatrix} = e^{i\mu x} \begin{pmatrix} \hat{U}(x) \\ \hat{V}(x) \end{pmatrix}, \quad \hat{U}(x+2T(k)) = \hat{U}(x), \quad \hat{V}(x+2T(k)) = \hat{V}(x), \tag{2.23}$$

with $\mu \in [-\pi/4T(k), \pi/4T(k))$. It follows from Floquet's theorem [3] that *all* bounded solutions of (2.22) are of this form. Here bounded means that $\sup_{x\in\mathbb{R}}\{|U(x)|, |V(x)|\}$ is finite. Thus $(U,V)^T \in C_b^0(\mathbb{R})$. On the other hand, we also have $(U,V)^T \in L^2_{per}(-T(k),T(k))$ (the square-integrable functions of period $2T(k)$) since the exponential factor in (2.23) disappears in the computation of the L^2-norm. Thus

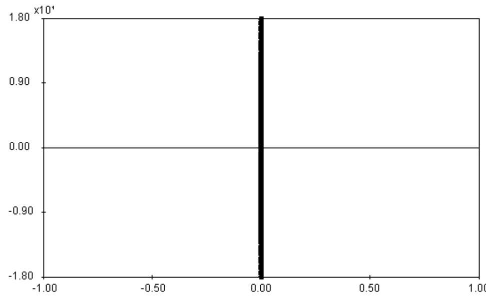
$$U, V \in C_b^0(\mathbb{R}) \cap L^2_{per}(-T(k), T(k)). \tag{2.24}$$

By a similar argument as that given at the end of Section 2.1, the typical eigenfunction (2.23) obtained this way is quasi-periodic, with periodic eigenfunctions ensuing when the two periods $2T(k)$ and $2\pi/\mu$ are commensurate. Specifically, our investigations include perturbations of an arbitrary period that is an integer multiple of $2T(k)$, *i.e.*, subharmonic perturbations.

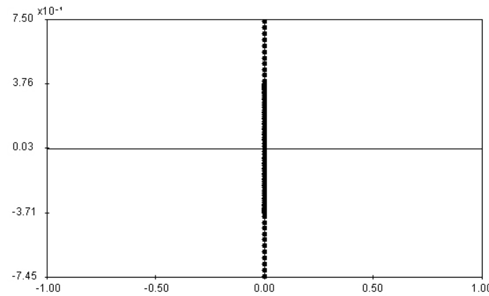
Figure 4.1 shows discrete approximations to the spectrum of (2.22), computed using SpectrUW 2.0 [25]. The solution parameters for the top two panels (a-b) are $b = 0$ (thus corresponding to a trivial-phase solution (2.10)) and $k = 0.8$. The numerical parameters (see [26, 25]) are $N = 20$ (41 Fourier modes) and $D = 40$ (39 different Floquet exponents). The right panel (b) is a blow-up of the left panel (a) around the origin. First, it appears that the spectrum is on the imaginary axis[1], indicating spectral stability of the snoidal solution (2.10). Second, the numerics shows that a symmetric band around the origin has a higher spectral density than does the rest of the imaginary axis. This is indeed the case, as shown in more detail in Fig. 4.3a, where the imaginary parts $\in [-1, 1]$ of the computed eigenvalues are displayed as a function of the Floquet parameter μ. This shows that λ values with Im$(\lambda) \in [-0.37, 0.37]$ (approximately) are attained for four different μ values in $[-\pi/4T(k), \pi/4T(k))$. The rest of the imaginary axis is only attained for two different μ values. This picture persists if a larger portion of the imaginary λ axis is examined. These numerical results are in perfect agreement with the theoretical results below.

The bottom two panels (c-d) correspond to a nontrivial-phase solution with $b = 0.2$ and $k = 0.5$. The numerical parameters are identical to those for panels (a-b). Again, the spectrum appears to lie on the imaginary axis, with a higher spectral density around the origin. A plot of the imaginary parts of the computed eigenvalues as a function of μ is shown in Fig. 4.3b. As for the trivial-phase case this shows the quadruple covering of the spectrum of a band around the origin of the imaginary axis, and the double covering of the rest of the imaginary axis. Due to the nontrivial-phase profile, the curves in Fig. 4.3b have lost some symmetry compared to those in Fig. 4.3a. Making the opposite

[1]The order of magnitude of the largest real part computed is 10^{-10}.



(a)



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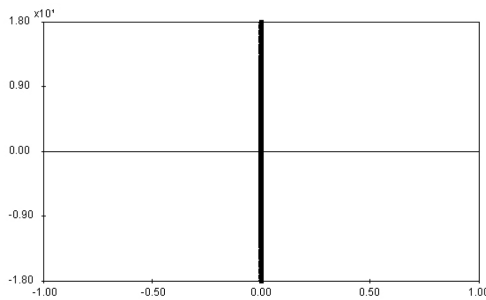
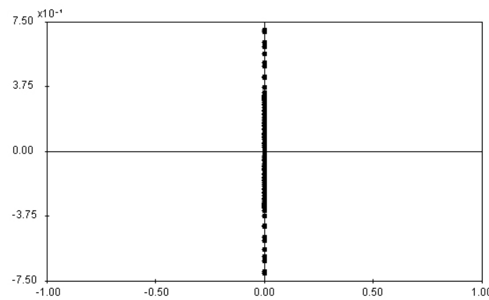


(c)

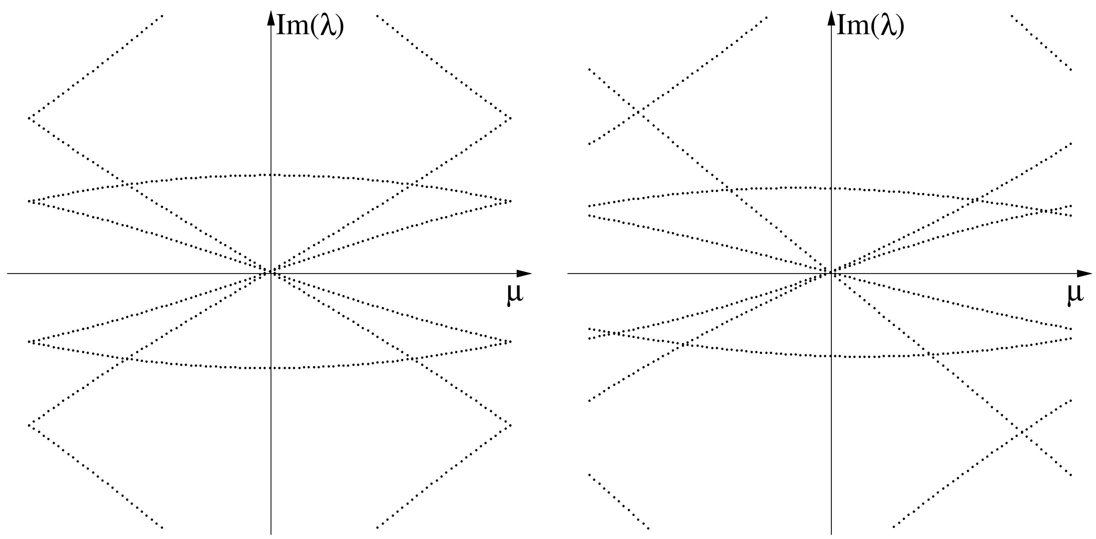


(d)

Figure 2.1: Numerically computed spectra of (2.22) for different solutions (2.2), with parameter values given below , using Hill's method with $N = 20$ (41 Fourier modes) and $D = 40$ (39 different Floquet exponents), see [26, 25]. (a) A trivial-phase sn-solution with $k = 0.5$. (b) A blow-up of (a) around the origin, showing a band of higher spectral density. (c) A nontrivial-phase solution with $b = 0.2$ and $k = 0.5$. (d) A blow-up of (c) around the origin, similarly showing a band of higher spectral density.

choice for the sign on c in (2.9) results in the figure being slanted in the other direction.

The above considerations remain true for different values of the offset $b \in \mathbb{R}^+$ and the elliptic modulus $k \in [0, 1)$, although the spectrum does depend on both, as we will prove in the following sections. Thus, for all values of $(b, k) \in \mathbb{R}^+ \times [0, 1)$, the spectrum of the elliptic solutions appears to be confined to the imaginary axis, indicating the spectral stability of these solutions. Similarly, for all these parameter values, the spectrum $\sigma(\mathcal{L})$ covers a symmetric interval around the origin four times, whereas the rest of the imaginary axis is double covered. The edge point on the imaginary axis where the transition from spectral density four to two occurs depends on both b and k and is



(a) (b)

Figure 2.2: The imaginary part of λ as a function of μ, demonstrating the higher spectral density (four *vs.* two) in Fig. 4.1b (left panel) and Fig. 4.1d (right panel). The parameter values are identical to those of Fig. 4.1.

denoted $\lambda_c(b,k)$. The k-dependence of $\lambda_c(b=0.2,k)$ is shown in Fig. 4.5. Again, both numerical and analytical results (see Section 2.5) are displayed. For these numerical results, Hill's method with $N=50$ was used.

2.4 Lax pair representation

Since our analytical stability results originate from the squared-eigenfunction connection between the defocusing NLS linear stability problem (2.16) and its Lax pair, in this section we examine this Lax pair, restricted to the elliptic solutions of the defocusing NLS.

As for the stability problem, we consider the generalized defocusing NLS (2.14). This equation is integrable, thus it has a Lax pair representation. Specifically, (2.14) is equivalent to the compatibility condition $\chi_{xt}=\chi_{tx}$ of the two first-order linear differential equations

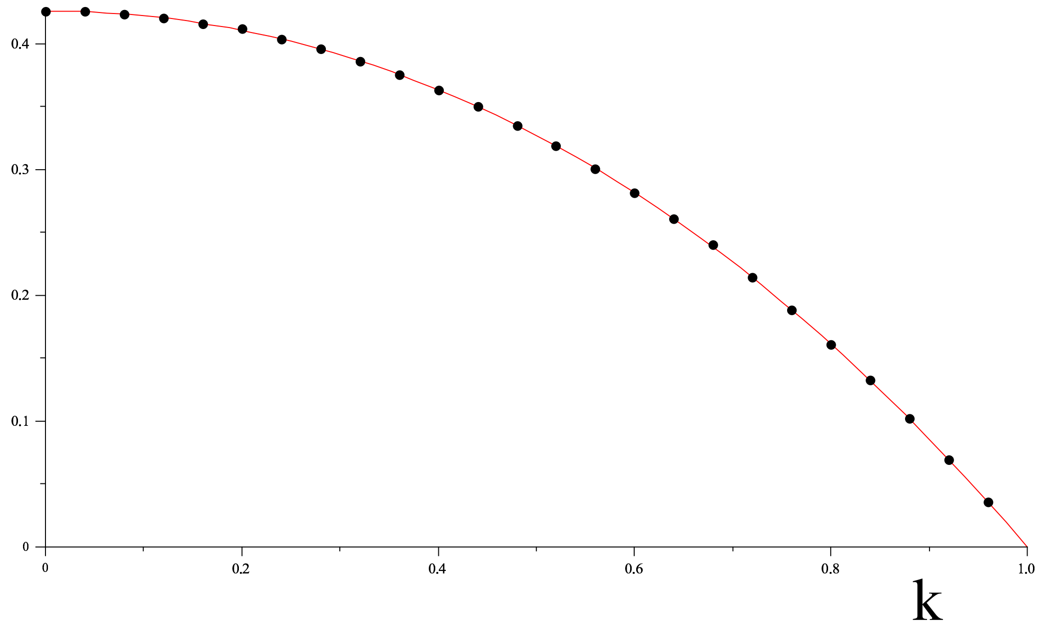


Figure 2.3: Numerical and analytical results for the imaginary part of the edge point $\lambda_c(b,k)$ of the quadruple-covered region as a function of the elliptic modulus k for $b = 0.2$. The solid curve displays the analytical result, the small circles are obtained numerically.

$$\chi_x = \begin{pmatrix} -i\zeta & \psi \\ \overline{\psi} & i\zeta \end{pmatrix} \chi, \qquad \chi_t = \begin{pmatrix} -i\zeta^2 - \frac{i}{2}|\psi|^2 + \frac{i}{2}\omega & \zeta\psi + \frac{i}{2}\psi_x \\ \zeta\overline{\psi} - \frac{i}{2}\overline{\psi_x} & i\zeta^2 + \frac{i}{2}|\psi|^2 - \frac{i}{2}\omega \end{pmatrix} \chi, \quad (2.25)$$

where $\overline{\psi}$ denotes the complex conjugate of ψ. Thus (2.14) is satisfied if and only if both equations of (2.25) are satisfiable. Note that the first equation may be rewritten as

$$\zeta\chi = \begin{pmatrix} i\partial_x & -i\psi \\ i\overline{\psi} & -i\partial_x \end{pmatrix} \chi. \quad (2.26)$$

The operator on the right-hand side is self adjoint, thus the spectral parameter ζ is confined to the real axis. Restricting to the elliptic solutions gives

$$\chi_x = \begin{pmatrix} -i\zeta & \phi \\ \overline{\phi} & i\zeta \end{pmatrix} \chi, \qquad \chi_t = \begin{pmatrix} -i\zeta^2 - \frac{i}{2}|\phi|^2 + \frac{i}{2}\omega & \zeta\phi + \frac{i}{2}\phi_x \\ \zeta\overline{\phi} - \frac{i}{2}\overline{\phi_x} & i\zeta^2 + \frac{i}{2}|\phi|^2 - \frac{i}{2}\omega \end{pmatrix} \chi. \tag{2.27}$$

We refer to the spectrum of the first equation of (2.27) as σ_L. It is the set of all ζ values for which this equation has a bounded in x (as in Section (2.3)) solution. As discussed above, $\sigma_L \subset \mathbb{R}$. The main goal of this section is the complete analytic determination of σ_L. For ease of notation, we rewrite the second equation of (2.27) as

$$\chi_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \chi. \tag{2.28}$$

Since A, B and C are independent of t, we may separate variables. Consider the ansatz

$$\chi(x,t) = e^{\Omega t}\varphi(x), \tag{2.29}$$

where Ω is independent of t. We refer to the set of all Ω such that χ is a bounded function of x as the t-spectrum σ_t. Substituting (2.29) into (2.28) and canceling the exponential, we find

$$\begin{pmatrix} A-\Omega & B \\ C & -A-\Omega \end{pmatrix} \varphi = \mathbf{0}. \tag{2.30}$$

This implies that the existence of nontrivial solutions requires

$$\Omega^2 = A^2 + BC = -\zeta^4 + \omega\zeta^2 - c\zeta + \frac{1}{16}\left(4\omega b - 3b^2 - k'^4\right). \tag{2.31}$$

where $k'^2 = 1 - k^2$. We have used the explicit form of $\phi(x)$, given in Section 2.1. This demonstrates that Ω is not only independent of t, but also of x. Such a conclusion could also be arrived at by expressing the derivatives of the operators of (2.27) as matrix commutators, and applying the fact that the trace of a matrix commutator is identically zero [8, 31].

Having determined Ω as a function of ζ for any given elliptic solution of defocusing NLS (*i.e.*, in terms of the parameters b and k), we now wish to do the same for the eigenvector $\varphi(x)$, determined by (2.30). Immediately,

$$\varphi = \gamma(x) \begin{pmatrix} -B(x) \\ A(x) - \Omega \end{pmatrix}, \tag{2.32}$$

where $\gamma(x)$ is a scalar function. Indeed, the vector part of (2.32) ensures that $\chi(x,t)$ satisfies the second equation of (2.27). Next, we determine $\gamma(x)$ so that $\chi(x,t)$ also satisfies the first equation. Substituting (2.32) in this first equation results in two homogeneous linear scalar differential equations for $\gamma(x)$ which are linearly dependent. Solving gives

$$\gamma(x) = \gamma_0 \exp\left(-\int \frac{(A-\Omega)\phi + B_x + i\zeta B}{B} dx\right). \tag{2.33}$$

For almost all $\zeta \in \mathbb{C}$, we have explicitly determined two linearly independent solutions of the first equation of (2.27). Indeed, for all ζ, there should be two such solutions, and two have been constructed for all $\zeta \in \mathbb{C}$ for which $\Omega \neq 0$: the combination of (2.32) and (2.33) gives two solutions, corresponding to the different signs for Ω in (2.31). These solutions are clearly linearly independent. For those values of ζ for which $\Omega = 0$, only one solution is generated. A second one may be found using the method of reduction of order.

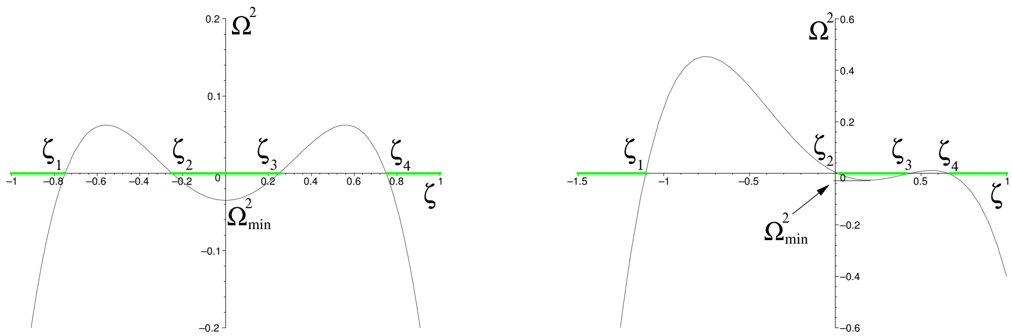
To determine the spectrum σ_L, we need to determine the set of all $\zeta \subset \mathbb{R}$ such that (2.32) is bounded for all x. Clearly, the vector part of (2.32) is bounded as a function of x. Thus, we need to determine for which ζ the scalar function $\gamma(x)$ is bounded. For this, it is necessary and sufficient that

$$\left\langle \Re\left(\frac{(A-\Omega)\phi + B_x + i\zeta B}{B}\right)\right\rangle = 0. \tag{2.34}$$

Here $\langle \cdot \rangle = \frac{1}{2T(k)}\int_{-T(k)}^{T(k)} \cdot\, dx$ is the average over a period and $\Re$ denotes the real part. The investigation of (2.34) is significantly simpler for the trivial-phase case $b = 0$ than for the general nontrivial-phase case. We treat these cases separately.

2.4.1 The trivial-phase case: $b = 0$

With $b = 0$ (2.31) becomes



(a) (b)

Figure 2.4: Ω^2 as a function of real ζ, for $k = 0.5$. The union of the thick line segments on the real axis is the Lax spectrum σ_L. The figure on the left shows the symmetric trivial-phase case with $b = 0$. The figure on the right illustrates a nontrivial-phase case, with $b = 0.2$.

$$\Omega^2 = -\zeta^4 + \omega\zeta^2 - \frac{k'^4}{16} = -(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4), \tag{2.35}$$

with

$$\zeta_1 = -\frac{1}{2}(1+k),\ \zeta_2 = -\frac{1}{2}(1-k),\ \zeta_3 = \frac{1}{2}(1-k),\ \zeta_4 = \frac{1}{2}(1+k). \tag{2.36}$$

The graph for Ω^2 as a function of ζ is shown in Fig. 2.4a.

The explicit form of (2.34) is different depending on whether Ω is real or imaginary. It should be noted that since $\zeta \in \mathbb{R}$, it follows from (2.35) that these are the only possibilities.

First, we consider Ω being imaginary or zero, requiring $|\zeta| \geq (k+1)/2$ or $|\zeta| \leq (1-k)/2$. It follows from the definitions of A and B that the integrand in (2.34) may be written as a rational function of the periodic function $\mathrm{sn}^2(x,k)$, multiplied by its derivative $2\mathrm{sn}(x,k)\mathrm{cn}(x,k)\mathrm{dn}(x,k)$. As a consequence the average of this integrand is zero. Thus all these values of ζ belong to the Lax spectrum. Extra care should be taken when $\zeta = 0$, in which the denominator in (2.34) is singular, and not integrable. This case may be dealt with separately. One finds that the vector part of (2.32) cancels the singularity in $\gamma(x)$. In fact, the two eigenfunctions of the first equation of (2.27) are

$(-\mathrm{dn}(x,k), k\mathrm{cn}(x,k))^T$ and $(-k\mathrm{cn}(x,k), \mathrm{dn}(x,k))^T$.

Next, we consider the case where Ω is real, requiring $(1-k)/2 < |\zeta| < (1+k)/2$. Similar to the above, the integrand contains many terms of the form $R(\mathrm{sn}^2(x,k))(\mathrm{sn}^2(x,k))'$, where R is a rational, nonsingular function. The average of such terms vanishes, leaving a single term $-4\Omega\zeta\mathrm{sn}^2(x,k)/(4\zeta^2\mathrm{sn}^2(x,k)+\mathrm{cn}^2(x,k)\mathrm{dn}^2(x,k))$. This term is of fixed sign and never results in zero average. The corresponding values of ζ are not in σ_L.

In summary, we have established that

$$\sigma_L = (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup [\zeta_4, \infty). \tag{2.37}$$

This set is indicated in Fig. 2.4a as a bold line. Furthermore, we find that the corresponding values of Ω are imaginary, covering the entire imaginary axis. Thus,

$$\sigma_t = i\mathbb{R}. \tag{2.38}$$

We may be more specific. The segment $\zeta \in (-\infty, \zeta_1]$ gives rise to a complete covering of the imaginary axis, as does $\zeta \in [\zeta_4, \infty)$. Next, the segment $\zeta \in [0, \zeta_3]$ gives rise to $\Omega \in [-i|\Omega_{\mathrm{min}}|, i|\Omega_{\mathrm{min}}|] = [-ik'^2/2, ik'^2/2]$, as does $\zeta \in [\zeta_2, 0]$. Thus, there is an interval on the imaginary axis around the origin that is quadruple covered, while the rest of the imaginary axis is double covered.

Thus

$$\sigma_t = (i\mathbb{R})^2 \cup \left[-\frac{ik'^2}{4}, \frac{ik'^2}{4}\right]^2, \tag{2.39}$$

where the exponents denote multiplicities.

2.4.2 The nontrivial-phase case: $b > 0$

The nontrivial-phase case is more complicated. First, note that the discriminant of (2.31) is $k^4k'^4 \neq 0$ for $k \neq 0, 1$. This implies that the four roots of the right-hand side of (2.31) are always real, for all values of $b > 0$. Indeed, complex roots would come about by the collision of real roots, which is not possible since the discriminant is never zero. In fact, the explicit expressions for these roots are quite simple:

$$\begin{aligned}
\zeta_1 &= \tfrac{1}{2}\left(-\sqrt{b}-\sqrt{b+k^2}-\sqrt{b+1}\right),\\
\zeta_2 &= \tfrac{1}{2}\left(\sqrt{b}+\sqrt{b+k^2}-\sqrt{b+1}\right),\\
\zeta_3 &= \tfrac{1}{2}\left(\sqrt{b}-\sqrt{b+k^2}+\sqrt{b+1}\right),\\
\zeta_4 &= \tfrac{1}{2}\left(-\sqrt{b}+\sqrt{b+k^2}+\sqrt{b+1}\right).
\end{aligned} \tag{2.40}$$

An indicative graph for Ω^2 as a function of ζ is given in Fig. 2.4b, using $k = 0.5$ and $b = 0.2$.

As for the trivial-phase case, we split the examination of the real ζ-axis in two parts: those ζ-values for which Ω is pure imaginary, and those for which Ω is real.

If $\zeta \in (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup [\zeta_4, \infty)$, then Ω is pure imaginary. As before, the integrand of (2.34) is of the form $\mathcal{R}(\mathrm{sn}^2(x,k))(\mathrm{sn}^2(x,k))'$, where $\mathcal{R}$ is a rational function of its argument, resulting in a zero average. Thus all these values of ζ are in the Lax spectrum. Again one has to consider the case where B might have zeros. It is easy to see that this occurs only when either $\zeta = c/2b$ or when $\zeta = c/2(b+k^2)$. Note that both values are in the specified ζ-range, as the corresponding values for Ω^2 are negative. Although the expressions of the corresponding eigenfunctions are not as compact as for the trivial-phase case, one easily shows that all singularities of $\gamma(x)$ cancel with roots of the vector part of (2.32). Thus these values are legitimate members of the Lax spectrum.

Next, if Ω is real, up to terms with zero average, the integrand may be written as

$$\frac{2\Omega}{P(\mathrm{sn}^2(x,k))}(k^s\mathrm{sn}^2(x,k)+b)(c-2\zeta(k^2\mathrm{sn}^2(x,k)+b)), \tag{2.41}$$

where $P(\mathrm{sn}^2(x,k))$ is a polynomial with no real roots. Unlike the trivial-phase case, the numerator of this expression has roots for $-K(k) < x < K(k)$, and it is not obvious to see that its average is nonzero. We use a more abstract argument. The left-hand side of (2.34) depends analytically on both b and ζ, at least for $\zeta \in (\zeta_3, \zeta_4)$ and $b > 0$. For convenience, we denote this left-hand side as $F(\zeta, b)$. Thus, elements of σ_L are real values of ζ for which $F(\zeta, b) = 0$. An identical argument holds for $\zeta \in (\zeta_1, \zeta_2)$. It should be noted that ζ_3 and ζ_4 depend on b (see (2.36)), but since ζ_3 and ζ_4 are always well separated, this is no cause for concern. For a fixed value of b, and using the analytical dependence of $F(\zeta, b)$ on ζ, it follows that $F(\zeta, b)$ is either identically zero, or has isolated zeros. If $F(\zeta, b)$ were to have isolated zeros, these would correspond to isolated points in

σ_L. Since σ_L is the spectrum of a period problem, this is not possible [81]. Thus we investigate the possibility that for a fixed value of $b > 0$, $F(\zeta, b)$ is identically zero for all $\zeta \in (\zeta_3, \zeta_4)$. We know this is not true for $b = 0$. Due to the analytic dependence on b, it follows that it is not true for $0 < b \le b_1$, for b_1 sufficiently small. The last possibility to examine is whether there can exist a value of $b > b_1$ for which $F(\zeta, b)$ is identically zero as a function of ζ. If we think of the spectra σ_L parameterized by increasing values of b, this would imply the sudden presence of a continuous subset of σ_L out of a vacuum: *i.e.*, this subset would not emerge from or be connected to other parts of σ_L. Since σ_L depends continuously on its parameters [49], this is not possible. We conclude that $F(\zeta, b)$ has no zeros if Ω is real.

It follows that our conclusions are identical to those for the trivial-phase case. Specifically, we have established (2.37) and (2.38). As before, the set σ_L is indicated in Fig. 2.4b as a bold line. Analogously to (2.39), we may write

$$\sigma_t = (i\mathbb{R})^2 \cup \left[-i\sqrt{|\Omega^2_{\min}|}, i\sqrt{|\Omega^2_{\min}|}\right]^2, \tag{2.42}$$

where the exponents denote multiplicities, as before. Here Ω^2_{min} is the minimal value of Ω^2 as a function of ζ. This value depends on the two parameters b and k. If desired, it can be calculated using Cardano's formulae, but we do not subject the reader to its explicit form. The perfect agreement between the numerics and this analytical result is illustrated in Fig. 4.5 for $b = 0.2$, for varying k.

2.5 Spectral stability

The connection between the eigenfunctions of the Lax pair (2.25) and the eigenfunctions of the linear stability problem (2.16) for the defocusing NLS equation (2.1) is well known [1, 2, 41, 72, 80]. It is convenient to phrase the result using the form (2.12) of the solutions. Letting

$$\Psi(x,t) = e^{-i\omega t + i\kappa x}\tilde{R}(x,t), \tag{2.43}$$

where κ is the average value of $c/R^2(x)$, with $R(x)$ the amplitude of the stationary solution under consideration, as before. We see that the periodic part $\hat{R}(x)$ of the considered elliptic solution (2.12) is a stationary solution of

$$i\tilde{R}_t = -\omega\tilde{R} - \frac{1}{2}\tilde{R}_{xx} - i\kappa\tilde{R}_x + \frac{\kappa^2}{2}\tilde{R} + \tilde{R}|\tilde{R}|^2. \tag{2.44}$$

To linearize around the elliptic solution with $\hat{R}(x) = \hat{R}_1(x) + i\hat{R}_2(x)$, we let

$$\tilde{R}(x,t) = \hat{R}(x) + \epsilon(w_1(x,t) + iw_2(x,t)) + \mathcal{O}(\epsilon^2), \tag{2.45}$$

which results in

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = J\hat{\mathcal{L}}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{2.46}$$

with

$$\hat{\mathcal{L}} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{1}{2}\kappa^2 + 3\hat{R}_1^2(x) + \hat{R}_2^2(x) - \omega & \kappa\partial_x + 2\hat{R}_1\hat{R}_2 \\ -\kappa\partial_x + 2\hat{R}_1\hat{R}_2 & -\frac{1}{2}\partial_x^2 + \frac{1}{2}\kappa^2 + \hat{R}_1^2(x) + 3\hat{R}_2^2(x) - \omega \end{pmatrix}. \tag{2.47}$$

It should be noted that although $\hat{R}(x)$ is a periodic solution of (2.44), it is not necessary for $\tilde{R}(x,t)$ to be periodic. Indeed, we wish to allow for infinitesimal perturbations (2.45) that are bounded and sufficiently smooth, but otherwise arbitrary. Noting the independence of $J\hat{\mathcal{L}}$ on t, we separate variables as before,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}, \tag{2.48}$$

to obtain the spectral problem

$$\lambda\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = J\hat{\mathcal{L}}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}. \tag{2.49}$$

We easily prove the following theorem.

Theorem 1. The vector $(w_1, w_2)^T = (e^{-i\kappa x}\chi_1^2 + e^{i\kappa x}\chi_2^2, -ie^{-i\kappa x}\chi_1^2 + ie^{i\kappa x}\chi_2^2)^T$ satisfies the linear stability problem (2.46). Here $\chi = (\chi_1, \chi_2)^T$ is any solution of (2.25) with the corresponding elliptic solution $\phi(x) = R(x)e^{i\theta(x)} = \hat{R}(x)e^{i\kappa x}$.

Proof. The proof is by direct calculation: calculate $\partial_t(w_1, w_2)^T$ using the product rule and the second equation of (2.25). Alternatively, calculate $(w_1, w_2)_t^T$ using (2.46), substituting $(w_1, w_2)^T = (e^{-i\kappa x}\chi_1^2 + e^{i\kappa x}\chi_2^2, -ie^{-i\kappa x}\chi_1^2 + ie^{i\kappa x}\chi_2^2)^T$. In both expressions so obtained, eliminate x-derivatives of w_1 and w_2 (up to order 2) using the first equation of (2.25). The resulting expressions are equal, finishing the proof.

Remarks.

- It is possible to repeat this proof for any solution $\Psi(x,t)$ of (2.1). It is not necessary that the solution is a stationary elliptic solution.

- Despite the different forms of the spectral stability problem (compare (2.49) with (2.22)), it is clear that they determine the same spectra, with different but equivalent eigenfunctions. Indeed, if an eigenfunction (W_1, W_2) corresponds to an element of the spectrum λ for (2.49), then there is a corresponding eigenfunction (U, V) with the same spectral element λ for (2.22). Thus, there is no confusion when we use (2.49) to determine the stability spectrum of an elliptic solution of (2.1).

To establish the spectral stability of the elliptic solutions of the defocusing NLS equation (2.1), we need to establish that *all* bounded solutions (W_1, W_2) of (2.49) are obtained through the squared-eigenfunction connection by

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = e^{2\Omega t} \begin{pmatrix} e^{-i\kappa x}\varphi_1^2 + e^{i\kappa x}\varphi_2^2 \\ -ie^{-i\kappa x}\varphi_1^2 + ie^{i\kappa x}\varphi_2^2 \end{pmatrix} \tag{2.50}$$

If we manage to do so then by comparing with (2.48) we immediately conclude that

$$\lambda = 2\Omega. \tag{2.51}$$

Since $\sigma_t = i\mathbb{R}$, we conclude that the stability spectrum is given by

$$\sigma(\mathcal{L}) = \sigma(\mathcal{L}_\kappa) = i\mathbb{R}. \tag{2.52}$$

In order to obtain this conclusion, we need the following theorem.

Theorem 2. *All but six* solutions of (2.49) are obtained through (2.50), where $\varphi = (\varphi_1, \varphi_2)^T$ solves the first equation of (2.27) and (2.30). Specifically, *all* solutions of (2.49) bounded on the whole real line are obtained through the squared eigenfunction connection (2.50), with one exception corresponding to $\lambda = 0$.

Proof. For any given value of $\lambda \in \mathbb{C}$, (2.49) can be written as four-dimensional first-order system of ordinary differential equations. Thus, for any value of $\lambda \in \mathbb{C}$, (2.49) has four linearly independent solutions. On the other hand, we have already shown (Theorem 1) that (2.50) provides solutions of this ordinary differential equation. Let us count how many solutions are obtained this way, for a fixed value of λ. For any value of $\lambda \in \mathbb{C}$, exactly one value of $\Omega \in \mathbb{C}$ is obtained through $\Omega = \lambda/2$. Excluding the six values of λ for which the discriminant of (2.31) as a function of ζ is zero (these turn out to be only the values of λ for which Ω^2 reaches its maximum or minimum value in Fig. 2.4), (2.31) gives rise to four values of $\zeta \in \mathbb{C}$. It should be noted that we are not restricting ourselves to $\zeta \in \sigma_L$ now, since the boundedness of the solutions is not a concern in this counting argument. Next, for a given pair $(\Omega, \zeta) \in \mathbb{C}^2$, (2.30) defines a unique solution of the system consisting of the first equation of (2.27) and (2.30). Thus, any choice of $\lambda \in \mathbb{C}$ not equal to the six values mentioned above, gives rise to exactly four solutions of (2.49), through the squared eigenfunction connection of Theorem 1. Before we consider the six excluded values, we need to show that the four solutions $(W_1(x), W_2(x))^T$ just obtained are linearly independent. As in [14], there are two parts to this.

1. If there is an exponential contribution to $(W_1, W_2)^T$ from $\gamma(x)$ then an argument similar to that given in [14] establishes the linear independence of the four solutions.

2. As in [14], the only possibility for the exponential factor due to $\gamma(x)$ not to contribute is for the integrand in that factor to be proportional to a logarithmic derivative. It is easily checked that this occurs only for $\lambda = 0 = \Omega$. It is a tedious calculation to verify that the four solutions $(W_1(x), W_2(x))^T$ obtained through the squared eigenfunction connection are linearly dependent. In fact, no two of them are linearly independent. Using the invariances of the equation, one can construct two bounded and two unbounded solutions. Unlike for the

KdV equation [14], no linear combination of the unbounded solutions is bounded. Thus, in this case, three of the solutions of (2.49) are not obtained through the squared eigenfunction connection, one of which is bounded.

For the six excluded values, three linearly independent solutions of (2.49) are found. The fourth one may be constructed using reduction of order, and introduces algebraic growth. Extra care is required for the trivial-phase case, for which both maxima are equal, but the same conclusion follows. For the two λ values for which Ω^2 reaches its minimum value, the two solutions obtained from (2.50) are bounded, thus these values of λ are part of the spectrum. The two values of λ for which Ω^2 reaches its maximum value only give rise to unbounded solutions and are not part of the spectrum.

We conclude that *all* but one of the bounded solutions of (2.48) are obtained through the squared eigenfunction connection. This finishes the proof.

Remark. It is important to remember that the algebraically growing solutions discussed above (corresponding to $\lambda = 0 = \Omega$) do not lead to solutions of (2.49) through the squared eigenfunction connection. Indeed, those solutions do not solve the second equation of (2.27), and therefore Theorem 1 does not apply to them. If it did, eight solutions would be obtained corresponding to $\lambda = 0$.

The above considerations are summarized in the following theorem.

Theorem 3. (Spectral Stability) The elliptic solutions of the NLS equation (2.1) are spectrally stable. The spectrum of their associated linear stability problem (2.49) (or (2.22)) is explicitly given by $\sigma(\mathcal{L}) = i\mathbb{R}$, or, accounting for multiple coverings,

$$\sigma(\mathcal{L}) = (i\mathbb{R})^2 \cup \left[-2i\sqrt{|\Omega^2_{\min}|}, 2i\sqrt{|\Omega^2_{\min}|}\right]^2, \tag{2.53}$$

where $|\Omega^2_{\min}|$ is as before.

If follows from Theorem 3 that the value of $\lambda_c(b, k)$ in Fig. 4.5 is given by

$$\lambda_c(b, k) = 2\sqrt{|\Omega^2_{\min}|}, \tag{2.54}$$

which is the expression for the solid curve in Fig. 4.5.

Similar to the calculations in [14], we could obtain parametric representations for the Floquet parameter and the imaginary part of the spectrum as a function of ζ. This would reproduce the curves in Fig. 4.3.

2.6 Nonlinear stability

In this section we consider the nonlinear stability of the elliptic solutions. To facilitate our study, we rewrite the NLS equation in real coordinates $(r(x,t), l(x,t))$, where $\Psi(x,t) = r(x,t) + il(x,t)$. Then (2.1) becomes

$$\begin{pmatrix} r \\ l \end{pmatrix}_t = \begin{pmatrix} -\frac{1}{2}l_{xx} + l(r^2+l^2) \\ \frac{1}{2}l_{xx} - r(r^2+l^2) \end{pmatrix}. \tag{2.55}$$

In the (r,l) coordinates we denote the stationary solution as $r^*(x) + il^*(x)$:

$$r^*(x) + il^*(x) = \phi(x) = e^{i\theta(x)}R(x). \tag{2.56}$$

This is the same simplification done in Section (2.5), but without the term $e^{i\kappa}$ factored out.

We allow for perturbations whose amplitude is periodic with period equal to an integer multiple of the minimal period $2T$ of $R(x)$, i.e., subharmonic perturbations. In order to properly define the higher-order equations in the NLS hierarchy that are necessary for our stability argument (see below), we need (r,l) and its derivatives of up to order three to be square-integrable. Therefore, we consider (2.55) on the function space

$$\mathbb{V} = \left\{ v \in H^3([-NT,NT]) \times H^3([-NT,NT]) : ||v(x+2NT)|| = ||v(x)|| \right\}, \tag{2.57}$$

for a fixed positive integer N, equipped with inner-product

$$\langle u, v \rangle = \int_{-NT}^{NT} \overline{u} \cdot v \, dx. \tag{2.58}$$

2.6.1 Hamiltonian structure and the NLS hierarchy

To begin, we reformulate the NLS equation as a Hamiltonian system. Using the real formulation above, we write (2.55) in Hamiltonian form [73]

$$\begin{pmatrix} r \\ l \end{pmatrix}_t = JH'(r,l) \tag{2.59}$$

on $\mathbb{V}$. Here J is the skew symmetric operator

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.60}$$

the Hamiltonian H is the functional

$$H(r,l) = \int_{-NT}^{NT} \left(\frac{1}{4}(r_x^2 + l_x^2) + \frac{1}{4}(r^2+l^2)^2 \right) dx, \tag{2.61}$$

and H' denotes the variational derivative of H

$$H'(r,l) = \sum_{i=0}^{\infty} \begin{pmatrix} (-1)^i \partial_x^i \frac{\partial H}{\partial r_{ix}} \\ (-1)^i \partial_x^i \frac{\partial H}{\partial l_{ix}}, \end{pmatrix} \tag{2.62}$$

where the sum in (4.11) terminates at the order of the highest derivatives involved.

Note: We can also write (4.1) in Hamiltonian form without switching to real coordinates with

$$H(\Psi) = \int_{-NT}^{NT} \left(\frac{1}{2}|\Psi_x|^2 + \frac{1}{4}|\Psi|^4 \right) dx \tag{2.63}$$

and $J = -i$. However, representation (4.8) is useful when examining the spectral properties of the linearization about an equilibrium solution. In what follows, by complex coordinates we mean the representation in terms of Ψ.

By virtue of its integrability, the NLS equation possesses an infinite number of conserved quantities $H_0, H_1, H_2, \ldots$, and just as the functional $H_2 = H$ defines the NLS equation, each H_i defines a Hamiltonian system with time variable τ_i through

$$\begin{pmatrix} r \\ l \end{pmatrix}_{\tau_i} = JH_i'(r,l). \tag{2.64}$$

This defines an infinite hierarchy of equations, the NLS hierarchy. It has the following properties:

- All the functionals H_i, $i = 0, 1, \ldots$, are conserved for each member of the NLS hierarchy (4.13).

- The flows of the NLS hierarchy (4.13) mutually commute, and we can think of Ψ as solving all of these equations simultaneously, i.e., $\begin{pmatrix} r \\ l \end{pmatrix} = \begin{pmatrix} r(\tau_0, \tau_1, \ldots) \\ l(\tau_0, \tau_1, \ldots) \end{pmatrix}$ [31].

As all the flows in the NLS hierarchy commute, we may take any linear combination of the above Hamiltonians to define a new Hamiltonian system. For our purposes, we define the *n-th NLS equation* with time variable t_n as

$$\begin{pmatrix} r \\ l \end{pmatrix}_{t_n} = J\hat{H}_n'(r,l), \tag{2.65}$$

where each $\hat{H}_n$ is defined as

$$\hat{H}_n := H_n + \sum_{i=0}^{n-1} c_{n,i} H_i, \quad \hat{H}_0 := H_0, \tag{2.66}$$

for constants $c_{n,i}$, $i = 0, \ldots, n-1$. For now these constants are undetermined. We later fix them as necessary.

Since every member of the nonlinear hierarchy (4.13) is integrable, each possesses a Lax pair, the collection of which is known as the *linear NLS hierarchy*. The first three members of the linear hierarchy are:

$$\chi_{\tau_0} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix} \chi, \tag{2.67}$$

$$\chi_{\tau_1} = \begin{pmatrix} \frac{\zeta}{2} & \frac{i}{2}(r+il) \\ \frac{i}{2}(r+il) & -\frac{\zeta}{2} \end{pmatrix} \chi, \tag{2.68}$$

$$\chi_{\tau_2} = \begin{pmatrix} -i(\zeta^2 + \frac{1}{2}(r^2+l^2)) & \zeta(r+il) + \frac{i}{2}(r_x + il_x) \\ \zeta(r-il) + \frac{i}{2}(r_x - il_x) & i(\zeta^2 + \frac{1}{2}(r^2+l^2)) \end{pmatrix} \chi. \tag{2.69}$$

We construct the Lax pair for the n-th NLS equation (4.23) by taking the same linear combination of the lower-order flows as we did for the nonlinear hierarchy, and define the *n-th linear NLS equation* [2] as

$$\chi_{t_n} = \hat{T}_n \psi = \begin{pmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n \end{pmatrix} \chi, \tag{2.70}$$

$$\hat{T}_n := T_n + \sum_{i=0}^{n-1} c_{n,i} T_i, \quad \hat{T}_0 := T_0. \tag{2.71}$$

2.6.2 Stationary solutions

Stationary solutions of the NLS hierarchy are defined as solutions such that

$$\begin{pmatrix} r \\ l \end{pmatrix}_{t_n} = 0 \tag{2.72}$$

for some integer n and constants $c_{n,0}, \ldots, c_{n,n-1}$ in (4.23-4.24). Thus, a stationary solution of the n-th NLS equation satisfies the ordinary differential equation

$$J\hat{H}_n'(r,l) = 0 \tag{2.73}$$

with independent variable x. Since J is invertible, this is equivalent to

[2] Not to be confused with the linear Schrödinger equation.

$$\hat{H}_n'(r,l) = 0. \tag{2.74}$$

The stationary solutions have the following properties:

- Since all the flows commute, the set of stationary solutions is invariant under any of the NLS equations, i.e., a stationary solution of the n-th equation remains a stationary solution of the n-th equation after evolving under any of the other flows.

- Any stationary solution of the n-th NLS equation is also stationary with respect to all of the higher order time variables t_m, $m > n$. In such cases, the constants $c_{m,i}$, $i \geq n$ are free parameters. We make use of this fact when constructing a Lyapunov function later.

Returning to solutions of the form $\Psi(x,t) = e^{-i\omega t}\phi(x) = e^{-i\omega t}(r^*(x) + il^*(x))$, we see that $\phi(x)$ satisfies the ordinary differential equation

$$-\frac{1}{2}\phi_{xx} + \phi|\phi|^2 - \omega\phi = 0. \tag{2.75}$$

However this is just the second stationary NLS equation (in complex coordinates)

$$\Psi_{t_2} = -i\hat{H}_2' = -i(H_2' + c_{2,1}H_1' + c_{2,0}H_0') = 0, \tag{2.76}$$

with

$$c_{2,1} = 0, \quad c_{2,0} = \omega = k^2/2 + 3b/2 + 1/2. \tag{2.77}$$

Furthermore, this solution is stationary with respect to all the higher-order NLS equations as well. For example, it is a stationary solution of the fourth NLS equation

$$\hat{H}_4'(r,l) = 0, \tag{2.78}$$

with

$$c_{4,1} \; = \; 4ic + c_{4,3}(k^2 + 1 + 3b) \tag{2.79}$$

$$c_{4,0} \; = \; (1/2(k^2+1)+3/2b)c_{4,2}+2icc_{4,3}-k^4/2-2k^2-5bk^2-1/2-15b^2/2-5b \tag{2.80}$$

for any values of $c_{4,3}$ and $c_{4,2}$.

2.6.3 Stability

First, let us express the linear stability results from the previous sections in the (r, l) coordinates. We linearize about the equilibrium solution (r^*, l^*)

$$r(x,t) = r^* + \epsilon w_1(x,t) + \mathcal{O}(\epsilon^2), \quad l(x,t) = l^* + \epsilon w_2(x,t) + \mathcal{O}(\epsilon^2), \tag{2.81}$$

resulting in the linear system

$$w_t = J\mathcal{L}w. \tag{2.82}$$

Here the symmetric differential operator $\mathcal{L} = \hat{H}_2''(r^*, l^*)$ is the Hessian of $\hat{H}_2$,

$$\hat{H}_2''(r,l) = \begin{pmatrix} -\frac{1}{2}\partial_{xx} + 3r^2 + l^2 - \frac{1}{2}\omega & 2rl \\ 2rl & -\frac{1}{2}\partial_{xx} + 3l^2 + r^2 - \frac{1}{2}\omega \end{pmatrix}, \tag{2.83}$$

evaluated at the stationary solution.

Again, we let $w(x,t) = e^{\lambda t}W(x)$ and consider the eigenvalue problem

$$\lambda W = J\mathcal{L}W. \tag{2.84}$$

By relating solutions of (2.82) to those of the stability problem considered in the previous sections, one can establish the squared eigenfunction connection

$$\lambda = 2\Omega, \quad W(x) = \begin{pmatrix} \chi_1^2 + \chi_2^2 \\ i(\chi_2^2 - \chi_1^2) \end{pmatrix}, \tag{2.85}$$

where χ_1, χ_2, and $\Omega \in i\mathbb{R}$ are defined as before.

Now, consider the problem of nonlinear stability. The NLS equation is invariant under rotation in the complex plane and translation in x. These symmetries are represented by the Lie group

$$G = \mathbb{R} \times S^1, \tag{2.86}$$

which acts on $\Psi(x,t)$ according to

$$T(g)\Psi(x,t) = e^{i\gamma}\Psi(x + x_0, t), \quad g = (x_0, \gamma) \in G, \tag{2.87}$$

or in real coordinates

$$T(g)\begin{pmatrix} r(x,t) \\ l(x,t) \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}\begin{pmatrix} r(x+x_0,t) \\ l(x+x_0,t) \end{pmatrix} \quad g = (x_0, \gamma) \in G. \tag{2.88}$$

Stability is considered modulo these symmetries. We use the following definition.

Definition: The stationary solution (r^*, l^*) is *orbitally stable* in $\mathbb{V}$ if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that if $(r(x,0), l(x,0)), (r^*(x), l^*(x)) \in \mathbb{V}$ then

$$||(r(x,0), l(x,0)) - (r^*(x), l^*(x))|| < \delta \Rightarrow \inf_{g \in G} ||(r(x,t), l(x,t)) - T(g)(r^*(x), l^*(x))|| < \epsilon,$$

where $||\cdot||$ is the norm obtained through $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$.

To prove orbital stability, we search for a Lyapunov function. For Hamiltonian systems, this is a constant of the motion, $K(r,l)$, for which (r^*, l^*) is an unconstrained minimum:

$$\frac{\partial}{\partial t}K(r,l) = 0, \quad K'(r^*, l^*) = 0, \quad \langle v, K''(r^*, l^*)v \rangle > 0, \quad \forall v \in \mathbb{V},\ v \neq 0. \tag{2.89}$$

We obtain an infinite number of candidate Lyapunov functions through the NLS hierarchy.

Linearizing (4.23) about the equilibrium solution (r^*, l^*) gives

$$w_{t_n} = J\mathcal{L}_n w, \tag{2.90}$$

where $\mathcal{L}_n$ is the Hessian of $\hat{H}_n$ evaluated at the stationary solution. Through the same squared eigenfunction connection we have

$$2\Omega_n W(x) = J\mathcal{L}_n W(x), \tag{2.91}$$

where Ω_n is defined through

$$\chi(x, t_n) = e^{\Omega_n t_n} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{2.92}$$

and due to the commuting property of the flows, the Lax hierarchy shares the common set of eigenfunctions $\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$ from before (still assuming the solution is stationary with respect to the second flow). Substituting (2.92) into the second equation in (4.57) determines a relationship between Ω_n and ζ, and in general Ω_n^2 defines a genus n Riemann surface. When evaluated at a stationary solutions of the NLS equation, Ω_n^2 takes a degenerate form.

Theorem. Let (r^*, l^*) be a stationary solution of the second NLS equation. Then for all $n > 2$, the n-th surface reduces to

$$\Omega_n^2(\zeta) = p_n(\zeta)^2\Omega^2(\zeta), \tag{2.93}$$

where $p_n(\zeta)$ is a polynomial of degree $n-2$ in ζ. Furthermore, $p_n(\zeta)$ depends on the free parameters $c_{n,2}, \ldots, c_{n,n-1}$ such that $c_{n,i}$ appears in the coefficients of ζ^{i-2} and lower. Therefore, the free parameters $c_{n,2}, \ldots, c_{n,n-1}$ give us total control over the roots of $p_n(\zeta)$.

Proof. The proof is a special case of the proof for the finite-genus solutions of the KdV equation in Chapter 4. When evaluated at a stationary solution of the NLS equation, all the higher-order flows become linearly dependent. The theorem is a consequence of this linear dependence and the functional form the Lax operators take as polynomials in ζ.

With the above facts established, we return to orbital stability. Just as we considered the norm of a solution modulo symmetries, we shall in effect do the same when considering a Lyapunov function. We have the following theorem due to [42, 68]:

Orbital Stability Theorem. Let (r^*, l^*) be a spectrally stable equilibrium solution of equation (4.8) such that the eigenfunctions W of the linear stability problem (2.84) form a basis for $\mathbb{V}$. Furthermore, suppose there exists an integer $n \geq 2$ and constants $c_{n,0}, \ldots, c_{n,n-1}$ such that the Hamiltonian for the n-th equation in the nonlinear hierarchy satisfies the following:

1. The kernel of $\hat{H}_n''(r^*, l^*)$ is spanned by the infinitesimal generators of the symmetry group G acting on (r^*, l^*).

2. For all eigenfunctions W corresponding to nonzero eigenvalues

$$K_n(W) := \langle W, \hat{H}_n''(r^*, l^*)W \rangle > 0.$$

Then (r^*, l^*) is orbitally stable in $\mathbb{V}$.

Let us consider the implications of this theorem for the problem at hand:

- It was established in [45] that the eigenfunctions W form a basis for

$$\mathbb{V} = \left\{ v \in H^3([-NT, NT]) \times H^3([-NT, NT]) : ||v(x + 2NT)|| = ||v(x)|| \right\}. \quad (2.94)$$

- The kernel of $\hat{H}_2''(r^*, l^*)$ has geometric multiplicity two when considered on $\mathbb{V}$ (see [38] for instance). In complex coordinates, the infinitesimal generators corresponding to phase invariance and translational invariance are i and ∂_x respectively. Therefore, the two linearly independent solutions iQ^*, Q_x^* span the two-dimensional null space of $\hat{H}_2''(Q^*)$. Furthermore, when evaluated at the equilibrium solution $\hat{H}_n''(Q^*)$, $n \geq 2$, and $\hat{H}_2''(Q^*)$ share the same kernel, which can be seen from the Riemann surface relations. Relating this back to r and l gives that the kernel of $\hat{H}_n''(r^*, l^*)$ is spanned by the vectors

$$\ker(\hat{H}_n''(r^*, l^*)) = Span\left\{\begin{pmatrix} -l^* \\ r^* \end{pmatrix}, \begin{pmatrix} r_x^* \\ l_x^* \end{pmatrix}\right\}, \tag{2.95}$$

What is left to verify is condition (2) in the nonlinear stability theorem, i.e., to prove orbital stability we need to find an n such that

$$K_n = \langle W, \mathcal{L}_n W \rangle = \int_{-NT}^{NT} \overline{W} \cdot \mathcal{L}_n W dx \geq 0, \tag{2.96}$$

with equality obtained only on the kernel of $\mathcal{L}_n$, i.e., only for $\Omega = 0$.

To calculate the higher order K_n, we make use of the following. Assume our solution is an equilibrium solution of the n-th flow. Then from equation (4.40) we have

$$\mathcal{L}_n W = 2\Omega_n J^{-1} W. \tag{2.97}$$

This gives

$$K_n = \int_{-NT}^{NT} \overline{W} \cdot \mathcal{L}_n W dx = 2\Omega_n \int_{-NT}^{NT} \overline{W} \cdot J^{-1} W dx. \tag{2.98}$$

Using that (r^*, l^*) is a stationary solution of the second flow and substituting for Ω_n in the above gives

$$K_n(\zeta) = \Omega_n(\zeta) \frac{K_2(\zeta)}{\Omega(\zeta)} = p_n(\zeta) K_2(\zeta). \tag{2.99}$$

Therefore, when considering stationary solutions of the defocusing NLS equation, one simply needs to calculate K_2 in order to calculate any of the higher order K_i. Let us do so. From (2.84) we have

$$\mathcal{L} W = 2\Omega J^{-1} W = 2\Omega \begin{pmatrix} -W_2 \\ W_1 \end{pmatrix}. \tag{2.100}$$

This gives

$$\overline{W} \cdot \mathcal{L}W = 2\Omega J^{-1}(-\overline{W}_1 W_2 + \overline{W}_2 W_1). \tag{2.101}$$

Using the explicit form of W gives

$$\overline{W} \cdot \mathcal{L}W = -i\left(|\chi_1|^4 - |\chi_2|^4 - (\overline{\chi_1}\chi_2)^2 + (\overline{\chi_2}\chi_1)^2\right), \tag{2.102}$$

where $\chi_1 = -\gamma \hat{B}_2$, $\chi_2 = \gamma(\hat{A}_2 - \Omega)$. First, calculate the norm of γ. Using $\phi = r^* + il^*$, we have

$$\gamma = \frac{1}{\hat{A}_2 - \Omega} \exp \int \left(-\frac{(\overline{\phi})\hat{B}_2}{\hat{A}_2 - \Omega} + i\zeta \right) \, dx, \tag{2.103}$$

up to a multiplicative constant. The above integrand simplifies to

$$\begin{aligned}
-\frac{(\overline{\phi})\hat{B}_2}{\hat{A}_2 - \Omega} &= i\frac{\mathrm{Re}(\overline{\phi}\hat{B}_2)}{\mathrm{Im}(\hat{A}_2 - \Omega)} - \frac{\mathrm{Im}(\overline{\phi}\hat{B}_2)}{\mathrm{Im}(\hat{A}_2 - \Omega)} \\
&= i\frac{\mathrm{Re}(\overline{\phi}\hat{B}_2)}{\mathrm{Im}(\hat{A}_2 - \Omega)} - \frac{\partial_x(\phi\overline{\phi})}{4(-\zeta^2 - \frac{1}{2}\phi\overline{\phi} + \frac{\omega}{2} - \mathrm{Im}(\Omega))} \\
&= i\frac{\mathrm{Re}(\overline{\phi}\hat{B}_2)}{\mathrm{Im}(\hat{A}_2 - \Omega)} + \frac{1}{2}\partial_x(\ln(\mathrm{Im}(\hat{A}_2 - \Omega))).
\end{aligned}$$

Therefore (2.103) becomes

$$\gamma = \frac{-i}{\sqrt{\Im(\hat{A}_2 - \Omega)}} \exp \int i \left(\frac{\Re(\overline{\phi}\hat{B}_2)}{\Im(\hat{A}_2 - \Omega)} + \zeta \right) \, dx, \tag{2.104}$$

giving

$$|\gamma|^2 = \frac{1}{\Im(\hat{A}_2 - \Omega)}. \tag{2.105}$$

Along with $\Omega^2 = \hat{A}_2^2 + |\hat{B}_2|^2$, the above implies

$$|\chi_2|^2 = \Im(\hat{A}_2 - \Omega), \quad |\chi_1|^2 = \Im(\hat{A}_2 + \Omega), \quad (\overline{\chi_1}\chi_2)^2 = -\overline{\hat{B}_2}^2. \tag{2.106}$$

Therefore, K_2 is given by

$$K_2 = -16\Omega^2 \int i\hat{A}_2 dx = -16\Omega^2 \int \left(\zeta^2 + \frac{1}{2}(r^{*2} + l^{*2}) - \frac{1}{2}c_{2,0}\right)\, dx, \tag{2.107}$$

with $c_{2,0} = \omega$.

Let us revisit the second condition of the nonlinear stability theorem. Using $\Omega^2 = \hat{A}_2^2 + |\hat{B}_2|^2$ we see that K_2 can be zero only if $\Omega \geq 0$. Also, we see that $K_2 \geq 0$ for all values of ζ or it changes signs at exactly two values $\pm\zeta_0 \notin \sigma_L$. If the first case is true, then we have proven orbital stability and we are done. In what follows, we assume the second case is true. Then $K_2 < 0$ for $\zeta \in \sigma_L$ such that $-\zeta_0 < \zeta < \zeta_0$ and $K_2 > 0$ for all other $\zeta \in \sigma_L$. Therefore, no conclusion with regard to stability can drawn from K_2. Let us go two flows higher. When evaluated at (r^*, l^*), a direct calculation shows that Ω_4^2 simplifies to

$$\Omega_4^2 = (4\zeta^2 + 2ic_{4,3}\zeta - c_{4,2} + 3b + k^2 + 1)^2\Omega_2^2, \tag{2.108}$$

with $c_{4,0}$ and $c_{4,1}$ as in (2.79). Choosing $c_{4,3} = 0$ and $c_{4,2} = 4\zeta_0^2 + 3b + k^2 + 1$ makes $K_4(\zeta) \geq 0$ for all $\zeta \in \sigma_L$ with equality obtained only when $\Omega = 0$.

We have proved the following theorem:

Theorem. There exist constants $c_{4,2}$, $c_{4,3}$, such that K_4 is positive on the Lax spectrum. Therefore, the equilibrium solution (r^*, l^*) is orbitally stable with respect to subharmonic perturbations, i.e., (r^*, l^*) is orbitally stable in the function space

$$\mathbb{V} = \left\{v \in H^3([-NT, NT]) \times H^3([-NT, NT]) : ||v(x + 2NT)|| = ||v(x)||\right\}.$$

Chapter 3

STABILITY OF TRAVELING WAVE SOLUTIONS OF THE MKDV EQUATION

The modified Korteweg-de Vries (mKdV) equation is given by

$$u_t + 6\delta u^2 u_x + u_{xxx} = 0, \tag{3.1}$$

where $\delta = -1$ corresponds to the defocusing case and $\delta = 1$ corresponds to the focusing case. It arises in many of the same physical contexts as the KdV equation, such as water waves and plasma physics, but in different parameter regimes.

It is well known that the mKdV equation possesses the periodic traveling wave solutions

$$u = k\mathrm{sn}(x - (-k^2 - 1)t, x), \tag{3.2}$$

in the defocusing case, and

$$u = k\mathrm{cn}(x - (2k^2 - 1)t, x), \quad u = \mathrm{dn}(x - (-k^2 + 2)t, x), \tag{3.3}$$

in the focusing case (though these do not constitute all periodic traveling wave solutions, see Section 3.1). The orbital stability of the dn solutions was first studied in [5], where they were proved to be orbitally stable with respect to periodic perturbations of the same period. However, as noted in [24], the proof fails for the other solutions mentioned above. More recently in [24], a modified version of the Bloch-decomposition and counting techniques in [45, 59] to Hamiltonian equations with a singular Poisson structure was developed. It is proven there that the sn and dn solutions are orbitally stable with respect to periodic perturbations of the same period for all values of elliptic modulus k. The dynamics of the cn solutions changes from from stable to unstable as the elliptic modulus passes through a fixed value k^*. However, the accompanying numerical investigation of the spectral stability of the cn solutions with respect to subharmonic perturbations suggests instability for *all* values of the elliptic modulus.

Stability results for other subclasses of solutions of the mKdV equation have been obtained in recent years as well. In [45], the spectral stability of small amplitude periodic traveling wave solutions with respect to periodic perturbations of the same period was established. This result was recently extended beyond spectral stability in the work of [56]. Through the use of the periodic instability index developed in [16] in combination with a periodic version of the Evans function technique employed in [76, 77], it was proven that the small amplitude solutions are orbitally stable with respect to periodic perturbations of the same period. The same result is also established for solutions in neighborhoods of homoclinic orbits.

There are two limitations in all the results discussed above: (i) They are restricted to special cases of traveling wave solutions. (ii) Only periodic perturbations of the same period are considered. Here we examine the spectral and (nonlinear) orbital stability of *all* traveling wave solutions of the mKdV equation with respect to *subharmonic* perturbations. Due to difficulties that arise with the spectral parameter in the Lax pair for the focusing mKdV equation (see Section 3.7), we first restrict ourselves to the defocusing case, setting $\delta = -1$. After deriving all traveling wave solutions of the defocusing mKdV equation in terms of the Weierstrass elliptic function (surprisingly, this result appears to be new), we analytically prove that all bounded traveling wave solutions are spectrally and orbitally stable with respect to subharmonic perturbations. We then return to the focusing case. We derive all periodic traveling wave solutions, and employ a combination of analytic and numerical techniques to study their stability

3.1 Traveling wave solutions

Here we derive all periodic traveling wave solutions of the defocusing mKdV equation (see Section 3.7 for their derivation in the focusing case). We employ a technique originally due to the work of Poincaré, Painlevé, Briot, and Bouquet [84], though most recently reformulated in [22, 23].

Note: A large class of solutions of the mKdV equation in terms of the Weierstrass elliptic function is derived using a different method in [88]. However, it is straightforward to check that they do not constitute the full set of periodic traveling wave solutions.

To examine traveling wave solutions, we change to a moving coordinate frame

$$y = x - Vt, \quad \tau = t. \tag{3.4}$$

In the (y, τ) coordinates the mKdV equation becomes

$$u_\tau - Vu_y - 6u^2 u_y + u_{yyy} = 0. \tag{3.5}$$

We look for stationary solutions $u_\tau = 0$, i.e, time-independent solutions of (3.5). Letting $u(y, \tau) = U(y)$, stationary solutions satisfy the ordinary differential equation

$$-VU_y - 6U^2 U_y + U_{yyy} = 0. \tag{3.6}$$

Integrating (3.6) gives

$$-VU - 2U^3 + U_{yy} = C, \tag{3.7}$$

for some constant C. Multiplying (3.7) by U_y and integrating a second time gives

$$-\frac{V}{2}U^2 - \frac{1}{2}U^4 + \frac{1}{2}U_y^2 - CU = E, \tag{3.8}$$

for some constant E. Therefore, all stationary solutions $U(y)$ satisfy the first-order ordinary differential equation (3.8).

Defining the new variable $v = U_y$, (3.8) becomes

$$(v)^2 - \frac{1}{2}U^4 - \frac{V}{2}U^2 - CU - E = 0. \tag{3.9}$$

This is a genus one algebraic curve [35], birationally equivalent to (using the normal form algorithm found in [85])

$$r^2 = 4s^3 - g_2 s - g_3, \tag{3.10}$$

where

$$U = R(s, r), \quad v = S(s, r), \tag{3.11}$$

and

$$g_2 = \frac{4}{3}V^2 + 32E, \quad g_3 = -\frac{8}{27}V^3 + \frac{64}{3}VE - 16C^2. \tag{3.12}$$

As the curve (3.10) is in Weierstrass form, it can be parameterized in terms of the Weierstrass $\wp$-function $\wp(x)$, with

$$r = \wp'(\omega x), \quad s = \wp(\omega x), \tag{3.13}$$

for some constant ω. Transforming back to our original variables gives

$$U = R(\wp(\omega x), \wp'(\omega x)), \quad v = S(\wp(\omega x), \wp'(\omega x)). \tag{3.14}$$

Imposing our original assumption, $U_y = v$, gives $\omega = \frac{1}{2}$. Thus our final solution is

$$U(y) = \frac{\pm\sqrt{2E}\wp'(\frac{1}{2}(y+y_0), g_2, g_3) + C(2\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{2}{3}V)}{\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} - 2\sqrt{2E}\right)\left(\wp(\frac{1}{2}(y+y_0), g_2, g_3) - \frac{V}{3} + 2\sqrt{2E}\right)}. \tag{3.15}$$

Here y_0 is an arbitrary shift in y determined by the initial conditions. These solutions are doubly periodic in the complex plane. When considered on the real line, they have period $2T$ determined by

$$2T = 4\int_{e_1}^{\infty} \frac{1}{\sqrt{4z^3 - g_2 z - g_3}}\, dz, \tag{3.16}$$

where e_1 is the largest root of the equation obtained by setting $r = 0$ in (3.10). This gives all periodic solutions due to a classic theorem by Briot and Bouquet [84].

We now determine which values of V, C, and E give rise to bounded periodic solutions. Letting $v = U_y$ in (3.7), we have the first-order two-dimensional system

$$U_y = v, \quad v_y = VU + 2U^3 + C. \tag{3.17}$$

All fixed points (U_0, v_0) satisfy

$$v_0 = 0, \;\; VU_0 + 2U_0^3 + C = 0. \tag{3.18}$$

After linearizing about $(U_0, 0)$, the resulting linear system has eigenvalues

$$\lambda = \pm\sqrt{V + 6U_0^2}. \tag{3.19}$$

We have two saddles and a center when the discriminant of the second equation in (3.18)

$$d = -8V^3 - 108C^2 \tag{3.20}$$

is greater than zero, and one saddle when the discriminant is less than zero. Therefore, we can only expect periodic solutions for $V < 0$ and $d > 0$ which gives

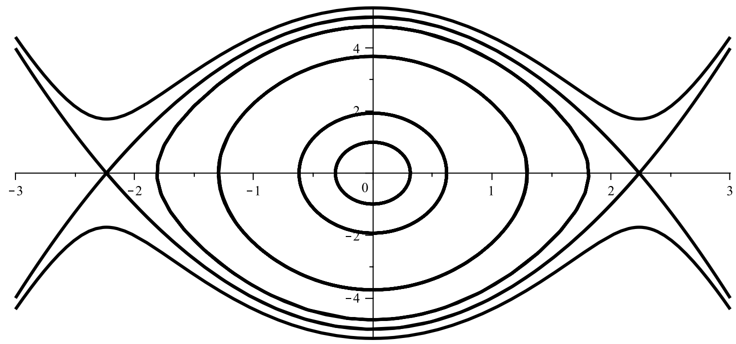
$$|C| < \sqrt{\frac{-8V^3}{108}}. \tag{3.21}$$

Using (3.8), we see that for fixed V and C the phase space is foliated by the family of curves

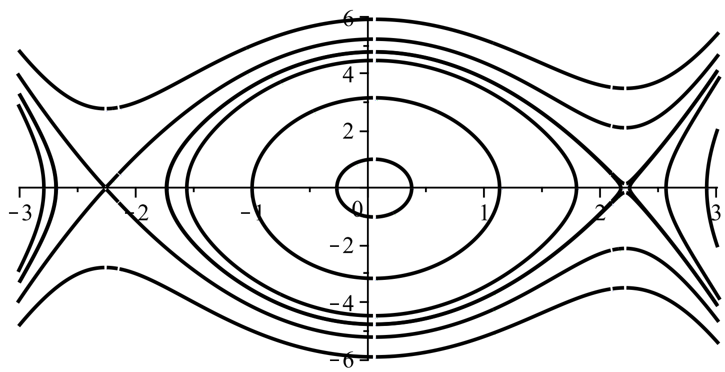
$$v^2 = U^4 + VU^2 + 2CU + 2E. \tag{3.22}$$

The parameter E is specified by the initial condition. Periodic solutions are separated from unbounded solutions by two heteroclinc orbits in the case $C = 0$, and by one homoclinc orbit in the case $C \neq 0$, see Fig. 3.1. All values of E which give rise to a solution lying inside the separatrix result in periodic solutions. Thinking of the right-hand side of (3.22) as a polynomial in U, all values of E which make its discriminant positive give rise to periodic solutions. For $C = 0$ we can write the solution in the particularly simple form

$$U(y) = \pm k\sqrt{\frac{-V}{1+k^2}}\mathrm{sn}\left(\sqrt{\frac{-V}{1+k^2}}y, k\right), \tag{3.23}$$



(a)



(b)

Figure 3.1: (a) Typical (U, v) phase plane in the defocusing case for $C = 0$ (here $V = -10$). (b) For $C \neq 0$ (here $C = 0.5$, $V = -10$), the heteroclinc orbits break into a single homoclinic orbit. The homoclinic orbit persists until $|C| = \sqrt{\frac{-8V}{108}}$.

where E is parameterized by the elliptic modulus k

$$E = \frac{k^2 V^2}{2(k^4 + 2k^2 + 1)}. \tag{3.24}$$

3.2 The linear stability problem

Before we study the orbital stability of the stationary solutions, we examine their spectral and linear stability. To this end, we consider perturbations of a stationary solution

$$u(y, \tau) = U(y) + \epsilon w(y, \tau) + \mathcal{O}(\epsilon^2), \tag{3.25}$$

where ϵ is a small parameter. Substituting this in (3.5) and ignoring higher-than-first-order terms in ϵ, we find

$$w_\tau = 6U^2 w_y + 12UU_y w - w_{yyy} + V w_y, \tag{3.26}$$

at first order in ϵ. The zeroth order terms vanish since $U(y)$ solves the mKdV equation. By ignoring the higher-order terms in ϵ, we are restricting our attention to examining linear stability. The traveling wave solution is defined to be *linearly stable* if for all $\epsilon > 0$, there is a $\delta > 0$ such that if $||w(y,0)|| < \delta$ then $||w(y,\tau)|| < \epsilon$ for all $\tau > 0$. This definition depends on our choice of the norm $||\cdot||$, to be determined later.

Next, since (3.26) is autonomous in time, we may separate variables. Let

$$w(y,\tau) = e^{\lambda\tau} W(y,\lambda), \tag{3.27}$$

then $W(y,\lambda)$ satisfies

$$-W_{yyy} + (V + 6U^2)W_y + 12UU_y W = \lambda W, \tag{3.28}$$

or

$$J\mathcal{L}W = \lambda W, \quad J = \partial_y, \quad \mathcal{L} = -\partial_{yy} + V + 6U^2. \tag{3.29}$$

In what follows, the λ dependence of W will be suppressed. To avoid confusion with other spectra arising below, we refer to $\sigma(J\mathcal{L})$ as the *stability spectrum*.

3.3 Numerical Results

Before we determine the spectrum of (3.29) analytically, we compute it numerically, using Hill's method [26]. Hill's method is ideally suited to a problem such as (3.29) with periodic coefficients. It allows us to compute all eigenfunctions of the form

$$W = e^{i\mu y}\hat{W}(y), \quad \hat{W}(y+2T) = \hat{W}(y), \tag{3.30}$$

with $\mu \in [-\pi/4T, \pi/4T)$. It follows from Floquet's theorem that *all* bounded solutions of (3.29) are of this form. Here bounded means that $\sup_{x\in\mathbb{R}} |W(x)|$ is finite. Thus $W \in C_b^0(\mathbb{R})$. On the other hand, we also have $W \in L^2(-T, T)$ (the square-integrable functions of period $2T$) since the exponential factor in (3.30) disappears in the computation of the L^2-norm. Thus

$$W \in C_b^0(\mathbb{R}) \cap L^2(-T, T). \tag{3.31}$$

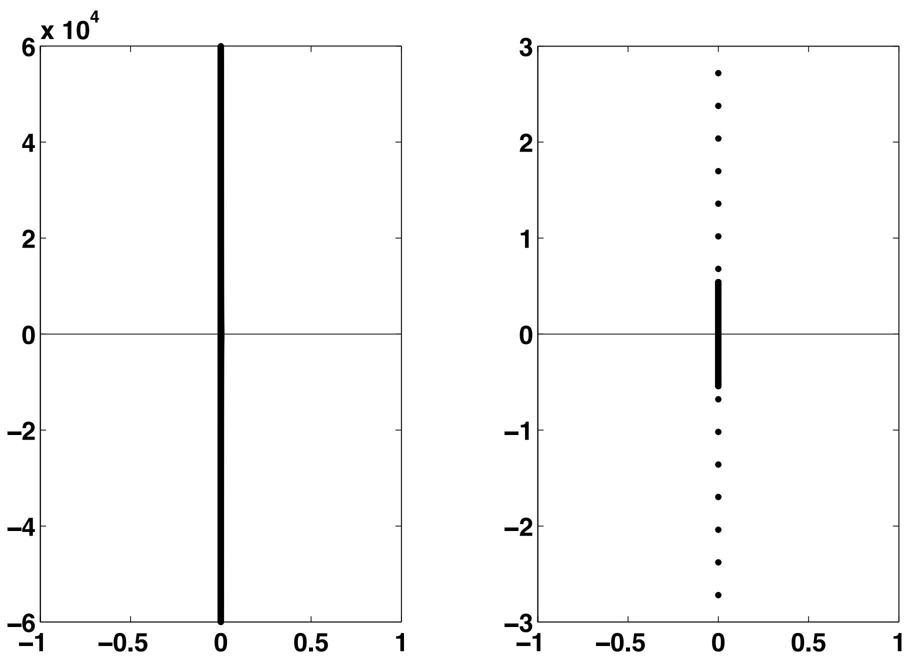
It should be noted that by this choice our investigations include perturbations of an arbitrary period that is an integer multiple of $2T$, *i.e.*, subharmonic perturbations.

Figure (3.2) shows discrete approximations to the spectrum of (3.29), computed using SpectrUW 2.0 [25]. The solution parameters are $V = -10$, $C = 0$, and $E \approx 11.9$ ($k = 0.8$). The numerical parameters (see [26, 25]) are $N = 40$ (81 Fourier modes) and $D = 80$ (79 different Floquet exponents). The right panel (b) is a blow-up of the left panel (a) around the origin. First, it appears that the spectrum is on the imaginary axis, indicating spectral stability of the solution. Second, the numerics shows that a symmetric band around the origin has a higher spectral density than does the rest of the imaginary axis. This is indeed the case, as shown in more detail in Fig. 3.4, where the imaginary parts $\in [-1.5, 1.5]$ of the computed eigenvalues are displayed as a function of the Floquet parameter μ (here 199 different Floquet exponents were used). This shows that λ values with $\mathrm{Im}(\lambda) \in [-0.54, 0.54]$ (approximately) are attained for three different μ values in $[-\pi/4T, \pi/4T)$. The rest of the imaginary axis is only attained for one μ value. This picture persists if a larger portion of the imaginary λ axis is examined. These numerical results are in perfect agreement with the theoretical results below.

Figure 3.3 shows discrete approximations to the spectrum for $C \neq 0$. The solution parameters are $V = -10$, $C = 10\sqrt{(}15)/9$, and $E \approx -1$. We see the same structure as the $C = 0$ case.

3.4 Lax pair representation

Equation (3.6) is equivalent to the compatibility of two linear ordinary differential systems:



(a) (b)

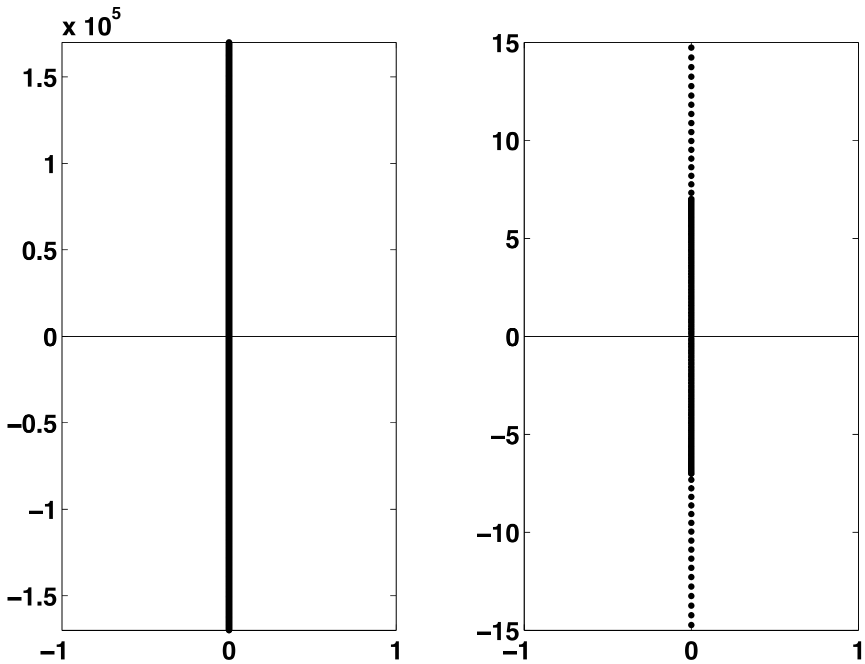
Figure 3.2: (a) The numerically computed spectrum for the traveling wave solution with $V = -10$, $C = 0$, and $E = 11.9$ ($k = .8$) using Hill's method with 81 Fourier modes and 79 different Floquet exponents, see [26, 25]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density;

$$\psi_y = \begin{pmatrix} -i\zeta & u \\ u & i\zeta \end{pmatrix} \psi, \tag{3.32}$$

$$\psi_\tau = \begin{pmatrix} (-V\zeta - 4\zeta^3 - 2\zeta u^2)i & Vu + 4\zeta^2 u + 2u^3 - u_{yy} + 2\zeta u_y i \\ Vu + 4\zeta^2 u + 2u^3 - u_{yy} - 2\zeta u_y i & -(-V\zeta - 4\zeta^3 - 2\zeta u^2)i \end{pmatrix} \psi. \tag{3.33}$$

In other words, the compatibility condition $\psi_{y\tau} = \psi_{\tau y}$ requires that u satisfies the defocusing mKdV equation. We can rewrite (3.32) as the spectral problem

$$\begin{pmatrix} i\partial_y & -iu \\ iu & -i\partial_y \end{pmatrix} \psi = \zeta\psi. \tag{3.34}$$



(a) (b)

Figure 3.3: (a) The numerically computed spectrum for the traveling wave solution with $V = -10$, $C = 10\sqrt{(15)}/9$, and $E \approx -1$ using Hill's method with 81 Fourier modes and 79 different Floquet exponents, see [26, 25]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density;

This problem is self-adjoint, therefore $\zeta \in \mathbb{R}$. Evaluating (3.32-3.33) at the stationary solution $u(y,\tau) = U(y)$, we find

$$\psi_y = \begin{pmatrix} -i\zeta & U \\ U & i\zeta \end{pmatrix} \psi, \tag{3.35}$$

$$\psi_\tau = \begin{pmatrix} (-V\zeta - 4\zeta^3 - 2\zeta U^2)i & VU + 4\zeta^2 U + 2U^3 - U_{yy} + 2\zeta U_y i \\ VU + 4\zeta^2 U + 2U^3 - U_{yy} - 2\zeta U_y i & -(-V\zeta - 4\zeta^3 - 2\zeta U^2)i \end{pmatrix} \psi. \tag{3.36}$$

We refer to the set of all ζ values such that (3.35-3.36) has bounded solutions as the *Lax spectrum* σ_L. Since the spectral problem (3.34) is self-adjoint, the Lax spectrum is a subset of the real line:

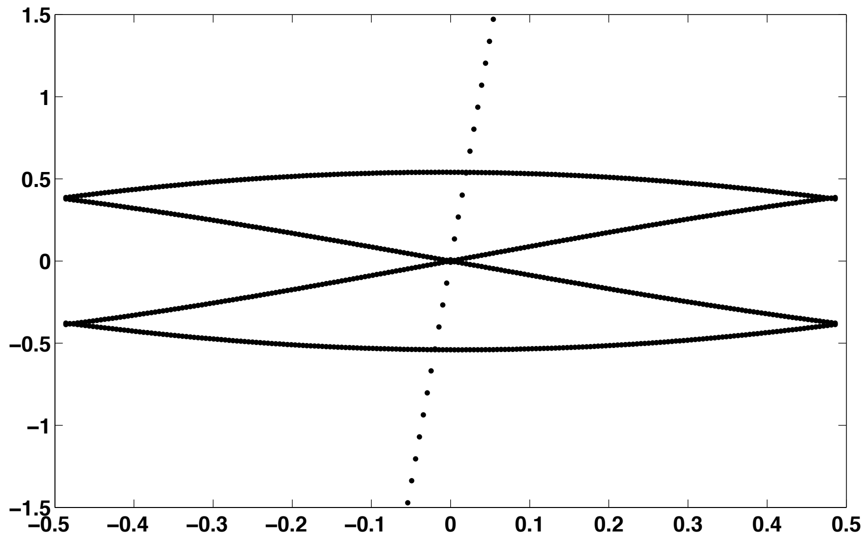


Figure 3.4: The imaginary part of λ as a function of μ, demonstrating the higer spectral density. The parameter values are identical to those of Fig. 3.2, except 199 different Floquet exponents were used here.

$\sigma_L \subset \mathbb{R}$. The goal of this section is to determine this subset explicitly. In the next section, we connect the Lax spectrum to its stability spectrum.

Equation (3.36) simplifies. Using (3.7) to eliminate U_{yy} gives

$$\psi_\tau = \begin{pmatrix} (-V\zeta - 4\zeta^3 - 2\zeta U^2)i & 4\zeta^2 U + C + 2\zeta U_y i \\ 4\zeta^2 U + C - 2\zeta U_y i & -(-V\zeta - 4\zeta^3 - 2\zeta U^2)i \end{pmatrix} \psi = \begin{pmatrix} A & B \\ \overline{B} & -A \end{pmatrix} \psi. \tag{3.37}$$

Since A and B do not explicitly depend on τ, we separate variables

$$\psi(y,\tau) = e^{\Omega\tau} \begin{pmatrix} \alpha(y) \\ \beta(y) \end{pmatrix}. \tag{3.38}$$

Substituting (3.38) into (3.37) and canceling the exponential, we find

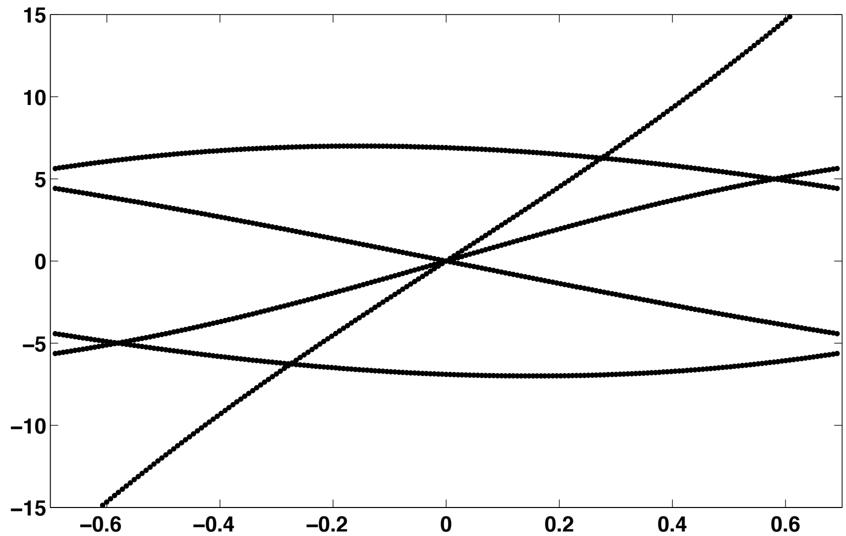


Figure 3.5: The imaginary part of λ as a function of μ, demonstrating the higer spectral density. The parameter values are identical to those of Fig. 3.3, except 199 different Floquet exponents were used here.

$$\begin{pmatrix} A & B \\ \overline{B} & -A \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0. \tag{3.39}$$

This implies that the existence of nontrivial solutions requires

$$\Omega^2 = A^2 + |B|^2 = -16\zeta^6 - 8V\zeta^4 - (V^2 + 8E)\zeta^2 + C^2 = 0, \tag{3.40}$$

where we have used the explicit form of the stationary solution $U(y)$ derived earlier. This determines Ω in terms of the spectral parameter ζ. Thinking of Ω^2 as a polynomial in ζ^2, one finds that the discriminant of (3.40) has the same sign as the discriminant of the polynomial (3.22). As discussed earlier, this discriminant is positive for periodic solutions. Also, Ω^2 is an even function of ζ. Therefore, for periodic stationary solutions, (3.40) can be written as

$$\Omega^2 = -16(\zeta - \zeta_1)(\zeta + \zeta_1)(\zeta - \zeta_2)(\zeta + \zeta_2)(\zeta - \zeta_3)(\zeta + \zeta_3) \tag{3.41}$$

for some positive constants $0 \leq \zeta_3 < \zeta_2 < \zeta_1$.

The eigenvector corresponding to the eigenvalue Ω is

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \gamma(y) \begin{pmatrix} -B \\ A - \Omega \end{pmatrix}, \tag{3.42}$$

where γ is a scalar function of x. It is determined by substitution of the above into the first equation of the Lax pair, resulting in a first-order scalar differential equation for γ. This equation may be solved explicitly giving

$$\gamma = \exp \int \left(i\zeta - \frac{A'}{A - \Omega} - \frac{UB}{A - \Omega} \right) dy, \tag{3.43}$$

up to a multiplicative constant. This simplifies to

$$\gamma = \frac{1}{A - \Omega} \exp \int \left(i\zeta - \frac{UB}{A - \Omega} \right) dy. \tag{3.44}$$

Each value of ζ results in two values of Ω (except for the six branch points $\pm\zeta_i$, $i = 1, 2, 3$, where $\Omega = 0$) and therefore (3.42) represents two eigenvectors. These solutions are clearly linearly independent. For those values of ζ for which $\Omega = 0$, only one solution is generated. A second one may be found using reduction of order, resulting in algebraically growing solutions.

To determine the Lax spectrum σ_L, we need to determine the set of all $\zeta \subset \mathbb{R}$ such that (3.42) is bounded as a function of x. Thus, we need to determine for which ζ the scalar function $\gamma(x)$ is bounded. First, one can readily check that the only values of ζ for which the denominator in (3.42) is singular are the branch points $\pm\zeta_i$, $i = 1, 2, 3$, where $\Omega = 0$. One finds that the vector part of (3.42) cancels the singularity in $\gamma(y)$. Thus, $\pm\zeta_i$, $i = 1, 2, 3$, are part of the Lax spectrum. For all other values of ζ, it is necessary and sufficient that

$$\left\langle \Re \left(-\frac{uB}{A - \Omega} \right) \right\rangle = 0. \tag{3.45}$$

Here $\langle \ \cdot \ \rangle = \frac{1}{T} \int_{-L}^{L} \ \cdot \ dy$ denotes the average over a period. The explicit form of the above depends on whether Ω is real or imaginary. It should be noted that since $\zeta \in \mathbb{R}$, it follows from (3.40) that these are the only possibilities. Let us investigate each case separately:

- If Ω is imaginary then

$$-\frac{UB}{A-\Omega} = \frac{-U\mathrm{Re}(B)}{\mathrm{Im}(A-\Omega)}i + \frac{\partial_y\left(\frac{1}{2}\mathrm{Im}(A-\Omega)\right)}{\mathrm{Im}(A-\Omega)}, \tag{3.46}$$

  where we used that

$$\partial_y\left(\frac{1}{2}(\mathrm{Im}(A)-\Omega)\right) = -2UU_y = -U\mathrm{Im}(B). \tag{3.47}$$

  The first term in (3.46) is imaginary and the second term is a total derivative, thus giving zero average. Therefore, all ζ values that make Ω imaginary are in the Lax spectrum.

- If Ω is real, then ignoring total derivatives one finds

$$\left\langle \mathrm{Re}\left(-\frac{UB}{A-\Omega}\right)\right\rangle = \left\langle \frac{U\mathrm{Re}(B)}{\Omega^2+(\mathrm{Im}(A))^2}\right\rangle\Omega = \left\langle \frac{4\zeta^2U^2+CU}{\Omega^2+(\mathrm{Im}(A))^2}\right\rangle\Omega = 0. \tag{3.48}$$

  The average term above is obviously non-zero for $C=0$. A similar argument as for the case of non-trivial phase in the defocusing NLS equation gives that this average term is never zero. Therefore, Ω must be identically zero, and all values of ζ for which Ω is real are not part of the Lax spectrum.

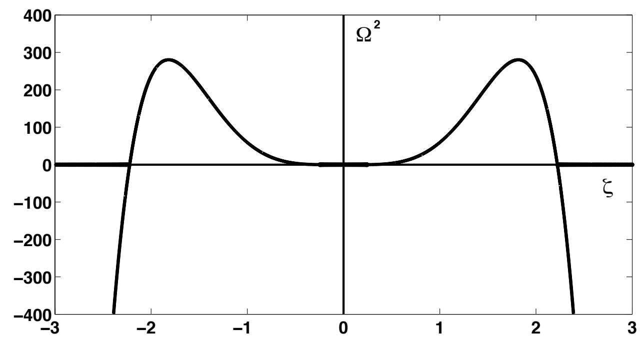
We conclude that the Lax spectrum consists of all ζ values for which $\Omega^2 \leq 0$:

$$\sigma_L = (-\infty,-\zeta_1]\cup[-\zeta_2,-\zeta_3]\cup[\zeta_3,\zeta_2]\cup[\zeta_1,\infty), \tag{3.49}$$

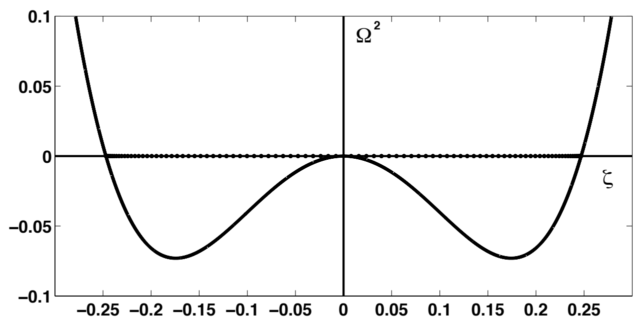
and Ω is purely imaginary

$$\Omega \in i\mathbb{R} \tag{3.50}$$

for all $\zeta \in \sigma_L$. In fact, Ω^2 takes on all negative values for $\zeta \in (-\infty,-\zeta_1]$, implying that $\Omega = \pm\sqrt{|\Omega^2|}$ covers the imaginary axis. The same is true of the segment $\zeta \in [\zeta_1,\infty)$. Furthermore, for



(a)



(b)

Figure 3.6: (a) Ω^2 as a function of real ζ, with the same paramter values as Fig. 3.2. The union of the dotted line segments is the numerically computed Lax spectrum (in the complex ζ-plane) with 81 Fourier modes and 49 different Floquet exponents. (b) A blow-up of (a) around the origin;

$\zeta \in [-\zeta_2, -\zeta_3]$, Ω^2 takes on all negative values in $[\Omega^2(\zeta^*), 0]$ twice, where $\Omega^2(\zeta^*)$ is the minimal value of Ω^2 attained for $\zeta \in [-\zeta_2, -\zeta_3]$. Since Ω^2 is an even function of ζ, the same is true of the segment $[\zeta_3, \zeta_2]$. Upon taking square roots, this implies that the interval on the imaginary axis $\left[-i\sqrt{|\Omega^2(\zeta^*)|}, i\sqrt{|\Omega^2(\zeta^*)|}\right]$ is covered six times, while the rest of the imaginary axis is double covered. Symbolically, we write [14]

$$\Omega \in (i\mathbb{R})^2 \cup \left[-i\sqrt{|\Omega^2(\zeta^*)|}, i\sqrt{|\Omega^2(\zeta^*)|}\right]^4, \tag{3.51}$$

where the exponents denote multiplicities (see Figs. 3.6 and 3.7).

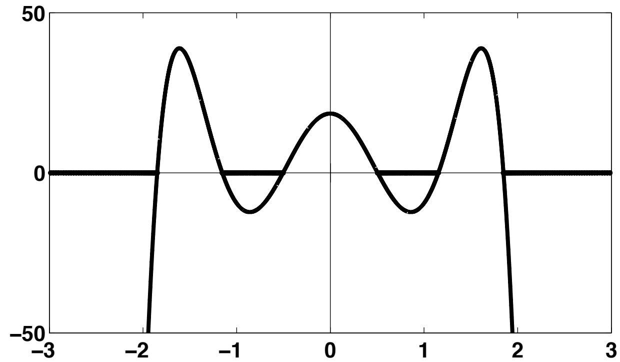


Figure 3.7: Ω^2 as a function of real ζ, for $C \neq 0$ (same paramter values as Fig. 3.3). The union of the dotted line segments is the numerically computed Lax spectrum (in the complex ζ-plane) with 81 Fourier modes and 49 different Floquet exponents.

3.5 Spectral stability

It is well known that there exists a connection between the eigenfunctions of the Lax pair of an integrable equation and the eigenfunctions of the linear stability problem for this integrable equation [1, 2, 41, 72, 80]. A direct calculation proves that the function

$$w(y,\tau) = \psi_1^2(y,\tau) + \psi_2^2(y,\tau) = \frac{1}{2i\zeta}\partial_y\left(\psi_2^2(y,\tau) - \psi_1^2(y,\tau)\right) \tag{3.52}$$

satisfies the linear stability problem (3.26). Here $\psi = (\psi_1, \psi_2)^T$ is any solution of (3.35-3.36) with the corresponding stationary solution $U(y)$.

In order to establish the spectral stability of equilibrium solutions of (3.6), we need to establish that *all* bounded solutions $W(y)$ of (3.28) are obtained through the squared-eigenfunction connection by

$$W(y) = \alpha^2(y) + \beta^2(y). \tag{3.53}$$

If we manage to do so then we may immediately conclude that

$$\lambda = 2\Omega. \tag{3.54}$$

Since $\Omega \in i\mathbb{R}$, we conclude that the stability spectrum is given by

$$\sigma(J\mathcal{L}) = i\mathbb{R}. \tag{3.55}$$

In order to obtain this conclusion, we need the following theorem.

Theorem. *All but six* solutions of (3.28) may be written as $W(y) = \alpha^2(y) + \beta^2(y)$, where $(\alpha, \beta)^T$ solves (3.35,3.36). Specifically, *all* solutions of (3.28) bounded on the whole real line are obtained through the squared eigenfunction connection, with one exception corresponding to $\lambda = 0$.

Proof. For any given value of $\lambda \in \mathbb{C}$, (3.28) is a third-order linear ordinary differential equation. Thus, it has three linearly independent solutions. On the other hand, we have already shown (see the previous theorem) that the formula

$$W(y) = \alpha^2(y) + \beta^2(y) \tag{3.56}$$

provides solutions of this ordinary differential equation. Let us count how many solutions are obtained this way, for a fixed value of λ. For any value of $\lambda \in \mathbb{C}$, exactly one value of $\Omega \in \mathbb{C}$ is obtained through $\Omega = \lambda/2$. Excluding the six values of λ for which the discriminant of (3.40) as a function of ζ is zero (these are the only values of λ for which Ω^2 reaches its maximum or minimum value, keeping in mind that Ω^2 is an even function of ζ), (3.40) gives rise to six values of $\zeta \in \mathbb{C}$. It should be noted that we are not restricting ourselves to $\zeta \in \sigma_L$ now, since the boundedness of the solutions is not a concern in this counting argument. Next, for a given pair $(\Omega, \zeta) \in \mathbb{C}^2$, (3.42) defines a unique solution of (3.35, 3.36). Thus, any choice of $\lambda \in \mathbb{C}$ not equal to the six values mentioned above, gives rise to exactly six solutions of (3.28), through the squared eigenfunction connection. Let us examine how many of these solutions are linearly independent.

- Since Ω^2 is an even function of ζ, the six values of ζ mentioned above come in pairs: $\pm\zeta_i$, $i = 1, 2, 3$. It can be checked that if ζ corresponds to the eigenfunction W, then $-\zeta$ corresponds to its complex conjugate $\overline{W}$. Therefore, when considering the general solution to the linear stability problem $w = a_1 e^{\Omega t} W + a_2 e^{-\Omega t} \overline{W}$, half of the ζ values provide no new solutions. Also, if there is an exponential contribution from $\gamma(y)$ then an argument similar to that in [14] establishes the linear independence of the remaining three solutions.

- As in [14], the only possibility for the exponential factor from $\gamma(y)$ not to contribute is if $\lambda = 0 = \Omega$. Only one linearly independent solution is obtained through the squared eigenfunction connection, corresponding to translational invariance, $W = U_y$. The other two can be obtained through reduction of order. Just as for the KdV equation in [14], this allows one to construct two solutions whose amplitude grows linearly in x. A suitable linear combination of these solutions is bounded. Thus, corresponding to $\lambda = 0$ there are two eigenfunctions. One of these is obtained through the squared eigenfunction connection.

Lastly, consider the six excluded values of λ. For the two λ values where Ω^2 reaches a local minimum, the two solutions obtained through the squared eigenfunction connection are bounded, thus, these values of λ are part of the spectrum. The third solution may be constructed using reduction of order, and introduces algebraic growth. For the two values of λ where Ω^2 obtains a global maximum, three solutions are obtained through the squared eigenfunction connection, all of which are unbounded. For the other two λ values where Ω^2 reaches a local maximum, two solutions are obtained through the squared eigenfunction connection, both of which are unbounded. The third solution may be constructed using reduction of order, and introduces algebraic growth. ■

We have established the following theorem.

Theorem (Spectral Stability). The periodic traveling wave solutions of the defocusing mKdV equation are spectrally stable. The spectrum of their associated linear stability problem is explicitly given by $\sigma(J\mathcal{L}) = i\mathbb{R}$, or, accounting for multiple coverings,

$$\sigma(J\mathcal{L}) = i\mathbb{R} \cup \left[-2i\sqrt{|\Omega^2(\zeta^*)|}, 2i\sqrt{|\Omega^2(\zeta^*)|}\right]^2, \tag{3.57}$$

where $|\Omega^2(\zeta^*)|$ is as before.

Remark: As previously mentioned, for a fixed value of Ω, only three of the six solutions (corresponding to the six different values of ζ) obtained through the squared eigenfunction connection contribute as independent solutions to the linear stability problem. Therefore, the double and sextuple coverings in the Ω representation (3.51) drop to single and triple coverings in (3.57).

3.6 Nonlinear stability

3.6.1 Hamiltonian structure

To begin, we reformulate the defocusing mKdV equation as a Hamiltonian system. We are concerned with the stability of $2T$-periodic traveling wave solutions of equation (3.1) with respect to subharmonic perturbations of period $2NT$ for any fixed positive integer N. Therefore, we naturally consider solutions u in the space of square-integrable functions of period $2NT$, $L^2_{per}[-NT, NT]$. In order to properly define the higher-order equations in the mKdV hierarchy that are necessary for our stability argument (see Section 3.6.2), we further require u and its derivatives of up to order two to be square-integrable as well. Therefore, we consider solutions of (3.5) defined on the function space

$$\mathbb{V} = H^2_{per}[-NT, NT], \tag{3.58}$$

equipped with natural inner product

$$\langle v, w \rangle = \int_{-NT}^{NT} \bar{v} w \, dx, \tag{3.59}$$

where $\bar{v}$ denotes the complex conjugate of v.

We write the mKdV equation in Hamiltonian form

$$u_\tau = JH'(u) \tag{3.60}$$

on $\mathbb{V}$. Here J is the skew symmetric operator

$$J = \partial_y, \tag{3.61}$$

the Hamiltonian H is the functional

$$H(u) = \int_{-NT}^{NT} \left(\frac{1}{2} u_y^2 + \frac{1}{2} V u^2 + \frac{1}{2} u^4 \right) dy, \tag{3.62}$$

and the notation G' denotes the variational derivative of G

$$G'(u) = \sum_{i=0}^{\infty} (-1)^i \partial_y^i \frac{\partial G(u)}{\partial u_{iy}}, \tag{3.63}$$

where the sum in (3.63) terminates at the order of the highest derivatives involved. For instance, in the computation of H' the sum terminates after accounting for first derivative terms.

We allow for perturbations in a function space $\mathbb{V}_p \subset \mathbb{V}$. In order to apply the stability result of [43], we follow [24] and restrict ourselves to the space of perturbations on which J has a well defined and bounded inverse. This amounts to fixing the spatial average of u on $H^2_{per}[-NL, NL]$, which poses no problem since it is a Casimir of the Poisson operator J, hence, it is conserved under the mKdV flow. Therefore, we consider perturbations in $\ker(J)^\perp$, i.e., zero-average subharmonic perturbations

$$\mathbb{V}_p = \left\{ v \in H^2_{per}([-NL, NL]) : \int_{-NL}^{NL} v \; dx = 0 \right\}. \tag{3.64}$$

Remark. Physically, requiring perturbations to be zero-average makes sense. It simply says that we do not consider perturbations which add mass to the system.

3.6.2 The mKdV hierarchy

By virtue of its integrability, the mKdV equation possesses an infinite number of conserved quantities $H_0, H_1, H_2, \ldots$, and just as the functional $H_1 = H$ defines the mKdV equation, each H_i defines a Hamiltonian system with time variable τ_i through

$$u_{\tau_i} = JH_i'(u). \tag{3.65}$$

This defines an infinite hierarchy of equations, the mKdV hierarchy. It has the following properties:

- All the functionals H_i, $i = 0, 1, \ldots$, are conserved for each member of the mKdV hierarchy (3.65).

- The flows of the mKdV hierarchy (3.65) mutually commute, and we can think of u as solving all of these equations simultaneously, i.e., $u = u(\tau_0, \tau_1, \ldots)$ [31].

As all the flows in the mKdV hierarchy commute, we may take any linear combination of the above Hamiltonians to define a new Hamiltonian system. For our purposes, we define the *n-th mKdV equation* with time variable t_n as

$$u_{t_n} = J\hat{H}_n'(u), \tag{3.66}$$

where each $\hat{H}_n$ is defined as

$$\hat{H}_n := H_n + \sum_{i=0}^{n-1} c_{n,i} H_i, \quad \hat{H}_0 := H_0, \tag{3.67}$$

for constants $c_{n,i}$, $i = 0 \dots n-1$.

Since every member of the nonlinear hierarchy (3.65) is integrable, each possesses a Lax pair, the collection of which is known as the *linear mKdV hierarchy*. We construct the Lax pair for the n-th mKdV equation (3.66) by taking the same linear combination of the lower-order flows as we did for the nonlinear hierarchy, and define the *n-th linear mKdV equation* as

$$\psi_{t_n} = \hat{T}_n \psi = \begin{pmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n \end{pmatrix} \psi, \tag{3.68}$$

$$\hat{T}_n := T_n + \sum_{i=0}^{n-1} c_{n,i} T_i, \quad \hat{T}_0 := T_0. \tag{3.69}$$

3.6.3 Stationary solutions

Stationary solutions of the mKdV hierarchy are defined as solutions such that

$$u_{t_n} = 0 \tag{3.70}$$

for some integer n and constants $c_{n,0}, \dots, c_{n,n-1}$ in (3.66-3.67). Thus, a stationary solution of the n-th mKdV equation satisfies the ordinary differential equation

$$J\hat{H}_n'(u) = 0 \tag{3.71}$$

with independent variable y.

The stationary solutions have the following properties:

- Since all the flows commute, the set of stationary solutions is invariant under any of the mKdV equations, i.e., a stationary solution of the n-th equation remains a stationary solution after evolving under any of the other flows.

- Any stationary solution of the n-th mKdV equation is also stationary with respect to all of the higher order time variables t_m, $m > n$. In such cases, the constants $c_{m,i}$, $i \geq n$ are undetermined coefficients. We make use of this fact when constructing a Lyapunov function later.

The traveling wave solution U is a stationary solution of the first mKdV equation with $c_{1,0} = V$. It fact, it is stationary with respect to all the higher-order flows. For example, it is a stationary solution of the second mKdV equation with

$$c_{2,0} = c_{2,1}(V^2 - 4E) + V^2 - 4E \tag{3.72}$$

for any value of $c_{2,1}$.

3.6.4 Stability

Now, consider the problem of nonlinear stability. The invariance of the mKdV equation under translation is represented by the Lie group

$$G = \mathbb{R}, \tag{3.73}$$

which acts on $u(y, \tau)$ according to

$$T(g)u(y, \tau) = u(y + y_0, \tau), \quad g = y_0 \in G. \tag{3.74}$$

Stability is considered modulo this symmetry. We use the following definition.

Definition. We say the equilibrium solution U is *orbitally stable* with respect to perturbations in $\mathbb{V}_p$ if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that if $u(y,0) - U(y) \in \mathbb{V}_p$ then

$$||u(y,0) - U(y)|| < \delta \Rightarrow \inf_{g \in G} ||u(y,\tau) - T(g)U(y)|| < \epsilon.$$

To prove orbital stability, we search for a Lyapunov function. For Hamiltonian systems, this is a constant of the motion, $K(u)$, for which U is an unconstrained minimum:

$$\frac{\partial}{\partial \tau} K(u) = 0, \quad K'(U) = 0, \quad \langle v, K''(U)v \rangle > 0, \quad \forall v \in \mathbb{V}_p,\ v \neq 0. \tag{3.75}$$

We obtain an infinite number of candidate Lyapunov functions through the mKdV hierarchy.

Linearizing (3.66) about the equilibrium solution U gives

$$w_{t_n} = J\mathcal{L}_n w, \tag{3.76}$$

where $\mathcal{L}_n$ is the Hessian of $\hat{H}_n$ evaluated at the stationary solution. Through the same squared eigenfunction connection we have

$$2\Omega_n W(x) = J\mathcal{L}_n W(x), \tag{3.77}$$

where Ω_n is defined through

$$\psi(x, t_n) = e^{\Omega_n t_n} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{3.78}$$

and due to the commuting property of the flows, the Lax hierarchy shares the common set of eigenfunctions $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ from before (still assuming the solution is stationary with respect to the first flow). Substituting (3.78) into the second equation in (3.68) determines a relationship between Ω_n and ζ, and in general, Ω_n^2 is a polynomial of degree $2n+1$ in ζ^2. When evaluated at a stationary solution of the mKdV equation, Ω_n^2 takes a degenerate form.

Theorem. Let U be a stationary solution of the first mKdV equation. Then for all $n > 1$, the n-th surface reduces to

$$\Omega_n^2(\zeta) = p_n(\zeta)^2\Omega^2(\zeta), \tag{3.79}$$

where $p_n(\zeta)$ is a polynomial of degree $n-1$ in ζ^2. Furthermore, $p_n(\zeta)$ depends on the free parameters $c_{n,1}, \ldots, c_{n,n-1}$ such that we have total control over the roots when considered as a function of ζ^2.

Proof. The proof is a special case of the proof for the finite-genus solutions of the KdV equation in Chapter 4. When evaluated at a stationary solution of the mKdV equation, all the higher-order flows become linearly dependent. The theorem is a consequence of this linear dependence and the functional form the Lax operators take as polynomials in ζ.

With the above facts established, we return to nonlinear stability. Just as we considered the norm of a solution modulo symmetries, we shall in effect do the same when considering a Lyapunov function. We have the following theorem due to [42, 68]:

Orbital Stability Theorem. Let U be a spectrally stable equilibrium solution of equation (3.60) such that the eigenfunctions W of the linear stability problem (3.28) form a basis for the space of allowed perturbations $\mathbb{V}_p$, on which the operator J has bounded inverse. Furthermore, suppose there exists an integer $n \geq 1$ and constants $c_{n,0}, \ldots, c_{n,n-1}$ such that the Hamiltonian for the n-th equation in the nonlinear hierarchy satisfies the following:

1. The kernel of $\hat{H}_n''(U)$ on $\mathbb{V}_p$ is spanned by the infinitesimal generators of the symmetry group G acting on U.

2. For all eigenfunctions not in the kernel of $\hat{H}_n''(U)$

$$K_n(W) := \langle W, \hat{H}_n''(U)W\rangle > 0.$$

Then U is orbitally stable with respect to perturbations in $\mathbb{V}_p$.

Let us consider the implications of this theorem for the problem at hand:

- An application of the SCS basis lemma in [45] establishes that the eigenfunctions W form a basis for $L^2_{per}([-NT, NT])$, for any integer N, when the potential U is periodic in y with period $2T$.

- Due to translation invariance, we know that U_y is in the kernel of $\hat{H}_1''(U)$. It is well established [24, 43] that the kernel of $\hat{H}_1''(U)$ is spanned by U_y when considered on $\mathbb{V}_p$. This is the infinitesimal generator of G, ∂_y, acting on U. Furthermore, it is a direct consequence of the Riemann surface relations that the kernel of $\hat{H}_1''(U)$ is equal to the kernel of $\hat{H}_n''(U)$, for all $n \geq 1$.

What is left to verify is condition (2) in the nonlinear stability theorem, i.e., to prove orbital stability we need to find an n such that

$$K_n = \langle W, \mathcal{L}_n W \rangle = \int_{-NT}^{NT} \overline{W} \mathcal{L}_n W dy \geq 0, \tag{3.80}$$

with equality obtained only on the kernel of $\mathcal{L}_n$, i.e., only for $\Omega = 0$.

To calculate the higher order K_n, we make use of the following. Assume our solution is an equilibrium solution of the n-th flow. Then from equation (3.77) we have

$$\mathcal{L}_n W = 2\Omega_n J^{-1} W. \tag{3.81}$$

This gives

$$K_n = \int_{-NT}^{NT} \overline{W} \mathcal{L}_n W dx = 2\Omega_n \int_{-NT}^{NT} \overline{W} J^{-1} W dy. \tag{3.82}$$

Using that U is a stationary solution of the second flow and substituting for Ω_n in the above gives

$$K_n(\zeta) = \Omega_n(\zeta) \frac{K_1(\zeta)}{\Omega(\zeta)} = p_n(\zeta) K_1(\zeta). \tag{3.83}$$

Therefore, when considering stationary solutions of the defocusing mKdV equation, one simply needs to calculate K_1 in order to calculate any of the higher order K_i. Let us do so. From (3.28) and the squared eigenfunction connection we have

$$\mathcal{L}W = 2\Omega J^{-1}W = \frac{\Omega}{i\zeta}\left(\beta^2 - \alpha^2\right). \tag{3.84}$$

This gives

$$\overline{W}\mathcal{L}W = \frac{\Omega}{i\zeta}\left(|\beta|^4 - |\alpha|^4 + (\overline{\alpha})^2\beta^2 - (\overline{\beta})^2\alpha^2\right). \tag{3.85}$$

Now

$$\alpha = -\gamma B, \quad \beta = \gamma(A - \Omega). \tag{3.86}$$

Using 3.46, we have (up to a multiplicative constant)

$$\gamma = \frac{1}{\sqrt{(\mathrm{Im}(A-\Omega))}} \exp\left(i\int \frac{u\mathrm{Re}(B)}{\mathrm{Im}(A-\Omega)}dy\right) \exp\left(\int i\zeta dy\right). \tag{3.87}$$

Therefore,

$$|\gamma|^2 = \frac{1}{\mathrm{Im}(A-\Omega)}. \tag{3.88}$$

Along with $\Omega^2 - A^2 - |B|^2 = 0$, the above implies

$$|\beta|^2 = \mathrm{Im}(A-\Omega), \quad |\alpha|^2 = \mathrm{Im}(A+\Omega), \quad \overline{\alpha}^2\beta^2 = -\overline{B}^2, \quad \overline{\beta}^2\alpha^2 = -B^2. \tag{3.89}$$

Therefore,

$$\int_{-NT}^{NT} \overline{W}\mathcal{L}_1 W dy = \int_{-NT}^{NT} \frac{\Omega}{i\zeta}\left(4A\Omega + 4\mathrm{Re}(B)\mathrm{Im}(B)i\right)dy. \tag{3.90}$$

Using that $\mathrm{Re}(B)\mathrm{Im}(B)$ is a total derivative gives

$$K_1 = 4\Omega^2 \int_{-NT}^{NT} \frac{A}{i\zeta}\, dy = 4\Omega^2 \int_{-NT}^{NT} \left(-V - 4\zeta^2 - 2u^2\right)\, dy. \tag{3.91}$$

Let us revisit the second condition of the nonlinear stability theorem. Using $\Omega^2 = A^2 + |B|^2$ we see that K_1 can be zero only if $\Omega \geq 0$. We also see that K_1 is linear in ζ^2, thus, it changes signs at some point ζ_0^2. Therefore, no stability conclusion can be drawn from K_1. However, let us go one flow higher and calculate K_2. A direct calculation gives

$$\Omega_2^2 = (-4\zeta^2 + V + c_{2,1})^2\Omega^2, \tag{3.92}$$

with $c_{2,0}$ as in (3.72). Therefore, choosing $c_{2,1} = 4\zeta_0^2 - V$ makes K_2 of definite sign.

We have proved the following theorem:

Theorem. There exists a constant $c_{2,1}$ such that K_2 is positive on the Lax spectrum. Therefore, all traveling wave solutions U of the defocusing mKdV equation are orbitally stable with respect to zero-average subharmonic perturbations, i.e., perturbations in the function space

$$\mathbb{V}_p = \left\{ v \in H^2_{per}([-NT, NT]) : \int_{-NT}^{NT} v \; dx = 0 \right\}, \tag{3.93}$$

where $2T$ is the period of the initial condition and N is any integer.

Remark. There is no restriction on the spatial average of the traveling wave solution, only on the spatial average of the perturbation.

3.7 Focusing case

We now examine the focusing mKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0. \tag{3.94}$$

3.7.1 Traveling wave solutions

We change to a moving coordinate frame

$$y = x - Vt, \quad \tau = t. \tag{3.95}$$

In the (y, τ) coordinates the focusing mKdV equation becomes

$$u_\tau - Vu_y + 6u^2 u_y + u_{yyy} = 0. \tag{3.96}$$

We look for stationary solutions $u_\tau = 0$. Letting $u(y,\tau) = U(y)$, stationary solutions satisfy the ordinary differential equation

$$-VU_y + 6U^2U_y + U_{yyy} = 0. \tag{3.97}$$

Integrating (3.97) gives

$$-VU + 2U^3 + U_{yy} = C, \tag{3.98}$$

for some constant C. Multiplying (3.98) by U_y and integrating a second time gives

$$-\frac{V}{2}U^2 + \frac{1}{2}U^4 + \frac{1}{2}\left(U_y\right)^2 - CU = E, \tag{3.99}$$

for some constant E. Therefore, all stationary solutions $U(y)$ satisfy the first-order ordinary differential equation (3.99). Following the same procedure as for the defocusing case, we find that all periodic solutions are given by

$$U(y) = \frac{\pm\sqrt{2E}\wp'(\frac{1}{2}(y+y_0),g_2,g_3) + C(2\wp(\frac{1}{2}(y+y_0),g_2,g_3) - \frac{2}{3}V)}{\left(\wp(\frac{1}{2}(y+y_0),g_2,g_3) - \frac{V}{3} - 2\sqrt{-2E}\right)\left(\wp(\frac{1}{2}(y+y_0),g_2,g_3) - \frac{V}{3} + 2\sqrt{-2E}\right)}. \tag{3.100}$$

Here y_0 is an arbitrary shift in y determined by the initial conditions.

We now determine which values of V, C, and E give rise to bounded periodic solutions. Letting $v = U_y$ in (3.7), we have the first-order two-dimensional system

$$U_y = v, \quad v_y = VU - 2U^3 + C. \tag{3.101}$$

All fixed points (U_0, v_0) satisfy

$$v_0 = 0, \;\; VU_0 - 2U_0^3 + C = 0. \tag{3.102}$$

After linearizing about $(U_0, 0)$, the resulting linear system has eigenvalues

$$\lambda = \pm\sqrt{V - 6U_0^2}. \tag{3.103}$$

We have two centers and a saddle when the discriminant

$$d = 8V^3 - 108C^2 \tag{3.104}$$

is greater than zero, and one center when the discriminant is less than zero. Consider the following cases

- $V < 0$. This implies $d < 0$, giving one center and periodic solutions for all values of C. For $C = 0$ the solution reduces to $U(y) = \pm k\sqrt{\frac{V}{2k^2-1}}\mathrm{cn}\left(\pm\sqrt{\frac{V}{2k^2-1}}y, k\right)$. This solution blows up as $k \to \frac{1}{\sqrt{2}}$ from below and is imaginary for $k > \frac{1}{\sqrt{2}}$.

- $V > 0$, $|C| < \sqrt{\frac{8V^3}{108}}$. There are two centers and one saddle. Periodic solutions exist for all values of E, except for one value giving rise to two homoclinic orbits, corresponding to the saddle, see Fig. 3.8 For $C = 0$ the solution reduces to $U(y) = \pm k\sqrt{\frac{V}{2k^2-1}}\mathrm{cn}\left(\pm\sqrt{\frac{V}{2k^2-1}}y, k\right)$. This solution is inside the homoclinic orbits for $|k| > 1$, and goes to zero as $k \to \infty$, using $\mathrm{cn}(y) = \mathrm{dn}(ky, \sqrt{\frac{1}{k^2}})$. It is outside the homoclinic orbits for $\frac{1}{\sqrt{2}} < k < 1$. For $k = 1$ it gives the soliton solution, which corresponds to the homoclinic orbit.

- $V > 0$, $|C| > \sqrt{\frac{8V^3}{108}}$. There is one center and the solutions are periodic solutions for all values of E.

3.7.2 Stability

The linear stability problem for the focusing mKdV equation takes the form

$$w_\tau = J\mathcal{L}w, \quad J\mathcal{L}W = \lambda W, \quad J = \partial_y, \quad \mathcal{L} = -\partial_{yy} + V - 6U^2. \tag{3.105}$$

The squared eigenfunction connection is given by

$$w(y,\tau) = \psi_1(y,\tau)^2 - \psi_2(y,\tau)^2 = -\frac{1}{2i\zeta}\partial_y\left(\psi_2(y,\tau)^2 + \psi_1(y,\tau)^2\right) \tag{3.106}$$

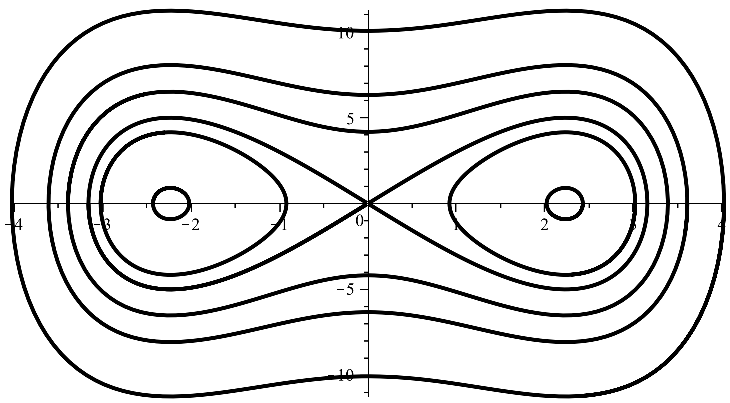


Figure 3.8: Typical (U, v) phase plane in the focusing case for $C = 0$. For $C \neq 0$, the homoclinic orbits change size relative to each other. Both homoclinic orbits persist until $|C| = \sqrt{\frac{8V^3}{108}}$, at which point only one center remains, surrounded by periodic orbits.

where ψ_1 and ψ_2 are obtained from the Lax pair representation

$$\hat{\psi}_y = \begin{pmatrix} -i\zeta & u \\ -u & i\zeta \end{pmatrix} \hat{\psi}, \quad \psi_\tau = \begin{pmatrix} (-V\zeta - 4\zeta^3 + 2\zeta u^2)i & 4\zeta^2 u + C + 2\zeta u_y i \\ -4\zeta^2 u - C + 2\zeta u_y i & -(-V\zeta - 4\zeta^3 + 2\zeta u^2)i \end{pmatrix} \psi. \quad (3.107)$$

However, unlike the defocusing case, the associated spectral problem for ζ

$$\begin{pmatrix} i\partial_y & -iu \\ -iu & -i\partial_y \end{pmatrix} \psi = \zeta\psi. \quad (3.108)$$

is not self-adjoint. Therefore, ζ is not restricted to the real axis. Several difficulties arise as a result:

- As in the defocusing case, we separate variables and find a relationship between Ω and ζ:

$$\Omega^2 = -16\zeta^6 - 8V\zeta^4 + (-V^2 + 8E)\zeta^2 - C^2. \quad (3.109)$$

  However, since ζ is not confined to the real axis, Ω is no longer restricted to $\mathbb{R} \cup i\mathbb{R}$.

- Looking for bounded eigenfunctions, one arrives at the necessary and sufficient condition

$$\left\langle \mathrm{Re}\left(i\zeta - \frac{A'}{A-\Omega} + \frac{uB}{A-\Omega} \right) \right\rangle = 0, \tag{3.110}$$

which is nearly identical to the condition obtained in the defocusing case. However, since ζ is no longer confined to the real axis, explicit analysis of (3.110) is much more difficult. This is the main stumbling block to examining stability in the focusing case. It should be noted that (3.110) still lends itself to numerical computation, which should be simpler than numerically tackling the original spectral problem since it does not involve solving any differential equations.

We make several observations. If ζ is real, the spectral problem for Ω is skew-adjoint in the focusing case. Therefore, for all ζ in the Lax spectrum and on the real axis $\Omega(\zeta)$ is imaginary. For such ζ values the squared eigenfunction connection immediately implies the corresponding solution to the linear stability problem is bounded. However, numerical results suggest that ζ is not confined to the real axis (see Figs. 3.9 and 3.13).

For solutions lying within the homoclinc orbits in Fig. 3.8 (the dn solutions when $C = 0$), it appears that the Lax spectrum is confined to the real and imaginary axis (see Fig. 3.9). We hypothesize that this is due to an underlying symmetry of the spectral problem

$$\begin{pmatrix} i\partial_y & -iu \\ -iu & -i\partial_y \end{pmatrix}^2 \psi = \zeta^2 \psi. \tag{3.111}$$

Though the above problem is not self-adjoint, it may have a PT-symmetry that would confine ζ^2 to be real (see [9], for instance), hence $\zeta \in \mathbb{R} \cup i\mathbb{R}$. We also see numerically that the dn solution appears to be stable to subharmonic perturbations. Furthermore, under such assumptions the analytic formula for $\Omega(\zeta)$ predicts the band of higher spectral density on the imaginary axis seen in the numerically computed spectrum (see Figs. 3.10-3.12). Also, if one assumes that $\zeta \in \mathbb{R} \cup i\mathbb{R}$ then the essential parts of the nonlinear stability calculations in the defocusing case carry over to the focusing case. In fact, one finds $c_{1,0} = V^2 + 4E$, $c_{2,0} = (V^2 + 4E)c_{2,1} + V^2 + 4E$, and

$$K_2 = (-4\zeta^2 + V + c_{2,1})K_1 = (-4\zeta^2 + V + c_{2,1}) \int_{-NT}^{NT} -V - 4\zeta^2 + 2u^2 dy \tag{3.112}$$

Therefore, we should expect spectrally stable solutions to also be nonlinearly (orbitally) stable.

For the solutions outside the homoclinic orbits in Fig. 3.8, (the cn solutions for $C = 0$), the Lax spectrum appears to be much more complicated (see Fig. 3.13), and numerical studies of the stability spectrum point to spectral instability (see Fig. 3.14 and Figure 1 in [24]). It is interesting to note that in numerical investigations the cn solutions appear stable with respect to periodic perturbations, but unstable with respect to subharmonic perturbations [24].

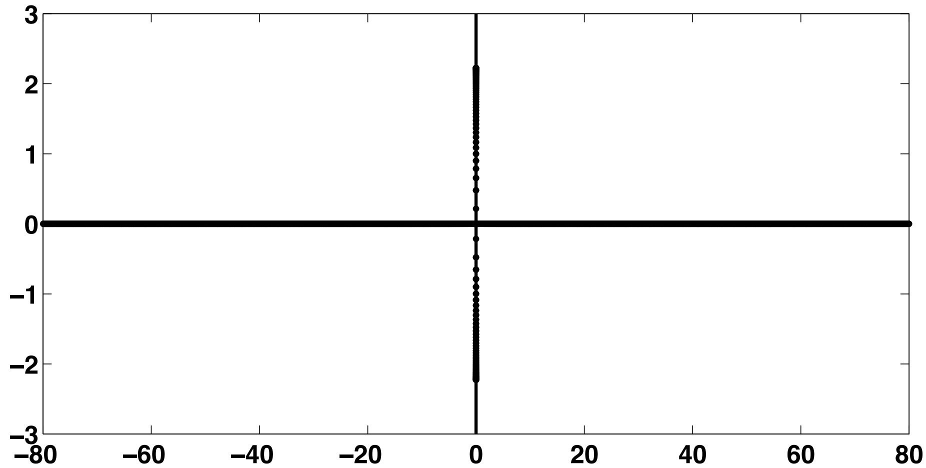
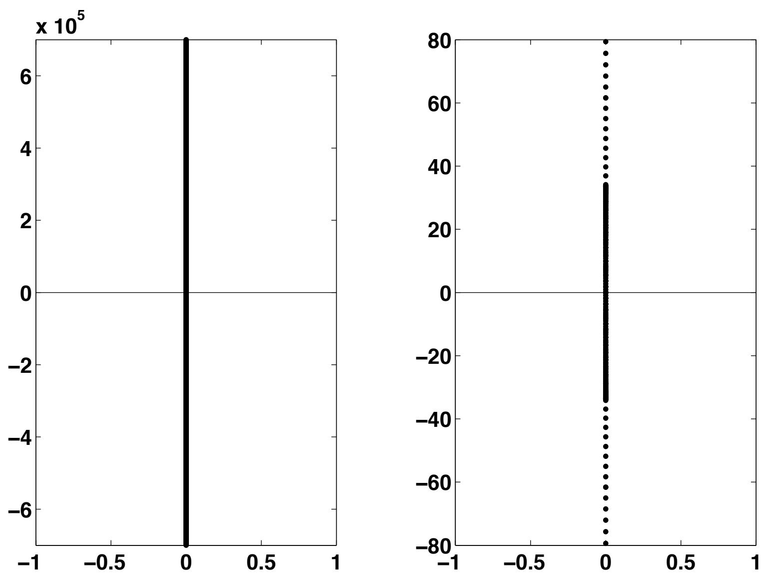


Figure 3.9: Numerically computed Lax spectrum for the traveling wave solution with $V = 10$, $C = 0$, and $k = 1.8$ using Hill's method with 81 Fourier modes and 49 different Floquet exponents.



(a) (b)

Figure 3.10: (a) The numerically computed spectrum for the traveling wave solution. The parameter values are identical to those of Fig. 3.9.; (b) A blow-up of (a) around the origin, showing a band of higher spectral density;

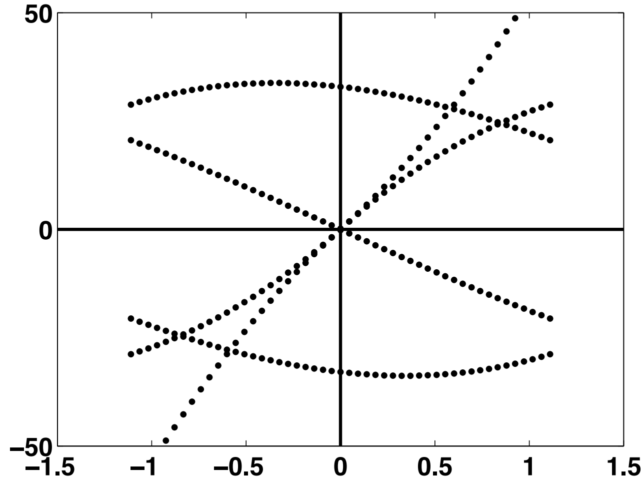
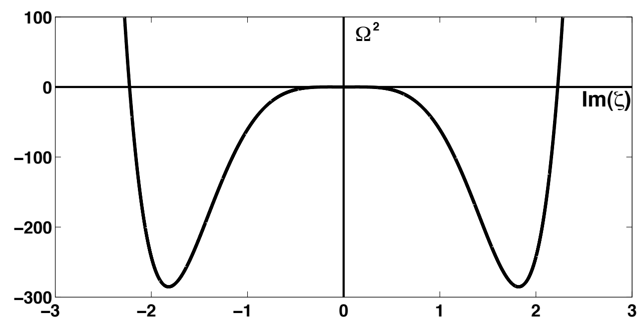
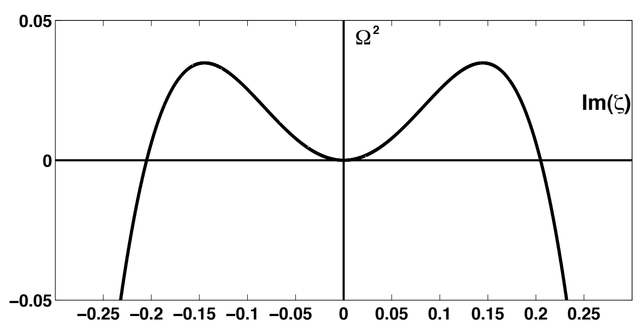


Figure 3.11: The imaginary part of λ as a function of μ, demonstrating the higher spectral density. The parameter values are identical to those of Fig. 3.9.



(a)



(b)

Figure 3.12: (a) Ω^2 as a function of $\mathrm{Im}(\zeta)$, with the same parameter values as Fig. 3.9. (b) A blow-up of (a) around the origin;

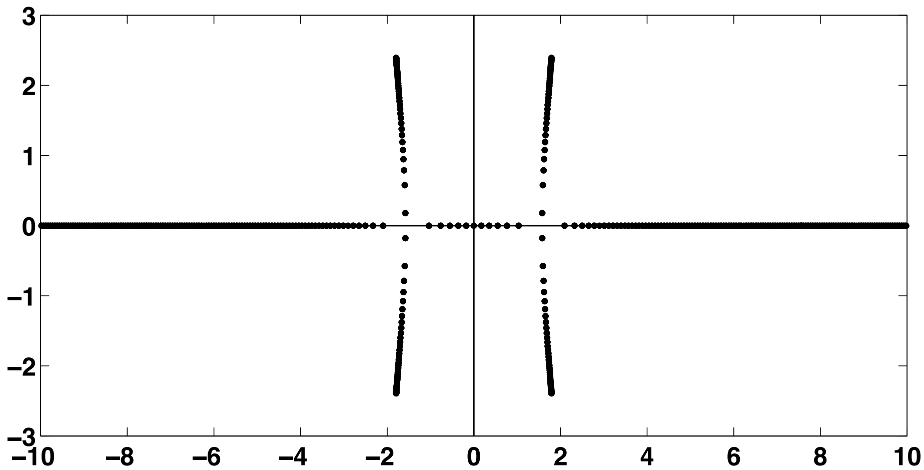


Figure 3.13: The imaginary part of λ as a function of μ, demonstrating the higher spectral density. for the traveling wave solution with $V = 10$, $C = 0$, and $k = .8$ using Hill's method with 81 Fourier modes and 49 different Floquet exponents.

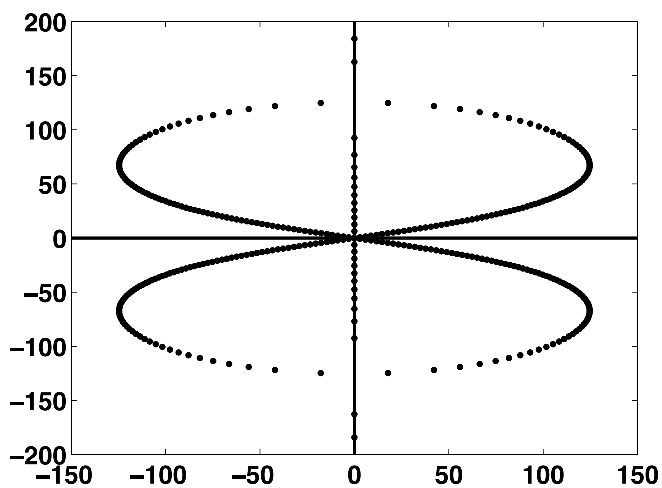


Figure 3.14: The numerically computed spectrum for the traveling wave solution. The parameter values are identical to those of Fig. 3.13.

Chapter 4

STABILITY OF FINITE-GENUS SOLUTIONS OF THE KDV EQUATION

The Korteweg-deVries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{4.1}$$

describes long, one-dimensional waves in weakly dispersive media and arises in a variety of physical settings ranging from water waves to plasma physics. It is characterized by its trademark soliton solutions and their quasi-periodic analogues. The most explicit of these are the one-soliton solution

$$u = u_0 + 12\kappa^2 \mathrm{sech}^2\left(\kappa(x - x_0 - (4\kappa^2 + u_0)t)\right), \tag{4.2}$$

and its periodic counterpart the *cnoidal wave* solution

$$u = u_0 + 12k^2\kappa^2 \mathrm{cn}^2\left(\kappa(x - x_0 - (8\kappa^2 k^2 - 4\kappa^2 + u_0)t), k\right), \tag{4.3}$$

both of which were written down by Korteweg and deVries [63]. Here u_0, κ, and x_0 are constants, and $\mathrm{cn}(\cdot, k)$ denotes the Jacobi elliptic cosine function [17, 66] with elliptic modulus $k \in [0, 1)$.

The stability problem for the above solutions has a rich history (a more detailed discussion is found in [14]), beginning with the works of Benjamin and Bona [10, 13], where the nonlinear orbital stability of the one-soliton solution (4.2) with respect to L^2 perturbations was established. Later, Maddocks and Sachs established the same result for general multi-soliton solutions [68]. More recently, the methods used by Benjamin and Bona were extended to the periodic problem, and the nonlinear orbital stability of cnoidal waves (4.3) with respect to periodic perturbations of the same period was verified [4]. Beyond periodic perturbations of the same period, Bottman and Deconinck proved the spectral stability of cnoidal waves with respect to bounded perturbations [14], and in a follow-up manuscript with Kapitula, the orbital stability of cnoidal waves with respect to *subharmonic perturbations* (periodic perturbations with period equal to any integer multiple of the period of the cnoidal wave) was established [24].

In this chapter we are concerned with the stability of the (quasi-)periodic analogs of the multi-soliton solutions, the *finite-genus* solutions. These are a large family of (quasi-)periodic solutions with n phases of the form [8, 32, 54, 64]

$$u(x,t) = u_0 + 2\partial_x^2 \ln \Theta(\phi_1, \dots, \phi_n), \tag{4.4}$$

where each phase ϕ_j is linear in x and t,

$$\phi_j = k_j x + \omega_j t + \phi_{0j}, \tag{4.5}$$

for constants u_0 and k_j, w_j, ϕ_{0j}, $j = 1, \dots, n$.[1] [2] Here Θ is the Riemann theta function [35], which is determined by a genus n compact connected Riemann surface generated by the initial condition $u(x, 0)$. Note that in the case $n = 1$ the solution (4.4) reduces to the cnoidal wave solution (4.3). The finite-genus solutions possess the following properties:

- They completely solve the initial-value problem for the KdV equation with periodic boundary conditions in the following sense: (i) They solve the initial-value problem for initial data that are periodic in x, and are of the form (4.4) [11, 33, 67, 74]. (ii) They are dense in the set of smooth periodic functions [71].

- Their stability was studied first by McKean [70]. He established that the tori on which the periodic finite-genus solutions lie are stable with respect to periodic perturbations. As noted in [68], McKean only briefly discusses the implications of his results concerning stability in a normed function space, such as L^2.

The object of this chapter is to establish the (nonlinear) orbital stability of periodic finite-genus solutions with respect to subharmonic perturbations. Extension beyond periodic perturbations of the same period to subharmonic perturbations is important in that they are a significantly larger class

---

[1] A genus n solution is periodic in x with period $2L$ if there exists n integers $N_1, \dots, N_n$ such that $2Lk_i = 2\pi N_i$ for $i = 1, \dots, n$, otherwise, they are quasi-periodic.

[2] We restrict ourselves to real-valued finite-genus solutions.

of perturbations than the periodic ones of the same period, while retaining our ability to discuss completeness and separability of a suitable function space. Note that this would not be the case for quasi-periodic or almost periodic perturbations.

The basis of our procedure is the Lyapunov method, which was first extended to infinite-dimensional systems (partial differential equations) by V.I. Arnold [6, 7] in his study of incompressible ideal fluid flows. Since its introduction, the Lyapunov method has formed the crux of subsequent nonlinear stability techniques (see [47, 50, 86] for instance). We build on the results recently obtained for the cnoidal wave (genus one) solutions in [24], and present a systematic generalization. As in [24], the method relies heavily on the integrability of the KdV equation. The outline is as follows:

- Each genus n solution (4.4) is a stationary solution of the n-th KdV equation (to be defined later) [64]. In turn, every bounded periodic stationary solution of the n-th KdV equation is a genus n solution of the form (4.4) [11, 33, 67, 74]. Our method does not require nor make use of the explicit form of the solution (4.4). We only require that it is stationary (with respect to the higher-order time variable t_n) and periodic (with respect to x).

- Since a genus n solution is not stationary with respect to the KdV flow, we cannot define spectral stability in the conventional sense. Instead we prove spectral stability with respect to bounded perturbations (not necessarily periodic) for the higher-order time variable t_n. We do this by generalizing the squared eigenfunction method used in [14] for the cnoidal wave solutions.

- We use the ideas in [24, 68] to construct a candidate Lyapunov function. We show that it is indeed a Lyapunov function using the squared eigenfunction connection and the spectral stability result. This establishes orbital stability from [42, 43].

We conclude by exploring the fact that the multi-soliton solutions can be obtained from (4.4) by taking certain limits [33]. In that case our method (informally) recovers the stability results in [68]. We also look at some explicit examples, including comparison with numerical results using Hill's method [25, 26].

Remark. Though we present the method for finite-genus solutions of the KdV equation, the ideas carry over to other integrable systems. This is discussed in Chapter 5.

4.1 Problem formulation

4.1.1 Hamiltonian structure

We are concerned with the stability of $2L$-periodic genus n solutions of equation (4.1) with respect to subharmonic perturbations of period $2NL$ for any fixed positive integer N. Therefore, we naturally consider solutions u in the space of square-integrable functions of period $2NL$, $L^2_{per}[-NL, NL]$. In order to properly define the Hamiltonian structure of the KdV equation and the corresponding KdV hierarchy (see Section 4.1.2), we further require u and its derivatives of order up to $2n$ to be square-integrable as well. Therefore, we consider solutions of (4.1) defined on the function space

$$\mathbb{V} = H^{2n}_{per}[-NL, NL], \tag{4.6}$$

equipped with natural inner product

$$\langle v, w\rangle = \int_{-NL}^{NL} \bar{v} w \, dx, \tag{4.7}$$

where $\bar{v}$ denotes the complex conjugate of v.

We write equation (4.1) in Hamiltonian form [40, 90]

$$u_t = JH'(u) \tag{4.8}$$

on $\mathbb{V}$. Here J is the skew symmetric operator

$$J = \partial_x, \tag{4.9}$$

the Hamiltonian H is the functional

$$H(u) = \int_{-NT}^{NT} \left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) dx, \tag{4.10}$$

and the notation G' denotes the variational derivative of G

$$G'(u) = \sum_{i=0}^{\infty} (-1)^i \partial_x^i \frac{\partial G(u)}{\partial u_{ix}}, \tag{4.11}$$

where the sum in (4.11) terminates at the order of the highest derivatives involved. For instance, in the computation of H' the sum terminates after accounting for first derivative terms.

We allow for perturbations in a function space $\mathbb{V}_p \subset \mathbb{V}$. In order to apply the stability result of [43], we follow [24] and restrict ourselves to the space of perturbations on which J has a well defined and bounded inverse. This amounts to fixing the spatial average of u on $H^{2n}_{per}[-NL, NL]$, which poses no problem since it is a Casimir of the Poisson operator J, hence, it is conserved under the KdV flow. Therefore, we consider perturbations in $\ker(J)^{\perp}$, i.e., zero-average subharmonic perturbations

$$\mathbb{V}_p = \left\{ v \in H^{2n}_{per}([-NL, NL]) : \int_{-NL}^{NL} v \, dx = 0 \right\}. \tag{4.12}$$

Remark. Physically, requiring perturbations to be zero-average makes sense. It simply says that we do not consider perturbations which add mass to the system.

4.1.2 The nonlinear KdV hierarchy

By virtue of its integrability, the KdV equation possesses an infinite number of conserved quantities [65] $H_0, H_1, H_2, \ldots$, and just as the functional $H_1 = H$ defines the KdV equation, each H_i defines a Hamiltonian system with time variable τ_i through

$$u_{\tau_i} = J H_i'(u). \tag{4.13}$$

The first few conserved quantities are

$$
\begin{aligned}
H_0 &= \int_{-NL}^{NL} \frac{1}{2}u^2 dx, && (4.14)\\
H_1 &= H = \int_{-NL}^{NL} \left(\frac{1}{2}u_x^2 - \frac{1}{6}u^3\right) dx, && (4.15)\\
H_2 &= \int_{-NL}^{NL} \frac{1}{2}\left(u_{xx}^2 - \frac{5}{6}uu_x^2 + \frac{5}{72}u^2\right) dx, && (4.16)
\end{aligned}
$$

with corresponding flows

$$
\begin{aligned}
u_{\tau_0} &= u_x \quad (\text{identifies } \tau_0 \text{ with } x), && (4.17)\\
u_{\tau_1} &= -uu_x - u_{xxx} \quad (\text{identifies } \tau_1 \text{ with } t), && (4.18)\\
u_{\tau_2} &= \frac{5}{6}u^2u_x + \frac{10}{3}u_xu_{xx} + \frac{5}{3}uu_{xxx} + u_{xxxxx}. && (4.19)
\end{aligned}
$$

Each of the higher-order flows can be explicitly calculated from the recursion formula (see [75], for instance)

$$u_{\tau_{i+1}} = \mathcal{R}u_{\tau_i}, \quad u_{\tau_0} = u_x, \tag{4.20}$$

where $\mathcal{R}$ is the operator

$$\mathcal{R} = -\partial_x^2 - \frac{2}{3}u - \frac{1}{3}u_x\partial_x^{-1} \tag{4.21}$$

(since each u_{τ_i} is a total derivative, the non-local term ∂_x^{-1} in (4.21) is well defined [67, 75]). This defines an infinite hierarchy of equations, the KdV hierarchy. It has the following properties:

- All the functionals H_i, $i = 0, 1, \ldots$, are conserved for each member of the KdV hierarchy (4.13) [67].

- The flows of the KdV hierarchy (4.13) mutually commute, and we can think of u as solving all of these equations simultaneously, i.e., $u = u(\tau_0, \tau_1, \ldots)$ [31, 67].

- A finite-genus solution (4.4) of the KdV equation is a simultaneous solution of all the flows of the KdV hierarchy [82], where now the phases depend on all the time variables

$$\phi_j = \sum_{i=0}^{\infty} k_{j,i}\tau_i. \tag{4.22}$$

As all the flows in the KdV hierarchy commute, we may take any linear combination of the above Hamiltonians to define a new Hamiltonian system. For our purposes, we define the *n-th KdV equation* with time variable t_n as

$$u_{t_n} = J\hat{H}_n'(u), \tag{4.23}$$

where each $\hat{H}_n$ is defined as

$$\hat{H}_n := H_n + \sum_{i=0}^{n-1} c_{n,i}H_i, \quad \hat{H}_0 := H_0, \tag{4.24}$$

for constants $c_{n,i}$, $i = 0, \ldots, n-1$.

Remarks.

- No constraints have been imposed on the parameters $c_{n,i}$. We use this freedom to our advantage below.

- There are two hierarchies of equations considered here. The hierarchy associated with the t_i time variables contains the one associated with the τ_i time variables as a special case.

- The time variable t_0 remains identified with x.

- We did not include the conserved quantity $H_{-1} = \displaystyle\int_{-NL}^{NL} u\,dx$ in any of the above. Its variational derivative is constant and is therefore in the kernel of J. This results in trivial dynamics (known as a Casimir). As discussed in [24] and seen in (4.12), the existence of such a functional restricts the space of allowed perturbations.

4.1.3 Stationary solutions of the KdV hierarchy

It was first shown in [33, 67, 74] that the KdV equation possesses a large class of dense periodic and quasi-periodic solutions by examining *stationary solutions* of the n-th KdV equation (4.23). These are solutions such that

$$u_{t_n} = 0, \tag{4.25}$$

for some integer n and constants $c_{n,0}, \ldots, c_{n,n-1}$ in (4.23-4.24). Thus, a stationary solution of the n-th KdV equation satisfies the ordinary differential equation

$$J\hat{H}_n'(u) = 0 \tag{4.26}$$

with independent variable x.

We are interested in the stability of the finite-genus solutions (4.4). This is equivalent to the study of the stationary solutions of the KdV hierarchy by the following theorem [67, 74, 82]:

Theorem. Each genus n solution (4.4) is a stationary solution of the n-th KdV equation (4.23) for some choice of the constants $c_{n,0}, \ldots, c_{n,n-1}$. In turn, every bounded stationary solution of the n-th KdV equation is a genus n solution of the form (4.4) (or the limit of one in the n-soliton case).

Here bounded means that $\sup_{x\in\mathbb{R}}|u(x)|$ is finite, i.e., $u(x) \in C_b^0(\mathbb{R})$.

Throughout the rest of the chapter we use the terms stationary solution of the n-th KdV equation and genus n solution interchangeably. Also, when referring to a genus n solution u we assume it is *non-degenerate*, i.e., u is a stationary solution of the n-th KdV equation and is not stationary with respect to any of the lower-order flows.

The stationary solutions of the KdV hierarchy have the following properties:

- Since all the flows commute, the set of stationary solutions is invariant under any of the KdV equations, i.e., a stationary solution of the n-th equation remains a stationary solution after evolving under any of the other flows [67].

- Any stationary solution of the n-th KdV equation is also stationary with respect to all of the higher order time variables t_m, $m > n$. In such cases, the constants $c_{m,i}$, $i \geq n$ are free parameters. We make use of this fact when constructing a Lyapunov function later.

Example. The first flow with time variable t_1 (4.23) is given by

$$u_{t_1} = -uu_x - u_{xxx} + c_{1,0}u_x. \tag{4.27}$$

Looking for stationary solutions, i.e., setting $u_{t_1} = 0$ results in the ordinary differential equation

$$-uu_x - u_{xxx} + c_{1,0}u_x = 0. \tag{4.28}$$

All periodic solutions of this equation can be written as [63]

$$u = c_{1,0} - 8\kappa^2 k^2 + 4\kappa^2 + 12k^2\kappa^2 \mathrm{cn}^2\left(\kappa(x - x_0), k\right), \tag{4.29}$$

where x_0 and κ are arbitrary constants due to translation and scaling symmetries. The period $2L$ is given by

$$2L = \frac{2\mathrm{K}(k)}{\kappa} = \frac{2}{\kappa}\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2 s}}\, ds, \tag{4.30}$$

where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind [17, 66]. Using the Galilean invariance of the KdV equation, we recover the well-known cnoidal wave solution (4.3).

To see the other side of the previous theorem, suppose we are given the genus one initial condition

$$u^*(x) = 12k^2\mathrm{cn}^2\left(x, k\right). \tag{4.31}$$

Imposing that u^* be stationary with respect to t_1 fixes $c_{1,0}$ as

$$c_{1,0} = 8k^2 - 4. \tag{4.32}$$

Furthermore, we can fix all constants $c_{m,0}$, $m \geq 1$, such that u^* is stationary with respect to all the higher-order time variables t_m. For example, imposing u^* is a stationary solution of the second KdV equation,

$$0 = u_{t_2} = \frac{5}{6}u^2 u_x + \frac{10}{3}u_x u_{xx} + \frac{5}{3}u u_{xxx} + u_{xxxxx} + c_{2,1}(-u u_x - u_{xxx}) + c_{2,0}u_x, \tag{4.33}$$

fixes $c_{2,0}$ as

$$c_{2,0} = c_{2,1}(8k^2 - 4) - 56k^4 + 56k^2 - 16. \tag{4.34}$$

Thus, $c_{2,1}$ is a free parameter, and u^* is a stationary solution of the second KdV equation for any value of $c_{2,1}$ with $c_{2,0}$ defined as above.

Remarks.

- In general, finite-genus solutions will be quasi-periodic in time as opposed to periodic, even if they are periodic in x [67].

- The soliton solutions are obtained as a special case of the finite-genus solutions [33]. For example, the one-soliton (4.2) is obtained form the cnoidal wave (4.3) by letting $k \to 1$.

4.1.4 Stability

We assume our solution is a stationary solution of the n-th flow

$$u(x, t_1, \ldots, t_{n-1}, t_n) = u^*(x, t_1, \ldots, t_{n-1}) \tag{4.35}$$

and consider various aspects of stability. Linearizing the n-th KdV equation (4.23) about u^*

$$u(x, t_1, \ldots, t_n) = u^* + \epsilon w(x, t_1, \ldots, t_n) + \mathcal{O}(\epsilon^2), \tag{4.36}$$

results in the linear system

$$w_{t_n} = J\mathcal{L}_n w. \tag{4.37}$$

Here the symmetric differential operator $\mathcal{L}_n := \hat{H}_n''(u^*)$ is the Hessian of $\hat{H}_n$,

$$\hat{H}_n''(u) = \sum_{i=0}^{\infty} \frac{\partial \hat{H}_n'(u)}{\partial u_{ix}} \partial_x^i, \tag{4.38}$$

evaluated at the stationary solution (the above sum terminates at the order of the highest derivatives involved). As $\mathcal{L}_n$ does not depend on t_n, we separate variables

$$w(x, t_n) = e^{\lambda_n t_n} W(x) \tag{4.39}$$

(since the operator $\mathcal{L}_n$ is expressed solely in terms of x, we have suppressed dependence on the lower-order time variables $t_1, \ldots, t_{n-1}$). This results in the spectral problem

$$J\mathcal{L}_n W(x) = \lambda_n W(x). \tag{4.40}$$

Definition. We say the solution u^* is t_n*-spectrally stable* with respect to perturbations in a function space $\mathbb{U}$ if $\mathrm{Re}(\lambda_n) \leq 0$ for all $W \in \mathbb{U}$. For Hamiltonian systems this is equivalent to $\lambda_n \in i\mathbb{R}$. We define the *stability spectrum* $\sigma(J\mathcal{L}_n)$ as the spectrum of the operator $J\mathcal{L}_n$.

Example. Linearizing (4.27) about $u = u^*$ from the previous example results in the spectral problem

$$J\mathcal{L}_1 W(x) = \lambda_1 W(x), \tag{4.41}$$

where $\mathcal{L}_1$ is given by

$$\hat{H}_1''(u^*) = -\partial_x^2 + (c_{1,0} - u^*)\partial_x - u_x^*, \tag{4.42}$$

with $c_{1,0}$ defined as in (4.32).

Now, consider the problem of nonlinear stability. The n-th KdV equation is invariant under translation with respect to any of the lower-order time variables. This is represented by the Lie group

$$G = \mathbb{R}^n, \tag{4.43}$$

which acts on $u(x, \ldots, t_n)$ according to

$$T(g)u(x, \ldots, t_i, \ldots, t_{n-1}, t_n) = u(x + t_{00}, \ldots, t_i + t_{0i}, \ldots, t_{n-1} + t_{0(n-1)}, t_n) \tag{4.44}$$

with $g = (t_{00}, \ldots, t_{0(n-1)}) \in G$. Stability is considered modulo this symmetry. We use the following definition:

Definition. A stationary solution u^* of the n-th KdV equation is *orbitally stable* with respect to perturbations in $\mathbb{V}_p$, under the t_i dynamics, if for a given $\epsilon > 0$ there exists a $\delta > 0$ such that if $u(x, \ldots, t_{i-1}, 0, t_{i+1}, \ldots) - u^*(x, \ldots, t_{n-1}) \in \mathbb{V}_p$, then

$$\begin{aligned} & \|u(x, \ldots, t_{i-1}, 0, t_{i+1}, \ldots) - u^*(x, \ldots, t_{n-1})\| < \delta \\ \Rightarrow \quad & \inf_{g \in G} \|u(x, \ldots, t_i, \ldots) - T(g)u^*(x, \ldots, t_{n-1})\| < \epsilon. \end{aligned}$$

One can think of the above definition as allowing for the optimal variation of the n phase constants in (4.4) before measuring the distance between functions. Thus, our definition of orbital stability of finite-genus solutions with periodic boundary conditions is equivalent to the analogous version in [68] for n-solitons with vanishing boundary conditions.

To prove orbital stability, we search for a Lyapunov function. For Hamiltonian systems, this is a constant of the motion, $K(u)$, for which u^* is an unconstrained minimum:

$$\partial_t K(u) = 0, \quad K'(u^*) = 0, \quad \langle v, K''(u^*)v \rangle > 0, \quad \forall v \in \mathbb{V}_p,\ v \neq 0. \tag{4.45}$$

The existence of such a functional implies formal stability [50]. Due to the work of Grillakis, Shatah, and Strauss [42], under extra conditions (see the orbital stability theorem below) this allows one to conclude orbital stability. Since all the KdV flows mutually commute, orbital stability with respect to any the time variables t_i implies stability with respect to all of the flows of the KdV hierarchy (most importantly the KdV flow) [24, 57, 68], as the Lyapunov function serves the same role for each equation in the hierarchy.

Just as the norm of a solution is considered modulo its symmetries, in effect the same is done when considering a Lyapunov function. Using the Lyapunov function construction techniques from [24, 68], we have the following stability theorem due to [42]:

Orbital Stability Theorem. Let u^* be a t_n-spectrally stable stationary solution of equation (4.23) such that the eigenfunctions W of the linear stability problem (4.40) form a basis for the space of allowed perturbations $\mathbb{V}_p$, on which J has bounded inverse. Furthermore, suppose there exists an integer $m > n$ and constants $c_{m,0}, \ldots, c_{m,m-1}$ such that the Hamiltonian for the m-th KdV equation satisfies the following:

1. The kernel of $\hat{H}''_m(u^*)$ on $\mathbb{V}_p$ is spanned by the infinitesimal generators of the symmetry group G acting on u^*.

2. For all eigenfunctions not in the kernel of $\hat{H}''_m(u^*)$

$$K_m(W) := \langle W, \hat{H}''_m(u^*)W\rangle > 0.$$

Then u^* is orbitally stable under the t_i dynamics, $i = 1, 2, \ldots$, with respect to perturbations in $\mathbb{V}_p$.

Remarks.

- The last condition $\langle W, \hat{H}''_m(u^*)W\rangle > 0$ can be replaced with $\langle W, \hat{H}''_m(u^*)W\rangle < 0$. In this case $-\hat{H}_m(u)$ serves as a Lyapunov function. Therefore, we only need that $\langle W, \hat{H}''_m(u^*)W\rangle$ is of definite sign.

- The sign of K_n is often called the Krein signature, see [58].

- For spectral stability we only require perturbations to be spatially bounded, thus $\mathbb{U} = C^0_b(\mathbb{R})$.

4.2 Spectral Stability

We prove that a periodic stationary solution of the n-th KdV equation is t_n-spectrally stable with respect to bounded perturbations. In order to accomplish this, we use the relationship between the solutions of the Lax pair equations and the solutions of the linear stability problem, the squared eigenfunction connection [2, 14, 67].

4.2.1 The Lax pair and the linear hierarchy

The KdV equation is equivalent to the compatibility of two linear ordinary differential systems:

$$\begin{aligned} \psi_x &= T_0\psi = \begin{pmatrix} 0 & 1 \\ \zeta - u/6 & 0 \end{pmatrix}\psi, && (4.46)\\ \psi_t &= T_1\psi = \begin{pmatrix} u_x/6 & -4\zeta - u/3 \\ -4\zeta^2 + (u^2 + 6\zeta u + 3u_{xx})/18 & -u_x/6 \end{pmatrix}\psi. && (4.47) \end{aligned}$$

In other words, the compatibility condition $\psi_{xt} = \psi_{tx}$ implies that u satisfies the KdV equation. We can write (4.46) in the form

$$\partial_{xx}\psi_1 + \frac{1}{6}u\psi_1 = \zeta\psi_1. \tag{4.48}$$

Therefore, ζ is the spectral parameter for the stationary Schrödinger equation which implies $\zeta \in \mathbb{R}$ for any bounded solution of (4.48).

Since every member of the nonlinear hierarchy (4.13) is integrable, each possesses a Lax pair (with first member (4.46)), the collection of which is known as the linear KdV hierarchy. For example, the Lax equation associated with the time variable τ_2 is

$$\psi_{\tau_2} = \begin{pmatrix} -\frac{1}{6}(uu_x + u_{xxx}) - \frac{2}{3}\zeta u_x & 16\zeta^2 + \frac{4}{3}\zeta u + \frac{1}{6}u^2 + \frac{1}{3}u_{xx} \\ \begin{array}{l} 16\zeta^3 - \frac{4}{3}\zeta^2 u - \zeta(\frac{1}{18}u^2 + \frac{1}{3}u_{xx}) \\ -\frac{1}{36}u^3 - \frac{1}{6}u_x^2 - \frac{2}{9}uu_{xx} - \frac{1}{6}u_{xxxx} \end{array} & \frac{1}{6}(uu_x + u_{xxx}) + \frac{2}{3}\zeta u_x \end{pmatrix}\psi. \tag{4.49}$$

All the higher order Lax operators are calculated in standard fashion (see [1, 2] for instance). We include some of the details here because our later calculations rely on them. Assume the time component of the Lax pair for the n-th flow with time variable τ_n (we are considering the first hierarchy (4.13)) takes the form

$$\psi_{\tau_n} = T_n \psi = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix} \psi. \tag{4.50}$$

Compatibility $\psi_{\tau_n x} = \psi_{x \tau_n}$ gives

$$A_n = -\frac{1}{2}\partial_x B_n, \tag{4.51}$$

$$C_n = \partial_x A_n + \left(\zeta - \frac{1}{6}u\right) B_n \tag{4.52}$$

$$u_{\tau_n} = (12\zeta - 2u)A_n - 6\partial_x C_n. \tag{4.53}$$

Solving the first two equations for A_n and C_n, we can express the last equation solely in terms of B_n

$$u_{\tau_n} = -12\zeta\partial_x B_n + 2u\partial_x B_n + u_x B_n + 3\partial_x^3 B_n. \tag{4.54}$$

The n-th member of the hierarchy is found by assuming an expansion of the form

$$B_n = \sum_{i=0}^{n} b_i(x)\zeta^i. \tag{4.55}$$

Plugging this assumption into (4.54) gives $b_n(x) = b_n$, a constant, and the recursive relationship

$$b_{i-1}(x) = \int \left(\frac{1}{4}\partial_x^3 b_i(x) + \frac{1}{6}u\partial_x b_i(x) + \frac{1}{12}u_x b_i(x)\right) dx, \tag{4.56}$$

which will be of use later (it can be shown that the integrand in (4.56) is always a total derivative, thus each T_n depends on u and its derivatives in a purely local fashion [67, 75]).

We construct the Lax pair for the n-th KdV equation (4.23) by taking the same linear combination of the lower-order flows as we did for the Hamiltonians, and define the *n-th linear KdV equation* as

$$\psi_{t_n} = \hat{T}_n \psi = \begin{pmatrix} \hat{A}_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n \end{pmatrix} \psi, \tag{4.57}$$

$$\hat{T}_n := T_n + \sum_{i=0}^{n-1} c_{n,i} T_i, \quad \hat{T}_0 := T_0. \tag{4.58}$$

Note: The extra constant b_n from the leading order term in the expansion for B_n (4.55) is chosen so that the compatibility condition gives (4.13). This amounts to rescaling τ_n.

4.2.2 The Lax spectrum

We assume $u = u^*$ is a $2L$-spatially periodic stationary solution of the n-th KdV equation. The object of this section is to explicitly determine all values of ζ that result in spatially bounded solutions of (4.46) and (4.57) (but not necessarily periodic).

Definition. We define the set of all ζ values such that (4.46) and (4.57) are spatially bounded as the *Lax spectrum* σ_{L_n} for the n-th flow.

As before, since ζ acts as the spectral parameter for the stationary Schrödinger equation, the Lax spectrum (for any of the time flows) is a subset of the real line: $\sigma_{L_n} \subset \mathbb{R}$.

Since $u = u^*$ is a stationary solution of the n-th KdV equation, $\hat{T}_n$ in (4.57) does not depend on t_n and we separate variables

$$\psi = e^{\Omega_n t_n} \begin{pmatrix} \alpha_n(x) \\ \beta_n(x) \end{pmatrix}. \tag{4.59}$$

Substitution into (4.57) gives

$$\begin{pmatrix} \hat{A}_n - \Omega_n & \hat{B}_n \\ \hat{C}_n & -\hat{A}_n - \Omega_n \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = 0. \tag{4.60}$$

Requiring a non-trivial solution of the above equation yields

$$\Omega_n^2 = \hat{A}_n^2 + \hat{B}_n \hat{C}_n. \tag{4.61}$$

It is easy to show that (4.61) is independent of x (and of any t_k) [64]. This determines a relationship between Ω_n and ζ. In general (4.61) is an algebraic curve representation of a genus n Riemann surface [8, 83]. Based on the expansion in ζ for B_n (4.55), we see that the right-hand side of (4.61) is a polynomial in ζ of degree $2n+1$. Furthermore, since u^* is a (non-degenerate) genus n potential it follows that all $2n+1$ roots of the aforementioned polynomial are real and distinct [67]. Therefore,

$$\Omega_n^2 = r_n(\zeta - \zeta_1)\cdots(\zeta - \zeta_{2n+1}), \tag{4.62}$$

for some constants $\zeta_1 < \cdots < \zeta_{2n+1}$ and positive constant r_n.

The eigenvector is given by

$$\begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} = \gamma_n \begin{pmatrix} -\hat{B}_n \\ \hat{A}_n - \Omega_n \end{pmatrix}, \tag{4.63}$$

where γ_n is a scalar function of x. It is determined by substitution of the above into the first equation of the Lax pair, resulting in two linear first-order scalar differential equations for γ_n which are linearly dependent. Solving gives

$$\gamma_n = \exp \int \left(\frac{-\partial_x \hat{B}_n - \hat{A}_n + \Omega_n}{\hat{B}_n} \right) dx, \tag{4.64}$$

up to a multiplicative constant. We simplify the above. Using $\hat{A}_n = -\frac{1}{2}\partial_x \hat{B}_n$, we find

$$\gamma_n = \frac{1}{\sqrt{\hat{B}_n}} \exp\left(\int \frac{\Omega_n}{\hat{B}_n} dx \right). \tag{4.65}$$

Each value of ζ results in two values of Ω_n (except for the $2n+1$ branch points $\zeta_1, \ldots, \zeta_{2n+1}$ which give $\Omega_n = 0$) and therefore (4.63) represents two eigenvectors. This explicitly verifies the results in [67, 69, 71], namely that in the spectral problem for the stationary Schrödinger equation with a genus n potential ζ is a double eigenvalue for all but $2n+1$ values.

Let us determine which values of ζ result in bounded eigenfunctions (4.63):

- Consider all values of ζ for which it is possible that $\hat{B}_n$ (when considered as a function of x) attains zero. From (4.61), we see that this can only happen for values of ζ such that $\Omega_n^2(\zeta) = \hat{A}_n^2 \geq 0$, since $\hat{A}_n \in \mathbb{R}$ for $\zeta \in \mathbb{R}$. For the branch points ζ_i where $\Omega_n(\zeta_i) = 0$, $i = 1, \ldots, 2n+1$, it is easily checked that the eigenfunction (4.63) is bounded, thus the zeros of Ω_n are part of the Lax spectrum. In fact, since $\hat{A}_n = -\frac{1}{2}\partial_x \hat{B}_n$, if $\Omega_n(\zeta) = 0$ then the derivative of $\hat{B}_n$ must also be zero at any value of x where $\hat{B}_n$ is zero. For any other values of ζ where $\hat{B}_n$ attains zero for some x, (4.63) is unbounded, thus these ζ values are not part of the Lax spectrum.

- Consider all values of ζ, $\Omega_n(\zeta) \neq 0$, such that $\hat{B}_n$ (when considered as a function of x) never attains zero (from our previous considerations, we know this is true for (at least) values of ζ where $\Omega_n^2(\zeta) < 0$). There is an exponential contribution from γ_n

$$\exp\left(\int \frac{\Omega_n}{\hat{B}_n} dx\right), \tag{4.66}$$

which we need to be bounded. To this end, it is necessary and sufficient that

$$\left\langle \mathrm{Re}\left(\frac{\Omega_n}{\hat{B}_n}\right)\right\rangle = 0, \tag{4.67}$$

where $\langle \cdot \rangle = \frac{1}{2L}\int_{-L}^{L} \cdot \, dx$ denotes the average over one period of u^*. This average is well defined since the denominator $\hat{B}_n$ is never zero by assumption. Now, it follows from (4.61) that Ω_n is either purely real or purely imaginary. We also have $\hat{B}_n \in \mathbb{R}$ for $\zeta \in \mathbb{R}$. The above condition becomes

$$\left\langle \frac{1}{\hat{B}_n} \right\rangle \mathrm{Re}(\Omega_n) = 0. \tag{4.68}$$

Therefore, we see that all values of ζ for which Ω_n is imaginary are part of the Lax spectrum. Now, suppose $\zeta \in \mathbb{R}$ is such that Ω_n is real and non-zero. Then the average term in (4.68) must be identically zero. However, since $\frac{1}{\hat{B}_n}$ is never zero it follows that it cannot have zero average. Therefore $\mathrm{Re}(\Omega_n)$ must be zero.

We conclude that the Lax spectrum consists of all ζ values for which $\Omega_n^2 \leq 0$:

$$\sigma_{L_n} = (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup \cdots \cup [\zeta_{2n}, \zeta_{2n+1}]. \tag{4.69}$$

This is a well known result, but it has been deduced here in more explicit terms. Furthermore, Ω_n is purely imaginary

$$\Omega_n \in i\mathbb{R} \tag{4.70}$$

for all values of $\zeta \in \sigma_{L_n}$. In fact, Ω_n^2 takes on all negative values for $\zeta \in (-\infty, \zeta_1]$, implying that $\Omega_n = \pm\sqrt{|\Omega_n^2|}$ covers the imaginary axis. Furthermore, for $\zeta \in [\zeta_2, \zeta_3]$, Ω_n^2 takes on all negative values in $[\Omega_n^2(\zeta_2^*), 0]$ twice, where $\Omega_n^2(\zeta_2^*)$ is the minimal value of Ω_n^2 attained for $\zeta \in [\zeta_2, \zeta_3]$. Upon taking square roots, this implies that the interval on the imaginary axis $\left[-i\sqrt{|\Omega_n^2(\zeta_2^*)|}, i\sqrt{|\Omega_n^2(\zeta_2^*)|}\right]$ is double covered in addition to the single covering already mentioned. Similarly, for $\zeta \in [\zeta_{2i}, \zeta_{2i+1}]$, $i = 1, \dots, n$, Ω_n^2 takes on all negative values in $[\Omega_n^2(\zeta_{2i}^*), 0]$ twice, where $\Omega_n^2(\zeta_{2i}^*)$ is the minimal value of Ω_n^2 attained for $\zeta \in [\zeta_{2i}, \zeta_{2i+1}]$. Symbolically, we write [14]

$$\Omega_n \in i\mathbb{R} \cup \left[-i\sqrt{|\Omega_n^2(\zeta_2^*)|}, i\sqrt{|\Omega_n^2(\zeta_2^*)|}\right]^2 \cup \dots \cup \left[-i\sqrt{|\Omega_n^2(\zeta_{2n}^*)|}, i\sqrt{|\Omega_n^2(\zeta_{2n}^*)|}\right]^2, \tag{4.71}$$

where the square is used to denote multiplicity, see Fig. (4.1).

4.2.3 The squared eigenfunction connection and spectral stability

It is well known [67, 2, 14] that the product

$$w(x, t_n) = \psi_1(x, t_n)\psi_2(x, t_n) = \frac{1}{2}\partial_x \psi_1^2(x, t_n) \tag{4.72}$$

satisfies the linear stability problem (4.37), coined as the *squared eigenfunction connection*. Using the results of the previous section, we see that the above takes the form

$$w(x, t_n) = e^{2\Omega_n t_n}\alpha_n(x)\beta_n(x) = \frac{1}{2}e^{2\Omega_n t_n}\partial_x \alpha_n^2(x), \tag{4.73}$$

where α_n, β_n are defined as in (4.63). Comparing the above with (4.39), we immediately conclude that

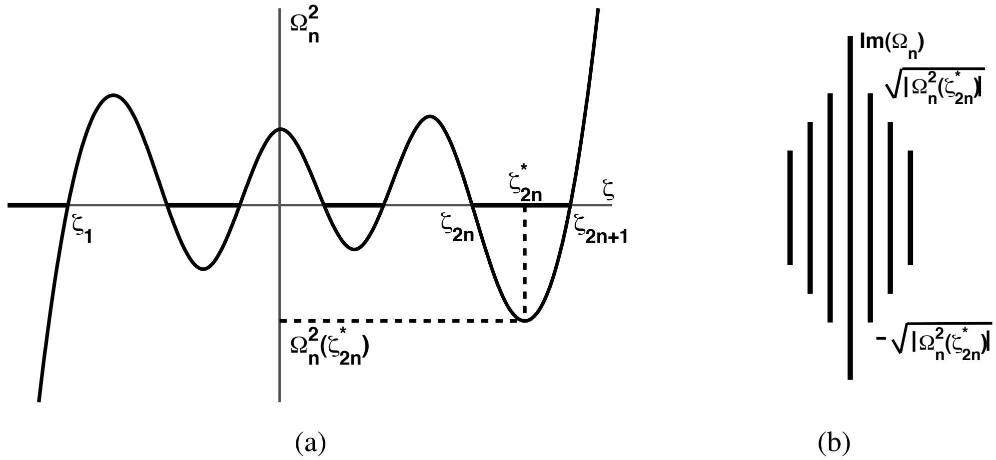


Figure 4.1: (a) Ω_n^2 as a function of real ζ. The union of the bold line segments is the Lax spectrum. (b) Corresponding plot of Ω_n (restricted to the Lax spectrum) in the complex plane. The vertical lines represent the multiple coverings in (4.71), which actually lie on the imaginary axis.

$$\lambda_n = 2\Omega_n, \quad W(x) = \alpha_n(x)\beta_n(x) = \frac{1}{2}\partial_x \alpha_n^2(x), \tag{4.74}$$

for all solutions obtained through the squared eigenfunction connection.

If we show that all solutions with $\mathrm{Re}(\lambda_n) > 0$ are unbounded in x, then spectral stability follows. To this end, let us examine which solutions of the linear stability problem are obtained through the squared eigenfunction connection. For any given $\lambda_n \in \mathbb{C}$, (4.40) is a $(2n+1)$-order differential equation. Thus, it has $2n+1$ linearly independent solutions.

Theorem. All spatially bounded solutions of the spectral problem (4.40) with $\lambda_n \neq 0$ are obtained through the squared eigenfunction connection (4.74). If $\lambda_n = 0$, then exactly n linearly independent spatially bounded solutions are obtained through (4.74).

Proof. First, let us count how many solutions are obtained from the squared eigenfunction connection for a fixed value of $\lambda_n \neq 0$. Exactly one value of Ω_n is obtained through $\Omega_n = \lambda_n/2$. Excluding the values of λ_n for which the discriminant of (4.61) as a function of ζ is zero, (4.61) gives rise to $2n+1$ values of ζ. Before we consider the excluded values separately, we need to show that the $2n+1$ functions $W(x)$ obtained as described are linearly independent.

From our previous calculations we see that

$$W = \alpha_n \beta_n = (\hat{A}_n - \Omega_n) \exp\left(\int \frac{\Omega_n}{\hat{B}_n} dx \right). \tag{4.75}$$

Therefore, as long as there is an exponential contribution, the $2n+1$ solutions W corresponding to the $2n+1$ values of ζ are linearly independent: indeed for different ζ different terms with singularities of different order in the complex x-plane are present with different coefficients. From the above we see that there is an exponential contribution if and only if $\Omega_n \neq 0$, which is true since $\Omega_n = \lambda_n/2 \neq 0$ by assumption. Furthermore, if $\mathrm{Re}(\lambda_n) > 0$ it follows that $\mathrm{Re}(\Omega_n) > 0$. This implies ζ is not in the Lax spectrum. Therefore, all $2n+1$ solutions obtained from the squared eigenfunction connection corresponding to $\mathrm{Re}(\lambda_n) > 0$ are unbounded in x.

For the values of λ_n at which the discriminant of (4.61) as a function of ζ is zero, only $2n$ solutions are obtained (see note below). The other solution can be obtained through reduction of order. This introduces algebraic growth, therefore it is unbounded in x. We have thus shown that all bounded solutions corresponding to $\lambda_n \neq 0$ are obtained through the squared eigenfunction connection.

Note: Extra care must be taken if the degeneracy is stronger, such as two local minima of Ω_n^2 being equal when the discriminant of (4.61) as a function of ζ is zero. In such cases less than $2n$ solutions are obtained. However, a simple perturbation argument resolves these higher degeneracies and unbounded eigenfunctions result.

Now, assume $\lambda_n = 0$. It follows that $\Omega_n = \lambda_n/2 = 0$. Substituting $\Omega_n = 0$ into (4.75) gives

$$W = \hat{A}_n. \tag{4.76}$$

Note that $\hat{A}_n$ is linearly related to the A_i from the τ_i-hierarchy. Using that $A_n = -\frac{1}{2}\partial_x B_n$, (4.56), and the expansion $A_n = \sum_{i=0}^{n} a_i(x)\zeta^i$ we find

$$a_{i-1} = \frac{1}{8}\left(-\partial_x^2 a_i - \frac{2}{3} u a_i - \frac{1}{3} u_x \partial_x^{-1} a_i \right). \tag{4.77}$$

The above is precisely the recursion operator (4.21) which generates the KdV hierarchy, i.e.,

$$a_{i-1} = \frac{1}{8}\mathcal{R}a_i \tag{4.78}$$

Using $a_n = 0$, $a_{n-1} = \frac{1}{24}b_n u_x = \frac{1}{24}b_n u_{\tau_0}$ gives

$$a_{i-1} = \frac{1}{8}\mathcal{R}a_i = d_i u_{\tau_{n-i}}, \quad i = n-1, \ldots, 1, \tag{4.79}$$

for some constants d_i. In other words, each $a_i(x)$ is a linear multiple of $u_{\tau_{n-i}}$, $i = 1, \ldots, n$. Since $\hat{A}_n$ is a linear combination of all the lower-order flows, the above result gives that $\hat{A}_n$ can be expressed as a linear combination of the $u_{t_{n-i}}$. Therefore, for $\lambda_n = 0$ we obtain n linearly independent solutions $u^*_{t_{n-i}}$, $i = 1, \ldots, n$, from the squared eigenfunction connection. Of course, each $u^*_{t_{n-i}}$, $i = 1, \ldots, n$, is expressed in terms of u^* and its x-derivatives through the KdV hierarchy (4.23).

■

As seen in this proof, there is no stability spectrum with $\mathrm{Re}(\lambda_n) > 0$. Therefore, we immediately conclude t_n-spectral stability. We summarize the above results:

Theorem (Spectral stability). All periodic genus n solutions of the KdV equation are t_n-spectrally stable with respect to spatially bounded perturbations. The spectrum of their associated linear stability problem (4.40) is explicitly given by $\sigma(J\mathcal{L}_n) = i\mathbb{R}$, or, accounting for multiplicities,

$$\sigma(J\mathcal{L}_n) = i\mathbb{R} \cup \left[-2i\sqrt{|\Omega_n^2(\zeta_2^*)|}, 2i\sqrt{|\Omega_n^2(\zeta_2^*)|}\right]^2 \cup \cdots \cup \left[-2i\sqrt{|\Omega_n^2(\zeta_{2n}^*)|}, 2i\sqrt{|\Omega_n^2(\zeta_{2n}^*)|}\right]^2. \tag{4.80}$$

4.3 Nonlinear stability

With spectral stability established, let us revisit the nonlinear stability theorem as applied to our problem. We have the following:

- It is an application of the SCS basis lemma in [45] that the eigenfunctions W form a basis for $L^2_{per}([-NL, NL])$, for any integer N, when the potential u is periodic in x with period $2L$.

- As seen in the proof of spectral stability, the infinitesimal generators of G:

$$\partial_{t_0}, \partial_{t_1}, \ldots, \partial_{t_{n-1}}, \tag{4.81}$$

acting on the solution u^* are in the kernel of $\hat{H}_n''(u^*)$. In fact, when restricted to the space $\mathbb{V}_p$, the infinitesimal generators of G span the kernel of $\hat{H}_n''(u^*)$ [24, 43]. As we see below, the kernel of $\hat{H}_m''(u^*)$ is equal to that of $\hat{H}_n''(u^*)$ for any $m \geq n$. It is interesting to note that the infinitesimal generators of G also span the tangent space of the abelian torus on which the Riemann theta function in (4.4) is defined [71], as they are linearly related to $\partial_{\phi_{01}}, \ldots, \partial_{\phi_{0n}}$.

What is left to verify is the last condition in the nonlinear stability theorem, i.e., we need to find an m such that

$$K_m(W) = \langle W, \hat{H}_m''(u^*)W \rangle = \int_{-NL}^{NL} \overline{W}(x)\mathcal{L}_m W(x)\ dx \geq 0 \tag{4.82}$$

for all $W \in \mathbb{V}_p$ with equality obtained only for W in the kernel of $\mathcal{L}_m = \hat{H}_m''(u^*)$.

Assume $m > n$ (what follows is trivial for $m = n$) and that u^* is a stationary solution of the n-th flow. It is a stationary solution of the m-th flow as well, for some choice of the constants $c_{m,0}, \ldots, c_{m,n-1}$. Now, consider the time component of the Lax hierarchy for the m-th flow

$$\psi_{t_m} = \hat{T}_m \psi. \tag{4.83}$$

Proceeding as we did for the n-th flow, we look for solutions of the form

$$\psi(x, t_m) = e^{\Omega_m t_m} W(x), \tag{4.84}$$

where due to the commuting properties of the flows, the Lax equations for $i \geq n$ share the same complete set of eigenfunctions

$$W(x) = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} \tag{4.85}$$

(for $i < n$ we can no longer separate variables). As before, this determines a relationship between Ω_m and ζ, and in general Ω_m^2 defines an algebraic curve corresponding to a genus m Riemann surface. However, when evaluated at a stationary solution of the n-th flow this curve is singular, and corresponds to a genus n surface.

Theorem. Let u^* be a stationary solution of the n-th KdV equation. Then for all $m > n$, the m-th surface reduces to

$$\Omega_m^2(\zeta) = p_m^2(\zeta)\Omega_n^2(\zeta), \tag{4.86}$$

where $p_m(\zeta)$ is a polynomial of degree $m - n$ in ζ. Furthermore, $p_m(\zeta)$ depends on the free parameters $c_{m,n}, \ldots, c_{m,m-1}$ such that $c_{m,i}$ appears in the coefficients of ζ^{i-n} and lower. Therefore, the free parameters $c_{m,n}, \ldots, c_{m,m-1}$ give us total control over the roots of $p_m(\zeta)$.

Proof. For $m > n$, we impose $\hat{H}_m'(u^*) = 0$. Without loss of generality, we assume that the free constants are chosen in such a way so that for all $m > n$ the m-th Hamiltonian takes the form

$$\hat{H}_m(u) = \tilde{H}_m(u) + \tilde{c}_{m,m-1}\hat{H}_{m-1}(u) + \cdots + \tilde{c}_{m,n}\hat{H}_n(u), \tag{4.87}$$

where $\tilde{H}_m'(u^*) = 0$ and each constant $\tilde{c}_{m,i}$ is expressed in terms of the constants $c_{m,j}$, $j \geq i$ (in practice this is not necessary, it only makes the proof more transparent). In this case, when evaluated at u^*, each $\hat{T}_i$, $i > n$, becomes a linear multiple of $\hat{T}_n$. Therefore,

$$\begin{aligned} \hat{T}_m &= \tilde{p}_m(\zeta)\hat{T}_n + \tilde{c}_{m,m-1}\tilde{p}_{m-1}(\zeta)\hat{T}_n + \cdots + \tilde{c}_{m,n}\tilde{p}_n(\zeta)\hat{T}_n && (4.88) \\ &= p_m(\zeta)\hat{T}_n && (4.89) \end{aligned}$$

where $p_m(\zeta) = \tilde{p}_m(\zeta) + \tilde{c}_{m,m-1}\tilde{p}_{m-1}(\zeta) + \cdots + \tilde{c}_{m,n}\tilde{p}_n(\zeta)$, and each polynomial $\tilde{p}_i(\zeta)$ is of degree $i - n$ in ζ. The existence of nontrivial solutions of the eigenvalue problem imposes

$$0 = \det(\hat{T}_m - \Omega_m I) = \det(p_m(\zeta)\hat{T}_n - \Omega_m I). \tag{4.90}$$

Therefore, $\Omega_m^2 = p_m^2(\zeta)\Omega_n^2$. Expressing $\tilde{c}_{m,i}$ in terms of $c_{m,i}$ completes the proof. ■

From the squared eigenfunction connection we have

$$2\Omega_m W(x) = J\mathcal{L}_m W(x), \tag{4.91}$$

so that

$$\mathcal{L}_m W(x) = 2\Omega_m J^{-1} W(x). \tag{4.92}$$

Therefore,

$$K_m(W) = \int_{-NL}^{NL} \overline{W}\mathcal{L}_m W \; dx = 2\Omega_m \int_{-NL}^{NL} \overline{W} J^{-1} W(x) \; dx = \Omega_m \frac{K_n}{\Omega_n}. \tag{4.93}$$

From the previous theorem, all values of ζ for which $\Omega_n = 0$ also give $\Omega_m = 0$, thus, these values of ζ pose no problem in (4.93). Substituting for Ω_m gives

$$K_m(\zeta) = p_m(\zeta)K_n(\zeta). \tag{4.94}$$

Therefore, when considering stationary solutions of the n-th flow, one simply needs to calculate K_n in order to calculate any of the higher-order K_i. Let us do so. We have

$$W(x) = \alpha_n(x)\beta_n(x) = \frac{1}{2}\partial_x \alpha_n^2(x). \tag{4.95}$$

Therefore, the integrand in K_n is given by

$$\overline{W}\mathcal{L}_n W = \overline{\alpha}_n \overline{\beta}_n \Omega_n \alpha_n^2 = \Omega_n |\alpha_n|^2 \overline{\beta}_n \alpha_n. \tag{4.96}$$

Now, on the Lax spectrum $|\gamma_n|^2 = 1/\hat{B}_n$, since the exponent in (4.65) is imaginary. This gives

$$|\alpha_n|^2 = |\gamma_n|^2 \hat{B}_n^2 = \hat{B}_n \tag{4.97}$$

and

$$\overline{\beta}_n \alpha_n = |\gamma_n|^2 \overline{(\hat{A}_n - \Omega_n)}(-\hat{B}_n) = -\overline{(\hat{A}_n - \Omega_n)} = -(\hat{A}_n + \Omega_n), \tag{4.98}$$

where, again, we used that $\hat{A}_n$ is real and Ω_n is imaginary on the Lax spectrum. This gives

$$\overline{W} \mathcal{L}_n W = -\Omega_n \hat{B}_n (\hat{A}_n + \Omega_n). \tag{4.99}$$

Therefore

$$K_n(\zeta) = -\Omega_n \int_{-NL}^{NL} \hat{B}_n \hat{A}_n \; dx - \Omega_n^2 \int_{-NL}^{NL} \hat{B}_n \; dx. \tag{4.100}$$

Using that $\hat{B}_n \hat{A}_n$ is a total derivative gives the final result

$$K_n(\zeta) = -\Omega_n^2 \int_{-NL}^{NL} \hat{B}_n \; dx. \tag{4.101}$$

Note that (4.101) is only valid on the Lax spectrum. However, we find it convenient to formally consider (4.101) as defining K_n as a function of all real ζ in our considerations below. This poses no problems in the application of the orbital stability theorem, as it is only concerned with the sign of K_n when evaluated at bounded eigenfunctions.

Now, consider the sign of $K_n(\zeta)$ on the Lax spectrum σ_{L_n}:

- Since u^* is t_n-spectrally stable, $\Omega_n^2 \leq 0$ on σ_{L_n}. Therefore, we only need to consider the sign of the integral term $\int_{-NL}^{NL} \hat{B}_n \, dx$ in (4.101).

- The Lax spectrum σ_{L_n} has $n+1$ components: $(-\infty, \zeta_1), [\zeta_2, \zeta_3], \ldots, [\zeta_{2n}, \zeta_{2n+1}]$. We previously saw that $\hat{B}_n$ never attains zero (as a function of x) for $\zeta \in \sigma_{L_n}$, except possibly at the endpoints where $\Omega_n(\zeta) = 0$. This implies that the integral term $\int_{-NL}^{NL} \hat{B}_n \, dx$ in (4.101) is never zero and has fixed sign on each component of the Lax spectrum. However, that sign may change from one component to the next. Therefore, $K_n(\zeta)$ can change sign only on the gaps where $\zeta \notin \sigma_{L_n}$ or on the band edges where $\Omega_n(\zeta) = 0$.

We see that $K_n(\zeta)$ it is not guaranteed to have fixed sign on the *entire* Lax spectrum, but only on each component separately. Thus, no stability conclusions can be drawn from $K_n(\zeta)$. However,

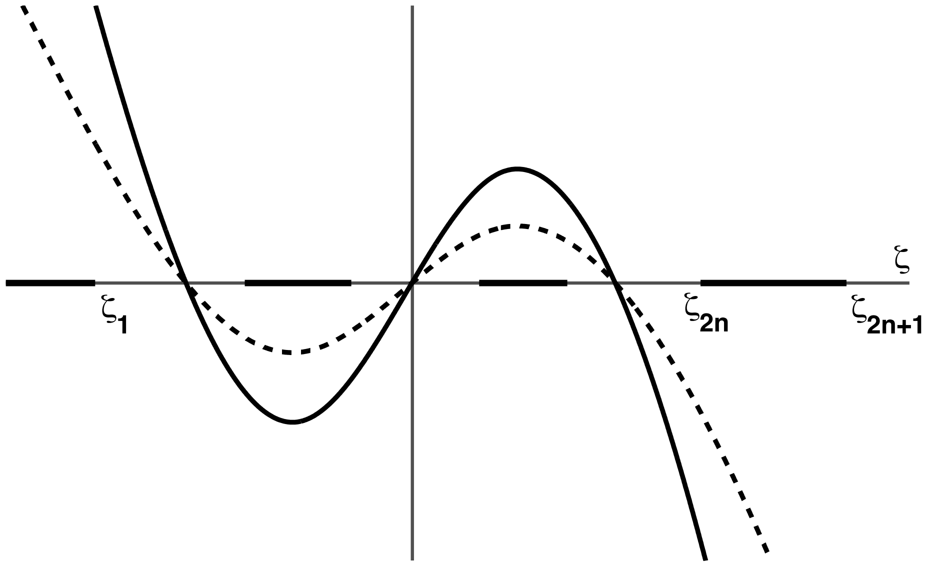


Figure 4.2: The bold and dashed curves represent the integral term $\int_{-NL}^{NL} \hat{B}_n \, dx$ in $K_n(\zeta)$ and $p_{2n}(\zeta)$ respectively, considered as functions of real ζ. The union of the thick line segments on the real axis is the Lax spectrum σ_{L_n}. Both curves have different signs on various components of the Lax spectrum, but their product has fixed sign on the entire Lax spectrum.

going n flows higher (calculating $K_{2n}(\zeta)$) provides the requisite number of constants to allow us to make $K_{2n}(\zeta)$ of definite sign on the entire Lax spectrum. We have

$$K_{2n}(\zeta) = p_{2n}(\zeta) K_n(\zeta), \tag{4.102}$$

where $p_{2n}(\zeta)$ is a polynomial in ζ of degree $2n - n = n$. Since we have total control over the roots of $p_{2n}(\zeta)$, we choose the n constants $c_{2n,n}, \ldots, c_{2n,2n-1}$ so that $p_{2n}(\zeta)$ changes sign whenever the integral term in $K_n(\zeta)$ changes sign, see Fig. (4.5). This is always possible since the integral term in $K_n(\zeta)$ is a polynomial in ζ of degree n. This makes $K_{2n}(\zeta)$ of definite sign on the entire Lax spectrum, establishing the last condition in the nonlinear stability theorem.

We have proved the following:

Theorem (Orbital stability). Spatially periodic genus n solutions of the KdV equation are orbitally stable (under the time dynamics of any of the KdV equations) with respect to perturbations in

$$\mathbb{V}_p = \left\{ v \in H^{2n}_{per}([-NL, NL]) : \int_{-NL}^{NL} v \; dx = 0 \right\}, \tag{4.103}$$

where $2L$ is the period of the initial condition and N is any positive integer.

Remarks:

- There is no restriction on the spatial average of the finite-genus solution, only on the spatial average of the perturbation.

- The choice of constants, $c_{2n,n}, \ldots, c_{2n,2n-1}$, that makes $K_{2n}(\zeta)$ of definite sign is not unique. For example, one could require $p_{2n}(\zeta)$ to have the same n zeros as the integral term in $K_n(\zeta)$. One could instead require $p_{2n}(\zeta)$ to have ζ_{2i} (or ζ_{2i-1}) as a zero if $K_n(\zeta)$ has an undesired sign on the band $[\zeta_{2i}, \zeta_{2i+1}] \subset \sigma_{L_n}$.

- In the soliton limit, the allowed bands in (4.69) collapse to single points [33]. Thus, the operator $\hat{H}''_n(u^*)$ may have up to n unstable directions and the theory of [68] applies. It is interesting to note that our formulation using $\hat{H}''_{2n}(u^*)$ eliminates the extra machinery (Theorem 2 of [42]) required to negotiate the unstable directions, and orbital stability seems to follow from Theorem 1 of [42]. To turn this into a formal proof for the soliton case, one needs to examine the interplay between the infinite period limit and the steps we take in our method.

4.4 Examples

4.4.1 Genus 1: cnoidal wave

We repeat the results of [14, 24] as an illustration of our general framework applied to the genus one case. Consider the genus one example (4.31)

$$u^* = 12k^2 \mathrm{cn}^2 (x, k), \tag{4.104}$$

with period $2L = 2\mathrm{K}(k)$ (see (4.30)).

We have

$$\hat{T}_1 = T_1 + c_{1,0} T_0, \tag{4.105}$$

where as before $c_{1,0} = 8k^2 - 4$. From

$$\det(\hat{T}_1 - \Omega_1 I) = 0, \tag{4.106}$$

a direct calculation gives

$$\Omega_1^2 = 16(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3), \tag{4.107}$$

where

$$\zeta_1 = k^2 - 1 < \zeta_2 = 2k^2 - 1 < \zeta_3 = k^2, \tag{4.108}$$

for $k \in (0, 1)$. Therefore, the Lax spectrum is

$$\sigma_{L_1} = (-\infty, k^2 - 1] \cup [2k^2 - 1, k^2]. \tag{4.109}$$

To examine orbital stability, let us calculate K_1. We have

$$\begin{aligned}
K_1(\zeta) &= -\Omega_1^2 \int_{-NL}^{NL} \hat{B}_1 \, dx && (4.110)\\
&= -\Omega_1^2 \int_{-NL}^{NL} (-4\zeta - \frac{1}{3}u^* + c_{1,0}) \, dx && (4.111)\\
&= -\Omega_1^2 \int_{-NL}^{NL} (-4\zeta - 4k^2 \mathrm{cn}^2\,(x,k) + 8k^2 - 4) \, dx. && (4.112)
\end{aligned}$$

There are two components to the Lax spectrum. We see that $K_1(\zeta) \geq 0$ on the first component $\zeta \in (-\infty, k^2 - 1]$, and $K_1(\zeta) \leq 0$ on the second component $\zeta \in [2k^2 - 1, k^2]$. In both cases equality is obtained only at the endpoitns, where $\Omega_1(\zeta) = 0$. Therefore, no stability conclusions can be drawn from $K_1(\zeta)$.

Let us calculate $K_2(\zeta)$. We have

$$\hat{T}_2 = T_2 + c_{2,1}T_1 + c_{2,0}T_0, \tag{4.113}$$

where as before $c_{2,0} = c_{2,1}(8k^2 - 4) - 56k^4 + 56k^2 - 16$ and $c_{2,1}$ is free. From

$$\det(\hat{T}_2 - \Omega_2 I) = 0, \tag{4.114}$$

another direct calculation gives

$$\Omega_2^2 = 16(8k^2 - 4 + 4\zeta - c_{2,1})^2(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3) = (4\zeta + 8k^2 - 4 - c_{2,1})^2\Omega_1^2. \tag{4.115}$$

We choose $c_{2,1}$ such that $4\zeta + 8k^2 - 4 - c_{2,1}$ has the same zero as the integral part of $K_1(\zeta)$. This choice of $c_{2,1}$ makes $K_2(\zeta)$ of definite sign on the Lax spectrum, and verifies orbital stability.

In fact, $4\zeta + 8k^2 - 4 - c_{2,1}$ is zero when $\zeta = 1 - 2k^2 + c_{2,1}/4$. Imposing that this sign change takes place in the gap (or on one of its edges) gives

$$1 - 2k^2 + c_{2,1}/4 \geq k^2 - 1, \tag{4.116}$$

and

$$1 - 2k^2 + c_{2,1}/4 \leq 2k^2 - 1. \tag{4.117}$$

This results in

$$4(3k^2 - 2) \leq c_{2,1} \leq 4(4k^2 - 2), \tag{4.118}$$

which has an entire interval of solutions for all values of the elliptic modulus $k \in (0, 1)$. Any choice of $c_{2,1}$ satisfying the above constraint makes $K_2(\zeta)$ of definite sign on the Lax spectrum.

Though not necessary for stability, let us calculate Ω_3 for illustrative purposes. We have

$$\hat{T}_3 = T_3 + c_{3,2}T_2 + c_{3,1}T_1 + c_{3,0}T_0. \tag{4.119}$$

Imposing u^* is stationary gives

$$c_{3,0} = c_{3,2}(-56k^4 + 56k^2 - 16) + c_{3,1}(8k^2 - 4) + 384k^6 - 576k^4 + 320k^2 - 64, \quad (4.120)$$

where $c_{3,2}$ and $c_{3,1}$ are free parameters. From

$$\det(\hat{T}_3 - \Omega_3 I) = 0, \quad (4.121)$$

another direct calculation gives

$$\Omega_3^2 = (16\zeta^2 + (32k^2 - 16 - 4c_{3,2})\zeta + 56k^4 - 56k^2 - 8k^2 c_{3,2} + 4c_{3,2} + 16 + c_{3,1})^2 \Omega_1^2. \quad (4.122)$$

Now, $c_{3,2}$ allows us to choose the coefficient of ζ and $c_{3,1}$ allows us to choose the constant term, hence, we have total control over the roots of the outside polynomial.

4.4.2 Genus 2: the Dubrovin-Novikov solution

Here we consider the genus two Lamé-Ince potential [30, 33, 51, 52]

$$u^* = -36\wp(x, g_2, g_3), \quad (4.123)$$

where $\wp(\cdot, g_2, g_3)$ is the Weierstrass elliptic function with invariants g_2 and g_3 [66]. Using the phase invariance of the KdV equation, we rewrite the above in the more convenient form [17, 66]

$$u^* = 36k^2 \mathrm{cn}^2(x, k). \quad (4.124)$$

Contrary to the genus one case, the solution $u(x,t)$ generated by u^* does not represent all periodic genus two solutions. In fact, it is considered the simplest periodic genus two solution, as noted by Dubrovin and Novikov, who integrated the KdV equation with u^* as an initial condition [33, 73]. It was later shown that the Dubrovin-Novikov solution is periodic in time as well [34]. We examine it here because it is a solution with genus greater than one for which explicit analysis is relatively

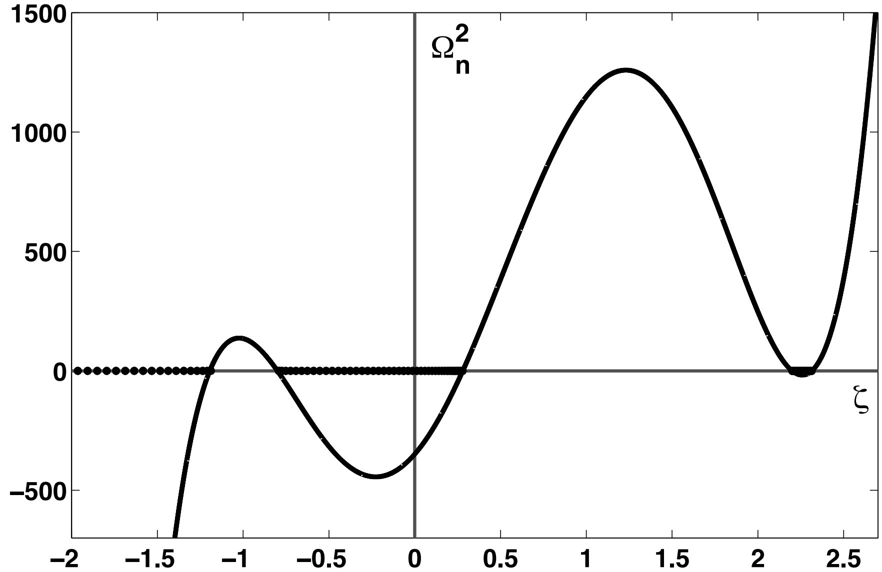


Figure 4.3: Ω_2^2 as a function of real ζ for the Dubrovin-Novikov solution with $k = 0.8$ in (4.127). The union of the dotted line segments on the horizontal axis is the numerically computed Lax spectrum using Hill's method with 81 Fourier modes and 49 different Floquet exponents, see [26, 25].

straightforward.

Imposing that u^* is a stationary solution of the second KdV equation gives

$$c_{2,0} = 424k^4 - 424k^2 + 64, \quad c_{2,1} = 40k^2 - 20. \tag{4.125}$$

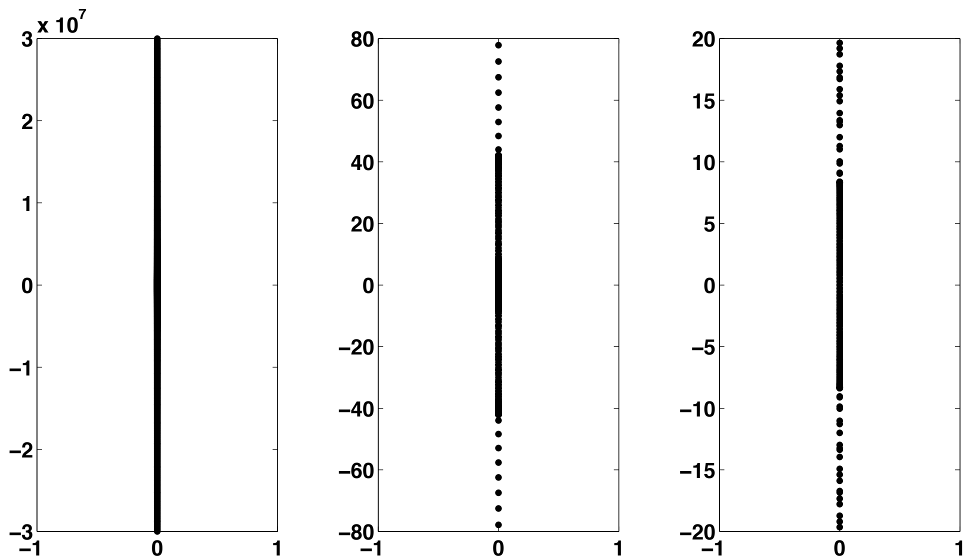
From

$$\det(\hat{T}_2 - \Omega_2 I) = 0, \tag{4.126}$$

we find

$$\Omega_2^2 = 256(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)(\zeta - \zeta_5), \tag{4.127}$$

where



(a) (b) (c)

Figure 4.4: (a) The numerically computed spectrum for the Dubrovin-Novikov solution with $k = 0.8$ using Hill's method with 81 Fourier modes and 49 different Floquet exponents, see [26, 25]; (b) A blow-up of (a) around the origin, showing a band of higher spectral density; (c) A blow-up of (b) around the origin, showing another band of even higher spectral density. The analytically predicted values for the band ends using (4.127) are $\pm 2i\sqrt{|\Omega_2^2(\zeta_2^*)|} \approx \pm 42.14i$ and $\pm 2i\sqrt{|\Omega_2^2(\zeta_4^*)|} \approx \pm 8.38i$, in agreement with the numerical results above.

$$(\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5) = (4k^2-2-2\sqrt{k^4-k^2+1}, 5k^2-4, 2k^2-1, 5k^2-1, 4k^2-2+2\sqrt{k^4-k^2+1}). \tag{4.128}$$

It is easily checked that all of the above roots are real and distinct for $k \in (0, 1)$. Therefore, the Lax spectrum is

$$\sigma_{L_2} = (-\infty, \zeta_1] \cup [\zeta_2, \zeta_3] \cup [\zeta_4, \zeta_5], \tag{4.129}$$

which is a confirmation of numerical results, see Fig. (4.3). Also, Ω_2 has two bands of increasingly higher density around the origin. This confirms the numerical results for the linear stability problem, see Figs. (4.4) and (4.5).

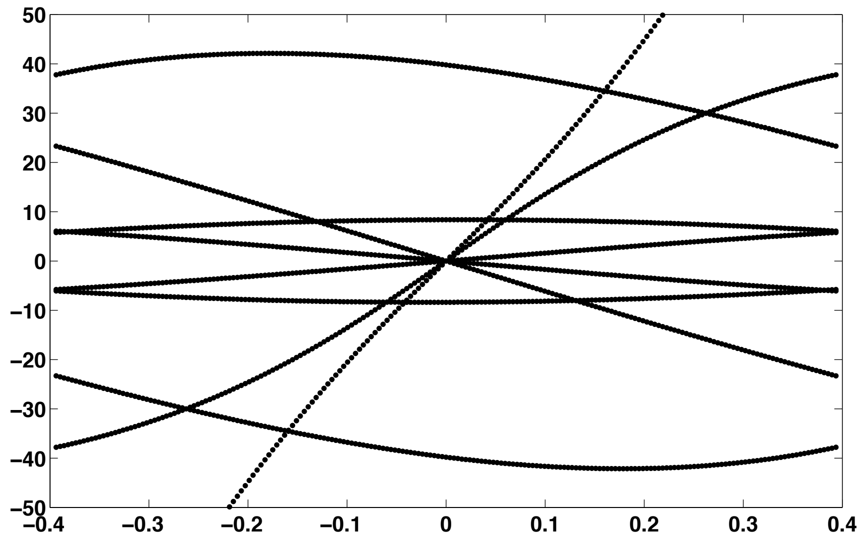


Figure 4.5: The imaginary part of λ_n as a function of μ, demonstrating the higher spectral density.

To examine nonlinear stability, we first calculate $K_2(\zeta)$:

$$K_2(\zeta) = -\Omega_2^2 \int_{-NL}^{NL} \hat{B}_2 \, dx \tag{4.130}$$

$$= -\Omega_2^2 \int_{-NL}^{NL} \left(16\zeta^2 + \left(\frac{4}{3}u^* - 4c_{2,1}\right)\zeta + \frac{1}{3}u^*_{xx} + \frac{1}{6}u^{*2} - \frac{1}{3}c_{2,1}u^* + c_{2,0}\right) dx. \tag{4.131}$$

As expected, the integral part of $K_2(\zeta)$ is a polynomial of degree two in ζ.

There are three components to the Lax spectrum. One can check that $K_2(\zeta) \geq 0$ on the first component $\zeta \in (-\infty, \zeta_1]$, $K_2(\zeta) \leq 0$ on the second component $\zeta \in [\zeta_2, \zeta_3]$, and $K_2(\zeta) \geq 0$ on the third component $\zeta \in [\zeta_4, \zeta_5]$. In all three cases equality is obtained only at the endpoints, where $\Omega_2(\zeta) = 0$. Therefore, no stability conclusions can be drawn from $K_2(\zeta)$.

For orbital stability we need to go two flows higher. Imposing that u^* is a stationary solution of the fourth KdV equation results in

$$
\begin{aligned}
c_{4,0} &= (-16192k^6 + 24288k^4 - 10656k^2 + 1280)c_{4,3} + (424k^4 - 424k^2 + 64)c_{4,2} \quad (4.132)\\
&\quad +467616k^8 - 935232k^6 + 677664k^4 - 210048k^2 + 21504,\\
c_{4,1} &= (-1176k^4 + 1176k^2 - 336)c_{4,3} + (40k^2 - 20)c_{4,2} \quad (4.133)\\
&\quad +30848k^6 - 46272k^4 + 26304k^2 - 5440,
\end{aligned}
$$

where $c_{4,2}$ and $c_{4,3}$ are free parameters.

From $\det(\hat{T}_4 - \Omega_4 I) = 0$ we find

$$\Omega_4^2 = p_4(\zeta)^2\Omega_2^2, \tag{4.134}$$

where

$$p_4(\zeta) = 16\zeta^2+(160k^2-4c_{4,3}-80)\zeta+1176k^4+(-40c_{4,3}-1176)k^2+20c_{4,3}+c_{4,2}+336. \tag{4.135}$$

Therefore, $c_{4,3}$ allows us to choose the coefficient of ζ and $c_{4,2}$ allows us to choose the constant term, giving us total control over the roots of $p_4(\zeta)$. Imposing that it has the same roots as the integral part of $K_2(\zeta)$ determines $c_{4,3}$ and $c_{4,2}$ such that $K_4(\zeta)$ is of definite sign, verifying orbital stability.

Chapter 5

ORBITAL STABILITY IN INTEGRABLE HAMILTONIAN SYSTEMS: GENERAL ALGORITHM

As we saw in our analysis of the finite-genus solutions of the KdV equation, the (nonlinear) orbital stability of a stationary solution of the n-th KdV equation depends heavily on its t_n-spectral stability. For a general integrable Hamiltonian system

$$u_t = JH'(u), \tag{5.1}$$

we consider the class of stationary solutions

$$J\hat{H}_n' = 0, \tag{5.2}$$

with all analogous quantities defined as for the KdV equation in Chapter 4. We outline the connection between t_n-spectral stability and orbital stability for finite-genus solutions of a general integrable system. We also comment on its implications for the general periodic initial-value problem for any integrable Hamiltonian system.

Note: We must keep in mind that when considering nonlinear stability, the space of allowed perturbations will depend upon the equation in question, for example, there are restrictions imposed by the kernel of the operator J.

Step 1: Determine the Lax spectrum

Since we have assumed that equation (5.1) is integrable, the n-th equation in the hierarchy is equivalent to the compatibility of two linear ordinary differential systems:

$$\psi_x = X\psi, \tag{5.3}$$

$$\psi_{t_n} = \hat{T}_n\psi. \tag{5.4}$$

Here, the above matrices can be of any size, say M. As u^* does not depend on t_n, we can separate variables in (5.4)

$$\psi(x,t_n) = e^{\Omega_n t_n}\Psi(x), \tag{5.5}$$

where Ω_n is determined by

$$\det(\hat{T}_n - \Omega_n I) = 0, \tag{5.6}$$

which specifies Ω_n as an M-valued function of ζ. The eigenvector Ψ is determined by

$$(\hat{T}_n - \Omega_n I)\Psi = 0. \tag{5.7}$$

To ensure that $\psi(x,t_n)$ is bounded as a function of x, one arrives at a condition

$$\mathrm{Re}\,\langle S(u,\gamma,\Omega_n)\rangle = 0, \tag{5.8}$$

with $\langle\cdot\rangle = \frac{1}{2L}\int_{-L}^{L}\cdot\ dx$. Here, $S(u,\gamma,\Omega_n)$ depends upon the elements of the matrices from the Lax pair. Equation (5.8) determines the Lax spectrum σ_{L_n} explicitly. Once σ_{L_n} is determined, the set of all Ω_n values follows from

$$\det(\hat{T}_n - \Omega_n I) = 0. \tag{5.9}$$

How readily condition (5.8) lends itself to investigation is determined by the equation in question.

Step 2: The squared eigenfunction connection

For all integrable equations, there exists a squared eigenfunction connection relating solutions to the linear stability problem for the n-th equation in the hierarchy, $w = e^{\lambda_n t_n} W(x)$, and ψ:

$$w = \sum_{i,j} A_{i,j} \psi_i \psi_j, \tag{5.10}$$

for some matrix A [2]. A can be found by direct substitution.

Step 3: Spectral stability

From the squared eigenfunction connection it follows that

$$\lambda_n = 2\Omega_n(\zeta), \quad W(x,\zeta) = \sum_{i,j} A_{i,j} \Psi_i(x,\zeta) \Psi_j(x,\zeta).$$

This provides a parameterization of the stability problem in terms of the Lax spectral parameter ζ. Therefore, if $\Omega_n \in i\mathbb{R}$ then u^* is t_n-spectrally stable. Of course, it must first be shown that all solutions to the linear stability problem are obtained through the squared eigenfunction connection. Again, this will depend upon the equation in question.

Step 4: Calculate K_n

If a completeness result such as the SCS basis lemma [45] exists for the equation in question, then using the squared eigenfunction connection, one can explicitly calculate K_n in terms of the Lax parameter ζ:

$$K_n(\zeta) = \left\langle W, \hat{H}_n''(u^*) W \right\rangle. \tag{5.11}$$

If the entries of the Lax operators are polynomials in ζ, then $K_n(\zeta)$ will be a polynomial in ζ as well. In general, $K_n(\zeta)$ will not be of fixed sign on the entire Lax spectrum.

Step 5: Riemann surface reductions

When evaluated at a stationary solution of a lower-order flow, all of the higher-flows become linearly dependent. Therefore, the essential parts of the Riemann surface calculations from the KdV equation carry over, and one obtains

$$\det(p_m(\zeta)\hat{T}_n - \Omega_m I) = 0 \Longrightarrow \Omega_m = p_m(\zeta)\Omega_n, \tag{5.12}$$

where $p_m(\zeta)$ is a polynomial of fixed degree in ζ. Furthermore, since each member of the Lax hierarchy depends on increasing powers of ζ, we should have some control over the form $p_m(\zeta)$ takes, using the free parameters $c_{m,n}, \ldots, c_{m,m-1}$.

Step 6: Higher-order K_m

From the Riemann surface relations one can show that for all $m > n$

$$K_m = p_m(\zeta)K_n. \tag{5.13}$$

Therefore, in order to conclude orbital stability, one must show that it is possible to go high enough in the hierarchy such that there exists constants $c_{m,n}, \ldots, c_{m,m-1}$ which make $K_m(\zeta)$ positive on the Lax spectrum. Again, the details will depend on the equation in question. One conclusion is immediately clear. If the spectral problem for ζ associated with the first member of the Lax hierarchy is self-adjoint, the analysis is much simpler. In that case $\sigma_{L_n} \subset \mathbb{R}$. Therefore, $K_n(\zeta)$ and $p_m(\zeta)$ are polynomials of the real variable ζ. Choosing the constants $c_{m,n}, \ldots, c_{m,m-1}$ such that $K_m(\zeta)$ is of fixed sign on the entire Lax spectrum now becomes a task in root finding, and orbital stability follows if we have sufficient control over $p_m(\zeta)$.

Let us consider some implications of the above method. Whether or not the set of stationary solutions is dense in a more general class of solutions depends on the equation in question. When such a relationship does exist, we expect the stability of the set of stationary solutions to suggest the stability of its closure. Making this statement more exact would require a diversion into the theory of infinite-genus solutions [36, 71]. Though an interesting path for future investigation, we do not pursue it here. In less rigorous terms, we make the following definition:

Definition. If the genus n solutions of a periodic initial-value problem are t_n-spectrally stable for all n, we say the periodic initial value problem with generic periodic initial data exhibits *implicit spectral stability*.

As we saw above, it is possible to carry out an analysis similar to what we did for the KdV equation on any integrable Hamiltonian system. In all such cases, there is a direct connection between the t_n-spectral stability of a genus n solution and the existence of a Lyapunov function. In finite dimensions, it is well known that all (non-degenerate) elliptic critical points are orbitally stable, since the Hamiltonian serves as a Lyapunov function [87]. Our analysis leads us to conjecture on the generalization of this idea to integrable Hamiltonian systems in infinite-dimensions:

Conjecture. Suppose (5.1) is a completely integrable Hamiltonian system and that the class of stationary solutions associated with the nonlinear hierarchy is dense in the set of periodic functions. Then the implicit spectral stability of the periodic initial-value problem implies the (nonlinear) orbital stability of the periodic initial-value problem, defined in an appropriate function space.

Of course, there is some ambiguity as to what is meant by orbital stability of a general periodic solution. One must first generalize the norm used for the finite-genus solutions to the infinite-genus case. Since each periodic solution can be obtained through an appropriate infinite-genus solution, orbital stability of a generic periodic solution could be defined through the orbital stability of the corresponding infinite-genus solution.

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